Homework #6
Economics D11-1, Fall 1999
Due Thursday, November 11.
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1. The following three sector exogenous growth model was recently proposed in Kongsamut, Rebelo and Xie ('Beyond Balanced Growth', National Bureau of Economic Research Working Paper number 6159, available on the web at http://www.nber.org/), to explain several key features of long-run (i.e., 1869-1990) growth: (i) K/Y is roughly constant, where K denotes the aggregate stock of capital, and Y denotes aggregate output, (ii) K grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable share in the economy.

   Consider the following technology. Agricultural output, A_t, is produced using the following production function:

   $$A_t = B_A K_{At}^{\alpha}(N_{At}z_t)^{1-\alpha},$$

   where $B_A > 0$, $0 < \alpha < 1$, z_t denotes the state of technology, and K_A and N_A denote capital and labor allocated to agriculture. The state of technology evolves according:

   $$z_t = exp(g)z_{t-1},\ g > 0.$$

   The manufacturing sector produces output that can be converted into consumption goods, C_t, or capital goods:

   $$C_t + K_{t+1} - (1-\delta)K_t = K_{Mt}^{\alpha}(N_{Mt}z_t)^{1-\alpha},$$

   in obvious notation. Finally, services, S_t, are produced according to:

   $$S_t = B_S K_{St}^{\alpha}(N_{St}z_t)^{1-\alpha}.$$

   Suppose that at a point in time, households supply K_t units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

   $$N_{At} + N_{Bt} + N_{Mt} = 1,\ K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be P_{At}. Let the price of a service be P_{St}. Finally, let r_t and w_t denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

(a) Show that competitive behavior by firms implies: $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$, $P_{At} = 1/B_A$, $P_{St} = 1/B_S$,

$$C_t + K_{t+1} - (1-\delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^{\alpha} z_t^{1-\alpha}. \quad (1)$$

One can treat the latter as a 'reduced form' expression for the economy's resource constraint.

(b) Derive an expression for the rate of return on capital. If K_t grows at the rate, g, will the rate of return on capital be constant?

(c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{\left[\left(A_t - \bar{A}\right)^{\eta} C_t^{\gamma} (S_t + \bar{S})^{\theta}\right]^{1-\sigma}}{1-\sigma}, \ \eta + \gamma + \theta = 1, \ \eta, \gamma, \theta, \sigma > 0.$$

i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.

ii. Derive the household's intertemporal Euler equation associated with capital.

iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \ \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \quad (2)$$

iv. Substitute out for $A_t - \bar{A}$ and $S_t + \bar{S}$ in terms of C_t in the household's intertemporal Euler equation, to get an expression in terms of the growth rate of C_t and the rate of return on capital alone.

v. Show that in general, there is no reason to expect the economy to converge to a 'balanced growth path', i.e., one in which A_t, C, S_t, K_t, Y_t all grow at a constant rate. (Here, $Y_t = K_t^{\alpha} {z_t}^{1-\alpha}$.) Hint: note that if you scale A_t, M_t, K_t, S_t by z_t and work with the scaled versions of (2) and (1) and the household's intertemporal Euler equation, you cannot get rid of z_t.

vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \tag{3}$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace A_t and S_t in (1) by $A_t - \bar{A}$ and $S_t - \bar{S}$, respectively without changing (1). Then, define $a_t = A_t - \bar{A}$ and $s_t = S_t - \bar{S}$ everywhere and impose (2), that a_t and s_t are each proportional to M_t.

vii. Explain why it is that under (3), the following are true:

$$\begin{aligned} \frac{K_t}{z_t} &\to k^*, \ \frac{A_t - \bar{A}}{z_t} \to a^*, \ \frac{S_t + \bar{S}}{z_t} \to s^* \\ \frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} &\to exp(g), \end{aligned}$$

where k^*, a^*, s^* are finite, positive, constants.

viii. Provide a simple formula for k^*.

ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?

x. What does the model imply for $P_{k'}$, the consumption price of capital, along the growth path.

2. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \;\; M_t > 0, \;\; 0 < \alpha < 1,$$

where M_t is a scalar such that for $i > M_t$, $x_t(i) = 0$. To produce $x_t(i)$ units of the i^{th} intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2)$$

units of capital if $x_t(i) > 0$ and zero units of capital if $x_t(i) = 0$. The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t,$$

where k_t is the beginning-of-period t aggregate stock of capital. The initial capital stock, $k_0 > 0$, is given. The resource constraint is:

$$c_t + I_t \le y_t,$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$, subject to the various constraints. You may assume that efficiency is consistent with $x_t(i) = \bar{x}_t$ for $i \in (0, M_t)$.

(a) Show that the planning problem for the Romer economy coincides with the planning problem for the Ak model. In particular, show that the problem can be written,

$$max_{\{k_{t+1} \in B(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k, k') = max_{c_t, \bar{x}_t, M_t} \frac{c_t^{1-\gamma}}{1-\gamma} = \frac{[(A + 1 - \delta) k - k']^{1-\gamma}}{1-\gamma},$$

and $A = (2 - \alpha)\left(\frac{\alpha}{2-\alpha}\right)^{\frac{\alpha}{2}}$. In addition to verifying the form of F, show what B is.

(b) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.

(c) The problem with monopoly power is that it results in an inefficiently low level of activity. In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let $p(i)x(i)$ be the revenues of the i^{th} monopolist in the absence of taxes or subsidies. A subsidy rate, τ_t, raises the revenues of the i^{th} monopolists to $p(i)x(i)(1+\tau_t)$. The total cost, G_t, to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose G_t is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1-\delta)k_t = r_t k_t + w_t n_t - T_t,$$

where T_t represents taxes paid by the representative household to the government. Suppose the government balances its budget period by period:

$$T_t = G_t.$$

Find the subsidy rate, τ_t, that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies induced by having to raise the revenues needed to finance the subsidy to monopolists. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. So, the problem of 'fixing' the inefficiency in the Romer model is actually more complicated than this question makes it out to be.