Christiano
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Introducing Hours Worked into the One-Sector Optimal Growth Model

1 The Canonical Model

Consider the following sequence problem (SP):

$$v^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}), \tag{1}$$

subject to $x_0 = x_0^* > 0$. Also,

$$\begin{aligned} F &: A \to \Re,\ \Gamma : X \to X,\ X \subseteq \Re^l, \\ A &= \{(x, x') \in X \times X : x' \in \Gamma(x)\}. \end{aligned} \tag{2}$$

The following functional equation (FE) is associated with the recursive formulation of this problem:

$$v(x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta v(x'),\ x \in X. \tag{3}$$

The associated policy rule is:

$$g(x) = \arg\max_{x' \in \Gamma(x)} F(x, x') + \beta v(x'),\ x \in X. \tag{4}$$

Stokey and Lucas (SL) make the following assumptions on (X, F, Γ, β) :

A4.3 $X \subseteq \Re^l$ is convex, and the correspondence, $\Gamma : X \to X$ is nonempty, compact-valued and continuous.

A4.4 The function $F : A \to \Re$ is bounded and continuous, and $0 < \beta < 1$.

A4.5 For each fixed y, $F(\cdot, y)$ is strictly increasing in its first arguments.

A4.6 Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.

A4.7 F is strictly concave.

A4.8 Γ is convex. That is, $x, x' \in X$, $y \in \Gamma(x)$, $y' \in \Gamma(x')$ implies $\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1-\theta)x')$ for all $0 \leq \theta \leq 1$.

A4.9 F is continuously differentiable on the interior of A.

SL establish the following: Under A4.3-A4.4, there exists a unique function v that solves (FE), and they provide an algorithm for finding it (see Theorem 4.6). Moreover, the solution to FE is identical to the solution to SP in the sense that (i) $v = v^*$ and (ii) the sequence $\{x_0 = x_0^*; x_{t+1}, t \geq 0\}$ satisfies $x_{t+1} = g(x_t)$, $t \geq 0$ if, and only if, it also attains the supremum in (SP). Under A4.3-A4.6, v is strictly increasing (Theorem 4.7); under A4.3-A4.4, A4.7-A4.8 v is strictly concave and g is continuous and single-valued (Theorem 4.8); under A4.3-4.4, A4.7-A4.9 v is continuously differentiable at all $x \in int(X)$ such that $g(x) \in int(\Gamma(x))$ (Theorem 4.11).

2 The Growth Model With Variable Hours Worked

These notes establish that the properties just described are robust to letting hours worked be a choice variable in the optimal growth model. The sequence representation of the problem is:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{5}$$

subject to:

$$\begin{aligned} c_t + k_{t+1} &\leq f(k_t, n_t) \\ k_{t+1}, c_t &\geq 0 \\ 0 &\leq n_t \leq 1 \\ k_0 &= k_0^* \end{aligned} \tag{6}$$

for $t = 0, 1, 2, ...$ The assumptions on preferences and technology are:

1. Preferences:

   P1 u is twice continuously differentiable, strictly concave, strictly increasing in c, decreasing in n for $c > 0$, $0 < n < 1$.

P2 $u_c \to \infty$ as $c \to 0$ for $0 \leq n < 1$

P3 $u_n \to -\infty$ as $n \to 1$ for $c > 0$

P4 $u_n \to$ finite as $c \to 0$ for $0 \leq n < 1$

P5 $u_c \to$ finite as $n \to 1$ for $c > 0$.

P6 if $c \to 0$ and $(-u_n/u_c)$ is decreasing, then $u_c \to \infty$.

2. Technology:

T1 f is strictly increasing, concave, homogeneous of degree one, twice continuously differentiable for $k > 0$, $0 < n < 1$,

T2 $f_k \to \infty$ as $k \to 0$, $0 < n \leq 1$,

T3 $f_n \to \infty$ as $n \to 0$, $k > 0$,

T4 $f_k \to < 1$ as $k \to \infty$, $0 < n \leq 1$,

T6 $f(0, n) = 0, 0 < n \leq 1$

T7 $f_k > 0$, $f_n > 0$, $f_{kk} < 0$, $f_{nn} < 0$, $f_{kn} > 0$, for $k > 0$, $n > 0$.

Assumption P6 is used to handle situations in which $c \to 0$ and $n \to 1$ simultaneously. This is not addressed, for example, by assumption P2. The following u satisfies assumptions P1-P6:

$$\begin{aligned} u(c,n) &= \begin{cases} \frac{[c(1-n)^{\sigma}]^{(1-\gamma)}}{1-\gamma}, \text{ for } \gamma \neq 1 \\ \log(c) + \sigma \log(1-n), \text{ for } \gamma = 1 \end{cases}, \\ \text{with } \sigma &> 0, \gamma > 0, \sigma(1-\gamma) < 1, \gamma - \sigma(1-\gamma) > 0. \end{aligned}$$

To see this, note that

$$\begin{aligned} u_c &= \frac{\tilde{u}}{c},\ u_{cc} = -\gamma\frac{\tilde{u}}{c^2},\ u_{cn} = -\sigma(1-\gamma)\frac{\tilde{u}}{c(1-n)}, \\ u_n &= -\sigma\frac{\tilde{u}}{1-n},\ u_{nn} = [\sigma(1-\gamma) - 1]\frac{\tilde{u}}{(1-n)^2}, \\ u_{cc}u_{nn} - u_{cn}^2 &= \left(\frac{\tilde{u}}{c(1-n)}\right)^2 \sigma[\gamma - \sigma(1-\gamma)], \end{aligned}$$

where

$$\tilde{u} = \begin{cases} (1-\gamma)u(c,n) > 0, \text{ for } \gamma \neq 1 \\ 1, \text{ for } \gamma = 1. \end{cases}$$

Strictly decreasing in n requires $\sigma > 0$. Concavity requires $u_{cc}, u_{nn} < 0$ and $u_{cc}u_{nn} - u_{cn}^2 > 0$, or,

$$\gamma > 0, \sigma(1-\gamma) < 1, \gamma - \sigma(1-\gamma) > 0,$$

respectively. Assumptions P1-P5 are easily verified. To establish P6, let $\omega = -u_n/u_c = \sigma c/(1-n)$, so that $1-n = \sigma c/\omega$. Then:

$$u_c = c^{-\gamma}\left[\frac{\sigma c}{\omega}\right]^{\sigma(1-\gamma)} = c^{[\sigma(1-\gamma)-\gamma]}\left(\frac{\sigma}{\omega}\right)^{\sigma(1-\gamma)},$$

which obviously goes to zero if $c \to 0$ with ω falling.

The following parametric production technology satisfies T1-T7:

$$f(k,n) = k^{\theta}n^{(1-\theta)} + (1-\delta)k,\ 0 < \delta, \theta < 1 \tag{7}$$

Problem (5)-(6) can be expressed in the form of (1)-(4) with the following interpretation of (X, Γ, F):

$$\begin{aligned}
X &\equiv K = [\varepsilon, \bar{k}],\ \Gamma(k) = \{k' \in K : \varepsilon \leq k' \leq f(k,1)\} \text{ for all } k \in K &(8)\\
k_0 &> \varepsilon > 0 &(9)\\
F(k,k') &= \max_{n \in N(k,k')} u[f(k,n) - k', n]\\
N(k,k') &= \{n : \bar{n}(k,k') \leq n \leq 1\}\\
\bar{k} &= \max(k_0, \tilde{k}),\ f(\tilde{k},1) = \tilde{k}
\end{aligned}$$

In (8), $\bar{n}(k,k')$ is the smallest value of $n \in [0,1]$ consistent with $f(k,n) - k' \geq 0$, and $\bar{k}$ represents the maximal feasible capital stock. The latter is guaranteed to exist by assumption T4. Also, c has been replaced by $f(k,n) - k'$. Imposing the resource constraint in (6) as a strict equality is nonbinding, in view of the assumption in P1 that u is strictly increasing in c. In (8), ε is introduced for technical reasons clarified below. We can think of ε as an extremely tiny number, not economically significantly different from zero. The objective is to show that (P1)-(P5), (T1)-(T4) imply A4.3-A4.9. In addition, if the assumption that leisure is a normal good is added, then $F_{12} > 0$.

Regarding assumption A4.3, K is obviously convex. $\Gamma(k)$ is non-empty for all $k \in K$ because T2 guarantees that, for ε sufficiently small, $\varepsilon < f(\varepsilon, 1)$, and because f is strictly increasing (T1). Γ is obviously compact and continuity (i.e., upper and lower hemicontinuity) follows from the continuity of $f(k,1)$ (assumption T1).

Regarding A4.4, F is bounded because u is. Continuity of u and f and the fact that N is compact-valued and continuous guarantee, by the Theorem of the Maximum (Theorem 3.6), that F is continuous too. The parameter $\varepsilon > 0$ was introduced to guarantee continuity of N. With $\varepsilon = 0$, N is not lower hemicontinuous at $(k, k') = (0, 0)$. To see this, consider the sequence $\{k_m, k'_m\} \in A$ with $k'_m = f(k_m, 1)$ and $k_m \to 0$. Then, $N(k_m, k'_m) = \{1\}$ for all finite m. But, since $N(0,0) = \{[0,1]\}$, there exist $n \in N(0,0)$ such that there is no sequence $\{n_m\} \in N(k_m, k'_m)$ with the property that $n_m \to n$. (For continuity of a correspondence, see page 56 of SL.)

Consider A4.6 and A4.8. Monotonicity of Γ follows trivially from the fact that $f(k,1)$ is strictly increasing. To establish convexity of Γ (for the definition of convexity of a correspondence, see page 80 in SL) choose θ such that $0 \leq \theta \leq 1$ and $k_1, k_2 \in K$. Let $k'_i \in \Gamma(k_i)$, $i = 1, 2$. Then, $\theta k'_1 + (1-\theta)k'_2 \geq \varepsilon$, since $k'_i \geq \varepsilon$, for $i = 1, 2$. Also, $\theta k'_1 + (1-\theta)k'_2 \leq \theta f(k_1, 1) + (1-\theta)f(k_2, 1) \leq f(\theta k_1 + (1-\theta)k_2)$, where the last inequality reflects the concavity of f. This establishes $\theta k'_1 + (1-\theta)k'_2 \in \Gamma(\theta k_1 + (1-\theta)k_2)$, which is the result sought.

Before establishing A4.7, we establish two facts about the employment decision that attains the maximum in (8). Let $h : A \to [0,1]$,

$$h(k, k') = \arg \max_{n \in N(k,k')} u[f(k,n) - k', n], \ (k, k') \in A. \tag{10}$$

Since $N : A \to \Re$ is nonempty, compact- and convex-valued, and continuous, and because $u[f(k,n) - k', n]$ is strictly concave in n for each $(k, k') \in A$, it follows that h is single-valued and continuous (Lemma 3.7). Next, we establish that $h(k, k') \in int(N(k, k'))$ for $k' \in int(\Gamma(k))$. To see that this is the case one simply verifies, using P2-P5 and T2-T3, that

$$\begin{aligned} \frac{d}{dn} u[f(k,n) - k', n] &\ \to\ \infty, \ n \to \bar{n}(k, k') \\ \frac{d}{dn} u[f(k,n) - k', n] &\ \to\ -\infty, \ n \to 1. \end{aligned} \tag{11}$$

To verify this, define:

$$r_n(n) = \frac{d}{dn} u[f(k,n) - k', n] = u_c[f(k,n) - k', n] f_n(k,n) + u_n[f(k,n) - k', n],$$

where r_n exists and is continuous by the assumptions placed on u and f (P1 and T1). Consider r_n as $n \to \bar{n}(k,k')$. Suppose $\bar{n}(k,k') > 0$. As $n \to \bar{n}(k,k')$, $f_n \to f_n(k, \bar{n}(k,k')) < \infty$ by hypothesis, and $u_n \to u_n(0, \bar{n}(k,k'))$, also finite by assumption. Since $u_c \to \infty$ as $n \to \bar{n}(k,k')$, it follows that $r_n \to \infty$ as $n \to \bar{n}(k,k')$. Suppose $\bar{n}(k,k') = 0$. The only modification required to the preceding argument is that $f_n \to \infty$ as $n \to \infty$. We conclude that for all $\bar{n}(k,k') \geq 0$, $r_n \to \infty$ as $n \to \bar{n}(k,k')$.

Now consider r_n as $n \to 1$. In this case, $u_c \to u_c[f(k,1) - k', 1]$ and $f_n \to f_n(k,1)$, both finite numbers. Therefore, since $u_n \to -\infty$ as $n \to 1$, it follows that $r_n \to -\infty$ as $n \to 1$. This establishes (11). Conclude that a zero for the derivative in (11) must exist for $n \in int(N(k,k'))$. Given strict concavity, that zero corresponds to the maximum in (11).

Establishing A4.7, A4.9, A4.5 and $F_{12} > 0$ is a little more involved, and so it is convenient to use the 'Proposition, Proof' format for these.

Proposition 1 *F is strictly concave, i.e., A4.7 is satisfied.*

Proof:

Let $(k_i, k_i') \in A$, $i = 1,2$, where $(k_1, k_1') \neq (k_2, k_2')$. Let $n_i = h(k_i, k_i')$ and $c_i = f(k_i, n_i) - k_i'$, $i = 1,2$. We first establish that $(c_1, n_1) \neq (c_2, n_2)$. Consider three cases:

(a) $k_1' \neq k_2'$, $k_1 = k_2$.

Suppose $n_1 = n_2$. Then $c_1 - c_2 = [f(k_1, n_1) - k_1'] - [f(k_2, n_2) - k_2'] = k_2' - k_1' \neq 0$. Suppose $c_2 = c_1$. Then $f(k_1, n_1) \neq f(k_2, n_2)$. But, given that f is strictly increasing in n, it follows that $n_1 \neq n_2$.

(b) $k_1' = k_2'$, $k_1 \neq k_2$.

Suppose $n_1 = n_2$. Then, $c_1 - c_2 = f(n_1, k_1) - f(n_2, k_2) \neq 0$, given that f is strictly increasing in k. Suppose $c_1 = c_2$. Then $f(n_1, k_1) = f(n_2, k_2)$ and the fact that f is strictly increasing in n and k implies that $n_1 \neq n_2$.

(c) $k_1' \neq k_2'$, $k_1 \neq k_2$.

Suppose $c_1 = c_2 = c$ and $n_1 = n_2 = n$. Because the maximum in (10) is interior, it must be that $w(c,n) = f_n(k_i, n)$, $i = 1,2$, where w is the marginal rate of substitution between consumption and leisure (leisure is $1-n$):

$$w(c,n)=\frac{-u_n(c,n)}{u_c(c,n)}. \tag{12}$$

But, this means that $f_n(k_1,n)=f_n(k_2,n)$, and this contradicts $f_{kn}>0$ in T7, which implies that f_n is strictly increasing in k.

We conclude from (a)-(c): $(k_1,k_1')\neq(k_2,k_2')\Rightarrow(c_1,n_1)\neq(c_2,n_2)$.

Now, let $k(\theta)=\theta k_1+(1-\theta)k_2$, $k'(\theta)=\theta k_1'+(1-\theta)k_2'$, and define $c(\theta)$ and $n(\theta)$ similarly. We know that $(k(\theta),k'(\theta))\in A$, because of convexity of K and Γ. To establish feasibility of $c(\theta)$ and $n(\theta)$, note first that $0\leq n(\theta)\leq 1$, $c(\theta)\geq 0$. Also,

$$\begin{aligned} f(n(\theta),k(\theta))-k'(\theta) &\geq \theta[f(n_1,k_1)-k_1']+(1-\theta)[f(n_2,k_2)-k_2'] \\ &\geq \theta c_1+(1-\theta)c_2=c(\theta), \end{aligned}$$

where the first inequality reflects the concavity of f and the second inequality reflects that c_1 and c_2 are feasible. Feasibility of $k(\theta),k'(\theta)$, $c(\theta)$ and $n(\theta)$ implies that

$$\begin{aligned} F(k(\theta),k'(\theta)) &\geq u(c(\theta),n(\theta)) \\ &> \theta u(c_1,n_1)+(1-\theta)u(c_2,n_2) \\ &= \theta F(k_1,k_1')+(1-\theta)F(k_2,k_2'), \end{aligned}$$

where the first inequality reflects the definition of F and feasibility of $c(\theta)$ and $n(\theta)$. The second inequality reflects strict concavity of u, and the equality reflects the definition of c_i,n_i, $i=1,2$. Thus, concavity of F is established. Q.E.D.

Proposition 2 *F is continuously differentiable in the interior of A (i.e., A4.9 holds).*

Proof:

Consider $(k_*,k_*')\in int(A)$, so that $k_*\in int(K)$ and $k_*'\in int(\Gamma(k_*))$. By interiority of $h(k_*,k_*')$ and continuity of $\bar{n}:A\rightarrow[0,1]$, it follows that there is an open interval, $D\subseteq A$, containing (k_*,k_*') in its interior, with the property that $h(k_*,k_*')$ is feasible for all $(k,k')\in D$. Define

$$F^*(k,k')=u[f(k,h(k_*,k_*'))-k',h(k_*,k_*')]$$

Then, by feasibility of $h(k_*, k'_*)$ given k, k':

$$F(k,k') \geq F^*(k,k'), \text{ for all } (k,k') \in D$$

with the strict equality holding for $(k,k') = (k_*, k'_*)$. Since F^* is differentiable and concave in (k,k') and F is concave and continuous, the conditions of the Benveniste and Scheinkman theorem (Theorem 4.10) are satisfied. Consequently, F is differentiable at k_*, k'_*. In addition, the Benveniste and Scheinkman theorem also implies:

$$F_1(k,k') = F_1^*(k,k') = u_c f_k, \;\; F_2(k,k') = F_2^*(k,k') = -u_c. \tag{13}$$

This establishes A4.9. Q.E.D.

Proposition 3 *$F(\cdot, y)$ is strictly increasing for each y (i.e., A4.5 is satisfied)*
Proof:

The proof to the previous proposition established that $F_1(k,k') = u_c f_k$. This term is strictly positive by T1 and P1. The result follows by application of the Mean Value Theorem. Q.E.D.

Proposition 4 *If, in addition to P1-P5, T1-T4, T7, leisure is a normal good (i.e., $w_c(c,n) > 0$) then $h_{k'}(k,k') > 0$ and $F_{12} > 0$.*
Proof:

By (13),

$$F_{12} = \frac{d}{dk'}\left(u_c[f(k,n) - k', n] f_k(k,n)\right). \tag{14}$$

To get $F_{12} > 0$, it is sufficient to establish two things: (i) $\frac{d}{dk'} f_k > 0$ and (ii) $\frac{d}{dk'} u_c > 0$. We get (i) from $f_{kn} > 0$ (see T7) and $h_{k'} > 0$. Conceivably, if leisure were an inferior good and (i) were sufficiently negative, then it could swamp (ii), making $F_{12} < 0$. Result (ii) follows from the strict concavity of the utility and production functions.

We focus on (i), i.e., $h_{k'}$, first. By interiority of the labor decision:

$$w(f(k,n) - k', n) = f_n(k,n). \tag{15}$$

According to the implicit function theorem, the partial derivative of h with respect to k', $h_{k'}$, can be obtained by totally differentiating (15) with respect to n and k'. Doing so, we find:

$$\frac{dn}{dk'} = h_{k'} = \frac{w_c}{\Delta} > 0, \tag{16}$$

by our assumption about w_c and since $\Delta \equiv w_c f_n + w_n - f_{nn} > 0$. (See Figure 1 for an illustration of this result.) To see that the latter is implied by strict concavity of u and f, first replace f_n by $-u_n/u_c$ and express w_c and w_n in terms of the first and second derivatives of u to get:

$$\begin{aligned} w_c f_n + w_n &= -\frac{1}{u_c} x'Ux, \text{ where } x' = [(u_n/u_c) \ \ -1], \\ U &= \begin{bmatrix} u_{cc} & u_{cn} \\ u_{cn} & u_{nn} \end{bmatrix}. \end{aligned}$$

By strict concavity of u, $w_c f_n + w_n > 0$. Since $f_{nn} < 0$, it follows that $\Delta > 0$ if $w_c > 0$. This establishes (i).

Now consider (ii).

$$\begin{aligned} \frac{d}{dk'} u_c[f(k,n) - k', n] &= u_{cc}(f_n h_{k'} - 1) + u_{cn} h_{k'} \\ &= \frac{1}{\Delta u_c}\{u_{cc} f_{nn} u_c - u_{cc} w_n u_c + u_c u_{cn} w_c\}, \end{aligned} \tag{17}$$

where the second equality makes use of the fact $h_{k'} = w_c/\Delta$ and the definition of Δ given above. Differentiating w :

$$\begin{aligned} w_c &= -\frac{u_{nc}}{u_c} + \frac{u_n}{u_c^2} u_{cc} \\ w_n &= -\frac{u_{nn}}{u_c} + \frac{u_n}{u_c^2} u_{cn}. \end{aligned} \tag{18}$$

Substituting (18) into (17):

$$\begin{aligned} \frac{d}{dk'} u_c &= \frac{1}{\Delta u_c}\{u_{cc} f_{nn} u_c - u_{cc} w_n u_c + u_c u_{cn} w_c\} \\ &= \frac{1}{\Delta u_c}\{u_{cc} f_{nn} u_c - u_{cc}(-u_{nn} + \frac{u_n}{u_c} u_{cc}) + u_{cn}(-u_{nc} + \frac{u_n}{u_c} u_{cc})\} \\ &= \frac{1}{\Delta u_c}\{u_{cc} f_{nn} u_c + (u_{cc} u_{nn} - u_{cn}^2)\} > 0. \end{aligned}$$

The last inequality reflects that the term in parentheses is positive by strict concavity of u. Similarly, $u_{cc} f_{nn}$ is positive by strict concavity of u, and f. We showed above that $\Delta > 0$. This establishes (ii), and, hence, the result. Q.E.D.