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Notes for MIDTERM Answers

1. This is straighforward.
2. The interiority assumption on x_t^* guarantees that $v'(x_t^*) = u_1(x_t^*, x_{t+1}^*)$. The rest is easy, once you know the tricks.
3. To show that $B(k, k')$ is non-empty requires verifying that $\underline{n}< 1$.

   (a) The Euler equation can be derived simply by working out the fact that $c_t^* + \varepsilon$, $c_{t+1}^* - \varepsilon A$ is a feasible deviation from the optimal path, for some $\varepsilon > 0$ (here, the Inada conditions are helpful). Then, differentiate the present discounted value of utility of the perturbed sequence with respect to ε and evaluate this at $\varepsilon = 0$. Optimality of the original sequence (and, differentiability of the objective) implies that this derivative must be zero.

   (b) Suppose
$$\lim_{t\to\infty} \beta^t u'(c_t^*) A k_t^* = \Delta > 0.$$
   (We need not consider the negative Δ case, because of the assumption that β, u' and k^* are non-negative.) The first order condition implies:
$$u_{c,0}^* = (\beta A)^t\, u_{c,t}^*,\ \ t = 0, 1, 2, ...$$
   Then,
$$\lim_{t\to\infty} \beta^t u'(c_t^*) A k_t^* = u_{c,0}^* A \lim_{t\to\infty} \frac{k_t^*}{A^t} = \Delta,$$
   or, for each $\delta > 0$ there exists a T such that
$$\left|\frac{k_t^*}{A^t} - \frac{\Delta}{u_{c,0}^* A}\right| < \delta,\ \text{ for all } t \geq T.$$
   This places a lower bound on k_t^* :
$$\frac{k_t^*}{A^t} > \frac{\Delta}{u_{c,0}^* A} - \delta,\ \text{ for all } t \geq T.$$

In particular, consider δ sufficiently small that the last expression is positive.

Consider the following pertubation on $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}, \{c_t, k_{t+1}\}_{t=0}^{\infty}.$ Let the two sequences coincide up to date T. Then, let $c_T = c_T^* + \varepsilon$, $c_t = c_t^*$ for all $t > T$. This sequence of consumptions obviously generates higher utility than the $\{c_t^*\}_{t=0}^{\infty}$ sequence. If the sequence is also feasible, then we have a contradiction to the notion that $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$ is optimal.

Note that with the deviation path, $k_{t+1} = k_{t+1}^*$ for $t = 0, 1, ..., T - 1$. Then,

$$\begin{aligned}
k_{T+1} &= k_{T+1}^* - \varepsilon \\
k_{T+2} &= k_{T+2}^* - A\varepsilon \\
&\vdots \\
k_{T+j} &= k_{T+j}^* - A^{j-1}\varepsilon.
\end{aligned}$$

The question is whether ε can be chosen sufficiently small, but positive, to guarantee that the k_{T+j}'s satisfy non-negativity. This is tantamount to the requirement that k_{T+j}^* grow sufficiently fast so that it is not 'overtaken' by $A^{j-1}\varepsilon$. From the above expression, we see that this is true, because we established that k_t^* eventually grows at a rate faster than A. To see this formally, subsitute

$$k_t^* > A^t \left[\frac{\Delta}{u_{c,0}^* A} - \delta\right] > 0,$$

into the last expression, to get:

$$\begin{aligned}
k_{T+j} &> A^{T+j} \left[\frac{\Delta}{u_{c,0}^* A} - \delta\right] - A^{j-1}\varepsilon \\
& A^{j-1} \left\{A^{T+1} \left[\frac{\Delta}{u_{c,0}^* A} - \delta\right] - \varepsilon\right\}.
\end{aligned}$$

So, fix a $\delta > 0$ such that the object in square brackets is positive, and identify the associated T. Then, it is always possible to find ε small enough, so that $k_{T+j} \geq 0$. This sequence of k_{T+j}'s is obviously feasible. (Note that if $\Delta = 0$, then a suitable $\delta > 0$ cannot be identified.)