Homework #7
Economics D11-1, Due Thursday, November 18
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1. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at t is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, w_t. They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \; c_{t+1}^t \leq r_{t+1} k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are r_t and w_t, respectively.

   (a) Define a sequence of markets equilibrium. Provide expressions for w_t and r_t in terms of k_t.

   (b) Consider a steady state equilibrium in which the aggregate stock of capital, the consumption of each period's young, and the consumption of each period's old are all constants. Time starts up in period 0, with the initial old generation owning the capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1+\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

   Interpret this expression. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

   (c) Show that, for parameter values where $r_{k,t} < 1$, the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibiurm consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?

2. (Matsuyama, *Econometrica* article in packet.) The model can be viewed as an extension of the Romer model. We will use the same strategy as the one used in class, by first studying the partial equilibrium of the firm sector, and then going to general equilibrium by bringing in households. As before, since the partial equilibrium of the firm sector is static, in this part of the analysis we do not use time subscripts.

   As in the Romer model, final good firms are competitive and make use of the linear homogeneous production function, $y = n^{1-\alpha} \int_0^M x(i)^\alpha di$, $0 < \alpha < 1$, where n is labor, $x(i)$ is the i^{th} intermediate input, and M is the variety of intermediate goods available. Final good firms take the wage rate, w, and the i^{th} intermediate good price, $p(i)$, as given and beyond their control. Profit maximization leads to first order conditions:

$$(1-\alpha)y/n = w, \ \alpha n^{1-\alpha} x(i)^{\alpha-1} = p(i), \ 0 \leq i \leq M.$$

   The differences between the Romer and the Matsuyama models center on the intermediate good firms. Matsuyama specifies that there are two types of intermediate good firms: those that produce old products, $x(i)$, $0 \leq i \leq M_{-1}$, and those that produce new products, $x(i)$, $M_{-1} < i \leq M$. Here, M_{-1} is the variety of goods in existence in the previous period. We suppose:

$$M \geq M_{-1}.$$

   When $M = M_{-1}$, then there are no new goods in the current period.

   Intermediate good firms producing old products are competitive (i.e., they take their output price as invariant to their own decisions) and firms producing new products (if there are any) produce monopolistically (i.e., they take into account the demand curve for their product).

To produce $x(i)$ units of the intermediate good, $h(x(i); M_{-1})$ units of capital are needed, where

$$h(x(i); M_{-1}) = \begin{cases} x(i) + F, \text{ if } M_{-1} < i \leq M \text{ and } x(i) > 0 \\ 0, \text{ if } x(i) = 0, \ 0 \leq i \leq M \\ x(i), \ 0 \leq i \leq M_{-1}. \end{cases}$$

Thus, to produce a new good requires paying a fixed cost, $F > 0$. We can think of the introduction of a new good as an innovation, so that F is the cost of innovating. The given specification of h indicates that the fixed cost only applies when a good is first introduced. As in the Romer model, monopoly power gives innovators the profits they need to cover the fixed costs of innovating.

We will consider a symmetric equilibrium, in which $x^c = x(i)$, $p^c = p(i)$ for $0 \leq i \leq M_{-1}$ and, *in case* $M > M_{-1}$, then $x^m = x(i)$, $p^m = p(i)$ for $M_{-1} < i \leq M$. Here, $x^m > 0$ is the profit maximizing level of output that a monopolist who ignores F would choose. The *actual* level of output that an innovator produces is either x^m or zero, whichever yields a higher level of profits.

Take M_{-1}, K, and n as given, where K denotes the capital stock. The variables in the firm sector that are to be determined are

$$[p^m, x^m, p^c, x^c, y, w, r, M - M_{-1}].$$

(a) Show:

$$x^m = \left[\frac{\alpha^2}{r}\right]^{\frac{1}{1-\alpha}} n.$$

(b) Show:

$$p^m = \frac{r}{\alpha}, \ p^c = r.$$

(c) Show, using the previous result and the demand curves for the intermediate inputs that,

$$\frac{x^c}{x^m} = \alpha^{\frac{1}{\alpha-1}}, \ \frac{p^c x^c}{p^m x^m} = \alpha^{\frac{\alpha}{\alpha-1}} \equiv \theta > 1.$$

(d) Because there is free entry of monopolists, monopoly profits cannot be positive in equilibrium. If a potential monopolist expects to make negative profits by innovating, then $M = M_{-1}$, i.e., there will be no innovation. If $M > M_{-1}$, so that there is innovation, then monopoly profits are zero because of free entry. Explain why this corresponds to

$$x^m \leq \frac{\alpha}{1-\alpha}F, \quad \left[x^m - \frac{\alpha}{1-\alpha}F\right][M - M_{-1}] = 0.$$

(e) Show that clearing in the capital market requires:

$$k = x^c + \frac{M - M_{-1}}{M_{-1}}(x^m + F).$$

where

$$k = \frac{K}{M_{-1}}.$$

(f) Show:

$$x^c = \alpha^{\frac{1}{\alpha-1}} x^m = \min\left\{k, \frac{\theta}{1-\alpha}F\right\}.$$

(Hint: the first equality was shown previously. The second equality is based on the first equality and on the previous two results that you derived.)

(g) Show:

$$\frac{M - M_{-1}}{M_{-1}} = \begin{cases} 0, \text{ if } k < \frac{\theta F}{1-\alpha} \\ \frac{1-\alpha}{F}\left[k - \frac{\theta F}{1-\alpha}\right], \text{ if } k > \frac{\theta F}{1-\alpha} \end{cases}$$

(Hint: use the previous result, the condition $M \geq M_{-1}$, and the capital market clearing condition.) Thus, innovation (i.e., $M - M_{-1} > 0$) will occur only if there is a lot of physical capital relative to the amount of variety of goods (i.e., k is large). No innovation will occur otherwise. This makes sense because when capital is relatively abundant, we can expect its rental rate to be relatively low, increasing the incentive to innovate.

(h) Show:

$$\frac{y}{M_{-1}} = \begin{cases} n^{1-\alpha}k^{\alpha}, \text{ if } k < \frac{\theta F}{1-\alpha} \\ n^{1-\alpha}Ak, \text{ if } k > \frac{\theta F}{1-\alpha} \end{cases}$$

where $A = (\theta F/(1-\alpha))^{\alpha-1}$. Note how this aggregate production relation is continuous in k.

(i) Show:

$$r = \begin{cases} \alpha n^{1-\alpha} k^{\alpha-1}, \text{ if } k < \frac{\theta F}{1-\alpha} \\ \alpha n^{1-\alpha} A, \text{ if } k > \frac{\theta F}{1-\alpha} \end{cases} \tag{1}$$

By hand, draw a graph with r on the vertical axis and k on the horizontal axis. Let k vary from nearly zero to a value exceeding $\theta F/(1-\alpha)$. Do the same for $\beta\left[r+1-\delta\right]$. You need only depict the qualitative behavior of these functions. Note how these functions are continuous in k.

(j) Suppose households supply $n_t = 1$ inelastically. Let their preferences be $\sum_{t=0}^{\infty} \beta^t \log(c_t)$, and let their budget constraint be

$$c_t + I_t \leq w_t + r_t K_t,$$

where $K_{t+1} = (1-\delta)K_t + I_t$. The resource constraint for this economy is:

$$c_t + I_t \leq y_t.$$

Define a sequence of markets equilibrium.

(k) For obvious reasons, Matsuyama refers to a state in which $k < \theta F/(1-\alpha)$ as a 'Solow regime' and a state in which $k > \theta F/(1-\alpha)$ as a 'Romer regime'. Let

$$G = \beta\left[\alpha\left(\frac{\theta F}{1-\alpha}\right)^{\alpha-1} + 1 - \delta\right].$$

i. Suppose $G < 1$. Show that there is a steady state value of k in the Solow regime, call it k^*. That is, for any $M_{-1} > 0$ if the initial stock of capital is $K_0 = k^* M_{-1}$, then there is a no growth equilibrium with

$$K_{t+1} = K_0, \text{ for } t = 0, 1, 2, \ldots .$$

Note that in this steady state equilibrium, there is never any innovation. This regime is more likely the larger is F, which makes sense because this represents the fixed cost of innovation.

Let $M_{-1} = 1$, $\beta = 1/1.03$, $\alpha = 0.36$, $F = 100$, so that $G = 0.8833$, after rounding. Compute k^*. If K_0 is in a sufficiently small neighborhood of $k^* M_{-1}$, show that there exists an equilibrium in which $lim_{t \to \infty} K_t = k^* M_{-1}$. (Hint: consider the household's intertemporal Euler equation after substituting out for the rental rate of capital using (1). Use the following facts: (i) for K_t sufficiently close to $k^* M_{-1}$, the Taylor series expansion of this equation about $K_t = K_{t+1} = K_{t+2} = k^* M_{-1}$ is an arbitrarily good approximation to this equation, and write this as

$$V_0 \tilde{K}_t + V_1 \tilde{K}_{t+1} + V_2 \tilde{K}_{t+2} = 0, \ t = 0, 1, 2, ...$$

where $\tilde{K}_t \equiv K_t - k^* M_{-1}$; (ii) the set of solutions to a linear difference equation like this is given by $\tilde{K}_t = (\tilde{K}_0 - a)\lambda_1^t + a\lambda_2^t$, for $t = 0, 1, 2, ...$, for arbitrary a, where the λ_i's solve:

$$V_0 + V_1 \lambda_i + V_2 \lambda_i^2 = 0, \ i = 1, 2;$$

(iii) an equilibrium must also satisfy a particular transversality condition.)

ii. Suppose $G > 1$. Show that there is a steady state value of k in the Romer regime, call it k^{**}. That is, given $M_{-1} > 0$ and $K_0 > 0$, there is an equilibrium in which

$$\frac{K_t}{M_{t-1}} = k^{**}, \ \frac{c_{t+1}}{c_t} = \frac{K_{t+1}}{K_t} = \frac{M_t}{M_{t-1}} = G, \ t = 0, 1, 2, ...$$

Provide a formula for computing k^{**} and verify $k^{**} > \theta F/(1-\alpha)$.

iii. Think about the possibility of equilibria that fluctuate between the Romer and Solow regimes: in a Solow regime the relatively low amount of physical capital results in a high rental rate on capital. This discourages innovation but encourages capital accumulation (just like in the neoclassical growth model when you are below steady state). When capital becomes relatively abundant (so that $k > \theta F/(1-\alpha)$)

then innovators have an incentive to enter: the Romer regime begins and M starts to grow. If M grows fast enough relative to K (this will depend upon parameter values) then k is driven back down towards the Solow regime, and the process starts all over again. Along such a growth path there will be alternating periods of fast growth during which there is no innovation and slow growth, during which there is a lot of innovation. Interestingly, the same conditions that encourage high growth in capital and output, i.e., a high rental rate of capital, discourage innovation. This model generates all sorts of empirical hypotheses that would be interesting to test (patent applications come in bursts, and at times of low growth?).