Homework #4
Economics D11-1
Due Thursday, October 21
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The planner maximizes, by choice of $\{k_{t+1}, c_t\}$, $t = 0, 1, \ldots$

$$\sum_{t=0}^{\infty} \tilde{\beta}^t u(\tilde{c}_t)$$

subject to

$$\begin{aligned} \tilde{c}_t + \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t &\leq \tilde{f}(z_t, \tilde{k}_t) = z_t^{(1-\theta)} \tilde{k}_t^{\theta}, \\ \tilde{c}_t &\geq 0, \ \tilde{k}_{t+1} \geq 0, \\ z_t &= \exp(x) z_{t-1}, \ x > 0, \end{aligned}$$

for all t, and $z_{-1}, \tilde{k}_0 > 0$, both given. Here, $\delta, \theta \in (0,1)$, and $u : K \to K$, where $K = \Re_+$,

$$\begin{aligned} u(\tilde{c}) &= \frac{\tilde{c}^{1-\gamma} - 1}{1-\gamma}, \ \gamma \neq 0, \\ &= \log(\tilde{c}), \ \gamma = 0. \end{aligned}$$

1. Under what restrictions on γ is u strictly concave? bounded above? bounded below?

2. Note that the production function shifts over time, exogenously (i.e., independent of anyone's decisions). But SL assume time-invariant functions (i.e., they include no time subscript on their functions, and no variable that moves exogenously with time, except on page 106). Show how, by adopting the following scaling of variables,

$$c_t = \tilde{c}_t / z_t, \ k_t = \tilde{k}_t / z_t,$$

and expressing the problem as one of choosing $\{c_t, k_{t+1}\}$, $t = 0, 1, \ldots$, it can be converted into the kind of problem covered by SL. Write this problem in recursive form:

$$v(k) = \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta v(k'), \text{ for all } k \in K$$

where $\Gamma : K \to K$. Be sure to define precisely Γ, β and f here.

(a) Suppose u is not bounded below, but is bounded above. Show that one way to make u bounded is to replace K by $\bar{K} = [\varepsilon, \infty)$, where ε is a small, but positive, number. Does this substantively modify the problem? Does this version of the problem satisfy assumptions A4.3-4.9?

(b) From here on, suppose u is bounded below, and work with $\hat{K} = [0, \bar{k}]$, where $\bar{k}$ is a suitably chosen upper bound. We know that k_t converges monotonically to a unique point, for all $\tilde{k}_0 > 0$. Why? Prove that the capital to output ratio, $\tilde{k}_t/\tilde{y}_t$, converges monotonically to a unique fixed constant for all $\tilde{k}_0 > 0$. Consider the growth rate of output and capital, $\tilde{k}_{t+1}/\tilde{k}_t$, and $\tilde{y}_{t+1}/\tilde{y}_t$. Show that these converge to a unique fixed constant for all $\tilde{k}_0 > 0$, but that convergence is *not* guaranteed to be monotonic. (That $\tilde{y}_{t+1}/\tilde{y}_t$ converges to a constant, greater than one, reflects that this is an *exogenous* growth model: fundamentally, the growth reflects the move up in z_t which just happens.) Let the (net) savings rate be defined as follows:

$$s_t = \frac{\tilde{k}_{t+1} - \tilde{k}_t}{\tilde{y}_t - \delta\tilde{k}_t}.$$

Prove that this converges to a unique point for all $\tilde{k}_0 > 0$, but that convergence is not guaranteed to be monotonic. An interesting feature of Japan's postwar economic performance is that output growth and saving display a hump shape: they are small in the early postwar period, then rise sharply to a peak and fall again. A quantitative question is whether the model account for this hump-shaped savings and growth rates, and for the timing of the hump.

(c) Let $\theta = .36$, $\delta = .07$, $x = .029$, $\tilde{\beta} = 1/1.03$, $u(\tilde{c}) = \log(\tilde{c})$. (Don't worry that u does not quite satisfy the assumptions made above.) Let

$$g(k) = \arg\max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ all } k \in \hat{K}.$$

Define g^l and g^q as the linear and quadratic Taylor series expansions of g about the steady state value of k, respectively. I suggest you calculate the parameters of these functions using a hand calculator. Then, do the following calculations using a spreadsheet program. Produce a table with 71 rows and 7 columns. Let the first row correspond to date 0. In column 1, report the date 0 to date 70 observations on $\tilde{k}_t$, assuming $\tilde{k}_0$ is on a steady state growth path, i.e., $\tilde{k}_0/z_0$ is a steady state. In columns 2,3,4, report $\tilde{k}_t$, s_t, $\tilde{y}_{t+1}/\tilde{y}_t$ for dates 0 to 70 assuming that in date 0, $\tilde{k}_0$ is only 12 percent of what it would be if it were on a steady state growth path. Compute these numbers using g^l. In columns 5, 6, 7 report the same numbers, except use g^q instead. Do you get a hump-shaped saving and output growth rate like that observed in postwar Japan? Is there any meaningful difference between your results in columns 2,3,4 and 5,6,7? What do you infer from this about how close g^l is to g?

3. Replace the utility function by $u(\tilde{c}_t - c_* z_t)$ (see the Japan article, for an interpretation of c_*.)

   (a) Suppose $u : \Re_+ \rightarrow \Re_+$ is bounded below. Scale the variables as before, and express the problem as a sequence problem. Show that there is a positive value of k, say k_*, such that if k ever slips below it, it is not feasible to maintain consumption at the minimum subsistence level, ie., the one required for utility to be defined: $c_t = c_*$. Explain why replacing K with $K' = [k_*, \bar{k}]$, where $\bar{k}$ is a suitably chosen finite number, does not substantively change the nature of the problem. Are all of A4.3-A4.7 satisfied with this K? Is the simple condition, $\tilde{k}_0$ enough to guarantee the existence of a solution to the sequence problem? Are the results in 2.3 above still valid for this modified economy?

   (b) Let $u(\cdot) = \log(\cdot)$. (Again, ignore the inconsistency with the boundedness assumption on u.) Let $c_* = .72$. Redo the calculations in 2.4 above. Do you get a hump? Is there a big difference between g^l and g^q?

   (c) In the experiments you did above, the capital stock always returned to the steady state growth path it would have been on had it started in steady state in date 0. Suppose you did the same experiment in the endogenous growth model in homework 5 (you are *not* expected to actually do this for this homework!). That is, you cut the physical capital stock to 12 percent of its steady state level in date 0. Do you think that, like in the exogenous growth model, the physical capital stock would eventually converge back to its unperturbed steady state growth path? You are only expected to use your intuition here, not actually to do any calculations.

(d) In my Japan article, I used much more computer intensive methods to work out the implications of the growth model (see the appendix to the article). Relative to the much simpler strategy of using g^l, was my effort worth it?