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D11-1, Fall 1999

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. Consider the canonical model in the following sequence representation:

$$\max_{x_{t+1}\in\Gamma(x_t)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),\ x_t \in X.$$

Suppose x_0 is given and $0 \in X$, $0 \in \Gamma(x)$ for all $x \in X$. Suppose x_1^*, x_2^*, ... where $x_t^* \in int(\Gamma(x_{t-1}^*))$ for $t = 1, 2, ...$ solve the maximization problem (here, $x_0^* = x_0$). Suppose that (β, Γ, F, X) satisfy conditions sufficient conditions for a strictly concave, differentiable and bounded function to exist, satisfying:

$$\begin{aligned} v(x) &= F(x, g(x)) + \beta v(g(x)), \text{ where} \\ g(x) &= \arg\max_{x'\in\Gamma(x)} F(x, x') + \beta v(x'). \end{aligned}$$

Suppose too that (β, Γ, F, X) satisfy assumptions that guarantee that the solution to this functional equation coincides with the solution to the sequence representation to the problem.

(a) State precisely what it means that the solution to the sequence and functional equations problems are 'the same'.

(b) Show that

$$v'(x) = u_1(x, x'),$$

where $x' = g(x)$. You may use the assumption that g is differentiable.

(c) Show that

$$\lim_{t\to\infty} \beta^t u_1(x_t^*, x_{t+1}^*) x_t^* = 0.$$

(Hint: exploit the fact that $x_t = 0$ is feasible; that v is a strictly concave and bounded; that the tangent line to a concave function lies weakly above the function; and make use of the fact that you know the derivative of v).

2. Suppose that the preferences of the representative agent over streams of consumption, c_t, and labor, n_t, has the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t).$$

Suppose that the resource constraint is, $c_t + k_{t+1} - (1-\delta)k_t \le f(k_t, n_t)$, where $f_x \to \infty$ as $x_t \to 0$, $f_x \to 0$ as $x_t \to \infty$ for $x = k, n$. In addition, $k_0 > 0$ is given, and the non-negativity constraints, $c_t \ge 0$ and $k_t \ge 0$ must be respected. The dynamic programming representation of this problem is:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} F(k, g(k)) + \beta v(g(k)), \\ g(k) &= \arg\max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'). \end{aligned}$$

Here,

$$\begin{aligned} F(k, k') &= u[f(k, h(k, k')) + (1-\delta)k - k', h(k, k')], \ (k, k') \in A \\ h(k, k') &= \arg\max_{n \in B(k,k')} u[f(k, n) + (1-\delta)k - k', n], \ (k, k') \in A \\ A &= \{k, k' : k \in K, \ 0 \le k' \le f(k, 1) + (1-\delta). \end{aligned}$$

and

$$\begin{aligned} B(k, k') &= \{n : \underline{n}(k, k') \le n \le 1\}, \\ \underline{n}(k, k') &= \begin{cases} 0, & \text{if } f(k, 0) + (1-\delta)k \ge 0 \\ n^*, & f(k, n^*) + (1-\delta)k - k', \text{ otherwise} \end{cases} \end{aligned}$$

(a) Show that $B(k, k')$ is non-empty for all $(k, k') \in A$.

(b) Define the function, $w(c, n) = -u_n(c, n)/u_c(c, n)$. Show, using graphs and intuition, that the following restriction on utility, $w_c > 0$, guarantees that h is increasing in k'. (A formal proof - which involves some algebra - is not required here.)

(c) Show that the cross derivative, F_{12}, is positive if h is increasing in k'. If you need differentiability of some function to establish this demonstration, but don't have time to prove it, just say so. To get full credit here requires sketching the argument, but formal proofs of differentiability are not required.

(d) Explain carefully why $F_{12} > 0$ implies that g is increasing in k. Give the economics and the formal reasoning, as far as you can go.

Note

Benveniste and Scheinkman theorem. Let $X \subseteq R^l$ be a convex set, let $V : X \to R$ be concave, let $x_0 \in int\ X$, and let D be a neighborhood of x_0. If there is a concave, differentiable function, $W : D \to R$, with $W(x_0) = V(x_0)$ and with $W(x_0) \leq V(x_0)$ for all $x \in D$, then V is differentiable at x_0, and $V_i(x_0) = W_i(x_0)$, $i = 1, 2, ..., l$.