1. Households

There is a continuum of households, indexed by $h \in (0, 1)$. The $h^{th}$ household’s preferences are:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ U \left( c_t^h - H_t^h \right) + f(L_t^h) + v \left( \frac{Q_t^h}{P_t} \right) \right],$$

where $c_t^h$ and $L_t^h$ denote the $h^{th}$ household’s levels of consumption and market work effort, respectively. Also, $Q_t^h / P_t$ is the real value of the part of its financial wealth that the household chooses to hold in non-interest bearing form. Finally, $H_t^h$ denotes the household’s ‘habit stock’:

$$H_t^h = \chi H_{t-1}^h + b \bar{c}_{t-1}^h.$$

We consider two cases. In one (‘external habit’), $\bar{c}_{t-1}^h$ is economy-wide average consumption. In the other (‘internal habit’), $\bar{c}_{t-1}^h = \bar{c}_{t-1}$ is the household’s own consumption. Also, $0 < \chi < 1$, and $b > 0$.

In the discussion that follows we define $u_{c,t}$ as the following partial derivative:

$$u_{c,t} = \frac{\partial \left[ \sum_{j=0}^{\infty} \beta^j U \left( c_{t+j}^h - H_{t+j}^h \right) \right]}{\partial c_t^h}.$$

The household faces two forms of uncertainty. There is aggregate uncertainty that stems from uncertainty in the growth rate, $\mu_t$, of nominal money balances:

$$\mu_t = \frac{\bar{M}_{t+1}}{M_t},$$

where $\bar{M}_t$ is the beginning of period aggregate, per capita, stock of money. In addition, the household faces two forms of idiosyncratic uncertainty. Being a monopoly supplier of its

\footnote{Alternative formulation follows McCallum and Nelson and others, in specifying utility like this:

$$u = \left( \frac{\tilde{Q}_t}{\tilde{P}_t} \right)^{1-\sigma}, \quad H_t = (H_{t-1})^\chi (\tilde{c}_{t-1})^b.$$}

The problem with this formulation is that habit is tied up with $\sigma$. Thus, if $\sigma = 1$, then habit has no effect on the marginal utility of consumption. When $\sigma < 1$, then habit has what seems like a perverse negative effect on the marginal utility of consumption.
own labor, it sets its wage rate. However, it can only adjust its wage at exogenously and randomly determined times. Similarly, the household can only adjust $Q_t^h$ at exogenously determined, random times. In modeling this, we follow Calvo. Thus, we seek to understand the consequences of frictions in the adjustment of these variables, without modeling explicitly what the source of those frictions is. A complete analysis would of course do so. Such an analysis would recognize that the timing of household $Q$ and wage decisions is endogenous. We hope that the outcome of our analysis is not distorted too much by our decision not to model this. In any case, our modeling choices are made by a desire to study, in an integrated framework, the operation of a variety of frictions studied in the literature. The Calvo approach has become the standard for modeling frictions in decisions.

We further restrict the analysis by making assumptions which guarantee that the frictions do not cause households to become heterogeneous in ways that are analytically difficult to model and (hopefully!) not important from the point of view of the phenomena we seek to understand. We do this by allowing households to enter into insurance markets against the outcomes of these frictions. Again, here we follow standard practice.

The timing of household decisions is like this. At the beginning of the period, before any uncertainty is realized, the households purchase consumption insurance. Then it is determined whether households can adjust their $Q$ and their wage. Households who can adjust $Q$ do so before the current period realization of $\mu$. Households who can adjust their wage do so after the current realization of $\mu$. Households also make an investment decision. This is made before the realization of all uncertainty.

1.1. Consumption Insurance

Insurance markets open at the beginning of the period, before that period’s uncertainty has been realized. Let $s = (s^q, s^w)$, where $s^q = 1$ indicates the household can change the current period value of $Q$, and $s^w = 0$ indicates that it cannot. Also, $s^w$ indicates whether it can or cannot change its posted nominal wage rate. We assume that the $s_l$’s are independent over time, with each other, and with all other random variables, with probability function, $\phi(s)$. We let

$$\xi_l = \text{prob}[s^q_l = 0], \ l = q, w.$$

We imagine that insurance against $h$ is offered by perfectly competitive insurance companies. For each possible value of $\mu = \bar{\mu}$ and $s = \bar{s}$, there is a representative, competitive insurance company. This company sells contracts which pay $\$1$ if $\bar{\mu}$ and $\bar{s}$ are realized and nothing otherwise. The quantity of such contracts purchased by the $h^{th}$ household is $B^h(\bar{s}, \bar{\mu})$. Regardless of the value of $s$ that is realized by the $h^{th}$ household, the company charges the $h^{th}$ household a premium, $\delta(\bar{s}, \bar{\mu})B^h(\bar{s}, \bar{\mu})$ if $\mu = \bar{\mu}$ and zero for all other values
of \( \mu \). We work with symmetric equilibria, in which all households buy the same amount of insurance, i.e.,

\[
B^h(\bar{s}, \bar{\mu}) = B^j(\bar{s}, \bar{\mu}) = B(\bar{s}, \bar{\mu}) \text{ for all } h, j,
\]
each \( \bar{s}, \bar{\mu} \).

Then, the payout of the \( \bar{s}, \bar{\mu} \) insurance company is

\[
\phi(\bar{s})B(\bar{s}, \bar{\mu}), \quad \mu = \bar{\mu},
\]

\[
0, \quad \mu \neq \bar{\mu}.
\]

The insurance company receives

\[
\delta(\bar{s}, \bar{\mu})B(\bar{s}, \bar{\mu}) \text{ if } \mu = \bar{\mu},
\]

\[
0 \text{ if } \mu \neq \bar{\mu}.
\]

The zero profit condition on the insurance company is equivalent with

\[
\delta(s, \mu) = \phi(s), \text{ for all } \mu.
\]

1.2. The Household’s Decisions

Suppose there is no aggregate uncertainty, just for a sec. Let the realization of Calvo uncertainty for the household in a particular date be denoted by \( s \). The vector \( s^t \) denotes \( s_0, s_1, \ldots, s_t \), i.e., the history of realizations of idiosyncratic uncertainty since date \( t = 0 \). At date \( t \), \( s^{t-1} \) is known, and the household has the opportunity to purchase Arrow securities contingent upon the realization of \( s_t, s_{t+1}, s_{t+2}, \text{ etc.} \). We denote all the possible histories in period \( t + \tau \) that are consistent with \( s^{t-1} \) by \( s^{t+\tau}|s^{t-1} \). Let \( B_{t,\tau}(s^{t+\tau}) \) denote a quantity of bonds which pay off 1 in the event \( s^{t+\tau} \) and 0 otherwise, \( \tau \geq 0 \). Let \( \delta_{t,\tau}(s^{t+\tau}) \) denote the corresponding price. Payment for these bonds occurs during period \( t + \tau \), after the realization of the current period shock. Consider a household in period \( t \) with a particular history \( \bar{s}^t \). The payments it owes in that history are:

\[
\sum_{j=0}^{t} \sum_{s^t|s^{t-j-1}} \delta_{t-j,j}(s^j)B_{t-j,j}(s^j),
\]

where \( \bar{s}^{-1} \) is empty. To understand this expression, note that the payments in a particular history, \( \bar{s}^t \), reflect Arrow securities purchased for period \( t \) at the beginning of periods \( t, t-1, \)

\[\text{Below, we verify that a symmetric equilibrium actually exists.}\]
\( t - 2, \ldots, 0 \). For example, the value of new purchases of securities at the beginning of \( t \), for delivery after the Calvo shocks in period \( t \) is:

\[
\sum_{s_t \mid \tilde{s}^{t-1}} \delta_{t,0}(s^t) B_{t,0}(s^t).
\]

The value of new purchases of securities at the beginning of \( t - 1 \), for delivery after the Calvo shocks in \( t \) is:

\[
\sum_{s_{t-1}, s_{t-1} \mid \tilde{s}^{t-1}} \delta_{t-1,1}(s^t) B_{t-1,1}(s^t).
\]

The household with history \( \tilde{s}^t \) receives the following payments:

\[
\sum_{j=0}^{t} B_{t-j,j}(\tilde{s}^t).
\]

The household’s overall flow budget constraint is:

\[
0 \leq R(\mu) \left[ M(s^{t-1}) - Q(s^t) + (\mu - 1)\bar{M} \right] + D(\mu) + \sum_{j=0}^{t} B_{t-j,j}(\tilde{s}^t) - \sum_{j=0}^{t} \sum_{s_t \mid \tilde{s}^{t-1}} \delta_{t-j,j}(s^t) B_{t-j,j}(s^t) + \left[ Q(s^t) + W(s^t)L(s^t) - P(\mu) \left( c(s^{t-1}) + I(s^{t-1}) \right) \right] + R^k(\mu)k(s^{t-2}) - M(s^t),
\]

for periods \( t = 0, 1, 2, \ldots \).

The Lagrangian representation of the problem is:

\[
\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t f(s^t) \left[ U \left( c(s^t) - bc(s^{t-1}) \right) + f(L(s^t)) + v \left( \frac{Q(s^t)}{P(\mu)} \right) \right] + \lambda(s^t) \left\{ R(\mu) \left[ M(s^{t-1}) - Q(s^t) + (\mu - 1)\bar{M} \right] + D(\mu) \right. \\
+ \sum_{j=0}^{t} B_{t-j,j}(s^t) - \sum_{j=0}^{t} \sum_{s_t \mid \tilde{s}^{t-1}} \delta_{t-j,j}(s^t) B_{t-j,j}(s^t) \\
+ \left. \left[ Q(s^t) + W(s^t)L(s^t) - P(\mu) \left( c(s^t) + I(s^t) \right) \right] + R^k(\mu)k(s^{t-2}) - M(s^t) \right\}
\]
We now work out the first order conditions corresponding to the optimal choice of \( B_{t, \tau}(s^{t-\tau}) \)’s. For \( B_{t,0}(s^{t-1}, s^0_t) \), the first order condition is:

\[
\sum_{s_t} F(s^{t-1}, s_t) \lambda(s^{t-1}, s_t) \delta_{t,0}(s^{t-1}, s^0_t) = F(s^{t-1}, s^0_t) \lambda(s^{t-1}, s^0_t).
\]

But, by free entry, perfect competition and zero profits, \( \delta_{t,0}(s^{t-1}, s^0_t) = F(s^{t-1}, s^0_t) \), so that:

\[
\sum_{s_t} F(s^{t-1}, s_t) \lambda(s^{t-1}, s_t) = \lambda(s^{t-1}, s^0_t).
\]

Note that the object on the left of the equality is independent of the value of \( s^0_t \). Therefore, the right side is too. But, since \( \lambda(s^t) \) corresponds to the marginal utility of consumption, it follows that consumption is not a function of \( s^0_t \).

Consider now the optimal choice of \( B_{t-1,1}(s^{t-2}, s^0_{t-1}, s^0_t) \). The first order condition is:

\[
\sum_{s_{t-1}, s_t} F(s^{t-2}, s_{t-1}, s_t) \lambda(s^{t-2}, s_{t-1}, s_t) \delta_{t-1,1}(s^{t-2}, s^0_{t-1}, s^0_t) = F(s^{t-2}, s^0_{t-1}, s^0_t) \lambda(s^{t-2}, s^0_{t-1}, s^0_t).
\]

Zero profits generates \( \delta_{t-1,1}(s^{t-2}, s^0_{t-1}, s^0_t) = F(s^{t-2}, s^0_{t-1}, s^0_t) \). Then,

\[
\sum_{s_{t-1}, s_t} F(s^{t-2}, s_{t-1}, s_t) \lambda(s^{t-2}, s_{t-1}, s_t) = \lambda(s^{t-2}, s^0_{t-1}, s^0_t).
\]

Note that the object on the left of the equality is independent of \( s^0_{t-1}, s^0_t \). So, the object on the right must be too. As a result, consumption in period \( t \) is independent of \( s^0_{t-1} \), in addition to being independent of \( s^0_t \).

Consider now the optimal choice of \( B_{t-j,j}(s^{t-j-1}, s^0_{t-j}, \ldots, s^0_{t-1}, s^0_t) \) for \( j = \tau \)

\[
\sum_{s_{t-1}, s_t} F(s^{t-2}, s_{t-1}, s_t) \lambda(s^{t-2}, s_{t-1}, s_t) \delta_{t-1,1}(s^{t-2}, s^0_{t-1}, s^0_t) = F(s^{t-2}, s^0_{t-1}, s^0_t) \lambda(s^{t-2}, s^0_{t-1}, s^0_t).
\]

\( \tau = 1 \) and \( s_{t+1} = s^0_{t+1} \), where \( s^0_{t+1} \) is a particular value of \( s_{t+1} \),

\[
-\delta_{t,1}(s^t, s^0_{t+1}) \lambda(s^t) F(s^t) \beta^t + \beta^{t+1} F(s^t, s^0_{t+1}) \lambda(s^t, s^0_{t+1}) = 0.
\]
For $\tau = 2$, and $s_{t+1} = s_{t+1}^0$, $s_{t+2} = s_{t+2}^0$

$$-\delta_{t,2}(s^t, s_{t+1}^0)\lambda(s^t)F(s^t)\beta^t + \beta^{t+2} F(s^t, s_{t+1}^0, s_{t+2}^0)\lambda(s^t, s_{t+1}^0, s_{t+2}^0) = 0.$$  

In general, for arbitrary $\tau > 0$ and $s^{t+\tau}|s^t$:

$$-\delta_{t,\tau}(s^{t+\tau})\lambda(s^t)F(s^t)\beta^t + \beta^{t+\tau} F(s^{t+\tau})\lambda(s^{t+\tau}) = 0.$$  

In addition, differentiating with respect to $c(s^t)$, we find:

$$u_c(s^t) = \lambda(s^t)P(\mu).$$

At the beginning of the period, before $s$ or $\mu$ are realized, the household makes its decision about $B^h(s, \mu)$. Then, $s$ is realized. If the realization of $s$ indicates that the household may change $Q^h$, then it does so. Otherwise, it goes with $Q_{t-1}^h$, where $Q_{t-1}^h$ denotes the previous period’s value of $Q^h$ and $\bar{\mu}$ is the mean money growth rate. Either way, it splits up $M^h$ into two components:

$$M^h = Q^h, Q^h,$$

where $M^h - Q^h$ goes to the financial intermediary, $M^h$ is the $h$th household’s beginning of period money balances and $Q^h$ is sent to the consumption goods market. It now goes to the goods market.

The asset evolution equation of the $h$th household conditional on a realization of $\mu$ and $s$ is:

$$M^{h,t}(\mu, s) = R(\mu) [M^h - Q^h(h) + (\bar{\mu} - 1)\bar{M}] + D(\mu)$$

$$+ B^h(s, \mu) - \sum_\delta \delta(\bar{s}, \mu)B^h(\bar{s}, \mu)$$

$$+ [Q^h(s) + W^h(s, \mu)L^h(s, \mu) - P(\mu) (c^h(s, \mu) + I)]$$

$$+ R^k(\mu)k^h,$$

where $D(\mu)$ denotes dividends received from firms and $\bar{M}$ is the economy-wide stock of money, $W^i$ is the wage rate and $P$ is the price level. In each case, we use the notation to indicate which shock the variable depends on.

We now consider the Euler equation for $B^h(s, \mu)$. This Euler equation allows us to prove that $u_c$ is independent of the realization of $s$. Fix $s_0, \mu_0$ at $\bar{s}_0$, $\bar{\mu}_0$. Now, differentiate the criterion with respect to $B^h_0(\bar{h}_0, \bar{\mu}_0)$:

$$\pi_0(\bar{\mu}_0)\phi(\bar{s}_0)\nu_0(\bar{s}_0, \bar{\mu}_0) + \delta(\bar{s}_0, \bar{\mu}_0)\pi_0(\bar{\mu}_0)\sum_{s_0} \phi(s_0)\nu_0(s_0, \bar{\mu}_0) = 0.$$
Here, \( v_0(\bar{s}_0, \bar{\mu}_0) \) is the marginal value of an extra dollar at the end of period 0 conditional on the realized value of \( s \) and of money growth being as indicated. That is, it is the multiplier on the household’s period 0 budget constraint in the Lagrangian formulation of its problem. Also, \( \pi_0(\bar{\mu}_0) \) is the probability that the realization of money growth is \( \bar{\mu}_0 \) at time zero. We have exploited the fact that \( B^s_t(\bar{s}_0, \bar{\mu}_0) \) appears as a cost in each state of nature, while it appears as a benefit in only the state of nature, \( \bar{s}_0, \bar{\mu}_0 \). Note that \( \pi_0(\bar{\mu}_0) \) appears on both sides of the minus sign, so it can be cancelled out. Substituting this and the zero-profit condition for insurance companies into this expression,

\[
v_0(\bar{s}_0, \bar{\mu}_0) = \sum_{s_0} \phi(s_0) v_0(s_0, \bar{\mu}_0).
\]

Since the right hand side is only a function of \( \bar{\mu}_0 \), it follows that the left hand side is only a function of \( \bar{\mu}_0 \) too. In particular, it is not a function of \( \bar{s}_0 \):

\[
v_0(s_0, \bar{\mu}_0) = v_0(\bar{\mu}_0), \text{ for all } s_0, \mu_0. \tag{1.2}
\]

Under additive separability, this implies that consumption is independent of \( s_0 \). (It is easy to verify from the Lagrangian representation of the household’s problem that \( u_c/P = v \).)

There is a simple (standard) explanation of why \( u_c \) is independent of the realization of \( s \). Optimization by the household implies that it equates the marginal rate of substitution in consumption between two different \( s \) states of nature to the associated relative price. The marginal utility of consumption is the probability of the realization of \( s \), times \( u_c \) for that state. At the same time, the price of consumption in that state is the probability of that state. Equating marginal rate of substitution to relative price there is a cancellation of the probability and price terms, leaving the implication that \( u_c(s')/u_c(s) = 1 \).

### 1.2.1. Fisher Equation

We now differentiate the household’s Lagrangian problem with respect to \( M^t_{t+1}(s_t, \mu_t) \). The notation here reflects that you can pick end-of-period \( t \) money, \( M^t_{t+1} \), contingent on the realized period \( t \) uncertainty. Fix \( \bar{s}_t, \bar{\mu}_t \) and differentiate with respect to \( M^t_{t+1}(\bar{s}_t, \bar{\mu}_t) \):

\[
\frac{u_{c,t}}{P_t} = \beta E_t \frac{u_{c,t+1}}{P_{t+1}} R_{t+1}.
\tag{1.3}
\]

We can derive this equation using a variational argument based on acquisition of state-contingent insurance. Thus, suppose the household buys one extra unit of insurance which pays off in state \( \bar{s}_t, \bar{\mu}_t \). That is, \( B^t_t(\bar{s}_t, \bar{\mu}_t) \) is increased by 1. The cost of this is \( \delta(\bar{s}_t, \bar{\mu}_t) \), to
be paid at the end of the period if $\mu_t = \tilde{\mu}_t$, regardless of the value of $s_t$. In utility terms this is $\delta(\tilde{s}_t, \tilde{\mu}_t) \sum s_t \phi(s_t) \frac{u(s_t)}{P_t} = \delta(\tilde{s}_t, \tilde{\mu}_t) \pi_t(\tilde{\mu}_t)u_{c,t}/P_t$. The benefit is that the extra dollar of insurance payoff that occurs in state $\tilde{s}_t$, $\tilde{\mu}_t$ can be invested in the next period’s financial market for a return of $R_{t+1}$ per dollar invested, which is valued in utility terms in the amount, $\pi_t(\tilde{\mu}_t)\phi(\tilde{s}_t)E_t u_{c,t+1}R_{t+1}/P_{t+1}$. The result follows from cancelling $\pi_t(\tilde{\mu}_t)$ in the cost and benefit, as well as from the observation (proved above) that $\delta(\tilde{s}_t, \tilde{\mu}_t) = \phi(\tilde{s}_t)$.

There is a different way to gain intuition into (1.3), based on thinking of consumption as a credit good. Suppose you have a dollar at the end of period $t$, and you’re thinking about it at the beginning of period $t$. You could use the dollar to purchase $1/P_t$ goods today. Or, you could invest the dollar in the morning of period $t + 1$, earn $R_{t+1}$ dollars by then end of period $t + 1$ and then use those dollars to finance the purchase of $R_{t+1}/P_{t+1}$ consumption goods during $t + 1$.

1.2.2. Household Cash Decision

The Euler equation for $Q$ under indexing scheme #1 is:

$$E_t \sum_{j=0}^{\infty} (\beta \xi_j)^j \left\{ v' \left( \frac{\tilde{Q}_t \tilde{\pi}^j}{P_t \pi_{t+1} \cdots \pi_{t+j}} \right) - u_{c,t+j} [R_{t+j} - 1] \right\} \frac{\tilde{\pi}^j}{\pi_{t+1} \cdots \pi_{t+j}} = 0, \quad (1.4)$$

where $\tilde{Q}_t$ is the value of $Q$ chosen by households who have the ability to choose at time $t$. Also (in an unfortunate switch of notation):

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}.$$ 

It is understood that when $j = 0$, $\pi_{t+1} \cdots \pi_{t+j} = 1$. Households that cannot change $Q$ must set it as follows:

$$Q_t = \bar{\pi}Q_{t-1}.$$ 

This Euler equation has a simple interpretation. Consider first the case where $\xi_j = 0$ and there is no uncertainty at all. In this case, the Euler equation says that you want to equate the marginal utility of an extra dollar spent on $Q$ (this operates via $v$ and via the additional money this gives you to augment your money next period) with the marginal utility of the extra dollars you would have at the end of the period if you had instead deposited the money with the financial intermediary. With uncertainty about $s^w$ and $\mu$, then this first order condition holds in expectation. Now, let $\xi_j > 0$. Then the marginal dollar in $q$ is committed probabilistically over many periods, and the Euler equation reflects this.
To understand this, remember that consumption is a credit good in this model. So, the gross earnings of a dollar in the bank during period 0 can be spent in the period 0 goods market. That’s one thing you can do with a dollar. That’s on the right side of the equation. Alternatively, you can hang the dollar on your rear view mirror and enjoy utility, while spending it at the same time. That’s what’s on the left side of the equation. Notice that when $u_c$ is high, then $q$ will be low (because $v'' < 0$). That’s because if you care a lot about consumption, you’ll maximize that by keeping your money in the bank so as to get as much money as possible. This seems like an unhappy feature of the model. It really draws attention to the fact that this is not a transactions-based model of money demand. If you really want to consume a lot, you’ll take that cash off the rear view mirror and stick it in the bank. Having the cash on the rear view mirror is interfering with your ability to enjoy consumption. In a transactions-based model, the more cash you hold on, the easier it is to have consumption. Interestingly, the sign of the income elasticity of $q$ demand is ‘right’. That is, the higher the level of consumption, the greater will be $q$. That’s because, with a higher consumption the marginal utility of consumption is lower, so you don’t mind having the cash hanging on your rear view mirror.

Let’s look some more at the static money demand equation:

$$v' \left( \frac{Q_t}{P_t} \right) + u_{c,t} = u_{c,t} R_t,$$

or

$$\psi_q \left( \frac{Q_t}{P_t} \right)^{-\sigma_q} + u_{c,t} = R_t.$$

Linearizing this around steady state (see below), with $\xi_q = 0$,

$$\hat{q}_t = -\frac{1}{\sigma_q} \left[ \frac{R}{R-1} \hat{R}_t + \hat{u}_{c,t} \right], \quad q_t = \frac{Q_t}{P_t},$$

$$\hat{u}_{c,t} = \frac{U_{cc} C_t [\hat{c}_t - b \hat{c}_{t-1}] - b \beta [\hat{c}_{t+1} - b \hat{c}_t]}{u_c},$$

when $\chi = 0$.

Under scheme #2 the household that cannot change $Q$ must set it as follows:

$$Q_t = \hat{\pi}_{t-1} Q_{t-1},$$

where $Q_{t-1}$ is its own past value of $Q$. Then, the objective becomes

$$E_t \sum_{j=0}^{\infty} (\beta \xi_q)^j \left\{ v' \left( \frac{\hat{Q}_t \hat{\pi}_t}{P_t \hat{\pi}_{t+j}} \right) - u_{c,t+j} [R_{t+j} - 1] \right\} \frac{\hat{\pi}_t}{\hat{\pi}_{t+j}} = 0.$$
1.2.3. Household Wage-Setting

The household is a monopoly supplier of its own (differentiated) labor. It sells labor to a firm which transforms household labor into a homogeneous input good, \( X \). That producer’s production function is:

\[
X = \left[ \int_{0}^{1} \left( L^h \right)^{\frac{1}{\lambda_w}} dh \right]^{\lambda_w}, \quad 1 \leq \lambda_w < \infty. \tag{1.5}
\]

The producer’s problem is to maximize (1.5) subject to:

\[
W_t X_t - \int_{0}^{1} W_t^h L_t^h dh, \tag{1.6}
\]

where \( W_t \) is the price of \( X_t \). The first order condition associated with this problem, which is the household’s demand for labor curve, is:

\[
L_t^h = \left[ \frac{W_t^h}{W_t} \right]^{\frac{1}{\lambda_w}} X_t. \tag{1.7}
\]

The household’s Euler equation for setting \( W_t^h \), when it is allowed to do so, is:

\[
0 = E_{t,w} \sum_{j=0}^{\infty} (\zeta_w \beta)^j \bar{L}_{t+j} \left[ \frac{\bar{W}_t W_t}{W_t} \frac{X_{t,j} u_{c,t+j}}{\lambda_w} + f_{L,t+j} \right], \tag{1.8}
\]

where \( \bar{W}_t \) is the value of \( W_t^h \) that it chooses. Here,

\[
\pi_{t+1} = \frac{P_{t+1}}{P_t}.
\]

Also,

\[
X_{t,j} = \frac{\bar{\pi}_j}{\pi_{t+1} \cdots \pi_{t+j}},
\]

under scheme #1 and

\[
X_{t,j} = \frac{\pi_t}{\pi_{t+j}},
\]

under scheme #2. In both cases, \( X_{t,j} = 1 \) for \( j = 0 \).
To interpret (1.7), consider first the case, $\xi_w = 0$. In this case, the above expression reduces to:

$$\frac{1}{\lambda_w} \frac{\tilde{W}_t}{P_t} u_{ct,t} + f_{L,t}^h = 0.$$ 

Another way to write this is:

$$-\lambda_w f_{L,t}^h = \frac{\tilde{W}_t}{P_t} u_{ct,t}.$$ 

The right side looks like the average revenue, in units of utility, associated with selling labor. You set that equal to a constant markup, $\lambda_w$, over the marginal cost of producing the good, i.e., the marginal utility of leisure, $-f_{L}^h$. With $\xi_w > 0$, you get a weighted average of these terms over each date.

1.2.4. Investment Decision

We adopt two different formulations of technology. The first is the Lucas-Prescott formulation, and the second is the cost-of-change formulation. We begin with the first formulation. The technology for producing end of period $t$ capital, $k^{h'}$:

$$k^{h'} \leq Q((1 - \delta)k^h, I),$$

and

$$Q(y, z) = \left[a_1 y^\psi + a_2 z^\psi\right]^{1/\psi},$$

for $\psi \leq 1$. In (1.8)–(1.9), $y$ denotes previously installed capital after depreciation and $z$ denotes new investment goods.$^3$ When $\psi = 1$, (1.9) corresponds to the conventional linear capital accumulation equation. In the case of adjustment costs, $\psi < 1$, the marginal product of new investment goods is decreasing in the flow of investment. We choose the constants, $a_1 > 0$ and $a_2 > 0$, to guarantee $Q_1 = Q_2 = 1$ when $k^{h'} = k^h$, so that $I = \delta k^h$. In this case, $y = (1 - \delta)k^h$ and $z = \delta k^h$. This has the effect of making the nonstochastic steady state properties of the model invariant to the value of $\psi$. We now work out the restrictions implied by this on $a_1$ and $a_2$.

$$Q_1 = a_1 \left(\frac{Q}{y}\right)^{1-\psi} = a_1 \left[a_1 + a_2 \left(\frac{\delta}{1 - \delta}\right)^\psi\right]^{1 - \psi} = 1$$

\[
Q_2 = a_2 \left( \frac{Q}{z} \right)^{1-\psi} = a_2 \left[ a_1 \left( \frac{1-\delta}{\delta} \right)^{\psi} + a_2 \right]^{1-\psi} = 1
\]

Then, \(Q_1/Q_2 = 1\) implies

\[
a_2 = a_1 \left( \frac{\delta}{1-\delta} \right)^{1-\psi}.
\]

Substituting this into the expression for \(Q_1\) and setting that to 1 implies:

\[
a_1 = (1-\delta)^{1-\psi}.
\]

Putting this into the expression for \(a_2\), we get:

\[
a_2 = \delta^{1-\psi}.
\]

The price of \(I\), in terms of consumption goods, is \(P_t = 1\). The period \(t\) marginal cost, to the household, of producing an extra unit of capital to be used in the beginning of the following period is denoted \(P_{k'}\). This too is denominated in consumption goods. We have the usual expression:

\[
P_{k'} = \frac{P_t}{Q_2} = \frac{1}{Q_2} = \frac{1}{a_2} \left[ a_1 \left( \frac{(1-\delta)k}{I} \right)^{\psi} + a_2 \right]^{\frac{1}{\psi}}.
\]  

(1.10)

By standard optimization logic, \(P_{k'}\) is also the value to the household of an extra unit of \(k'^{h'}\). If there were a competitive market in \(k'^{h'}\), it would sell for \(P_{k'}\). Since \(P_t = 1\), and Tobin’s \(q\) is \(P_{k'}/P_t\), we can interpret \(P_{k'}\) as Tobin’s \(q\). If the household (or, an entrepreneur) issued equity to finance the purchase of a unit of \(k'^{h'}\), it would sell for \(P_{k'}\). So, \(P_{k'}\) can be thought of as a stock price too. Note the model’s prediction that a positive monetary shock, by driving up \(I\), drives \(P_{k'}\) up.

We now derive the household’s first order condition for capital. Suppose they increase \(k'^{h'}\) by one unit in period \(t\). The cost, in consumption goods, is \(P_{k'}\). In utility terms, it is \(u_c P_{k'}\). There are two gains which occur in the next period, \(t+1\). First, the household receives \(R_{t+1}/P_t\) extra rent on the additional capital. The utility value of this is \(\beta u_{c,t+1} R_{t+1}/P_t\). Second, at the end of next period, the household has \((1-\delta)\) units of extra capital left over. The marginal product of each extra unit of capital in the next period, in producing new capital, is \(Q_{1,t+1}\). But, this is valued, in the next period at \(P_{k',t+1}\). So, a marginal increase in new capital today has a residual value of \((1-\delta)Q_{1,t+1}P_{k',t+1}\) in the next period. In today’s utility terms, this corresponds to \(\beta u_{c,t+1}(1-\delta)Q_{1,t+1}P_{k',t+1}\). In sum:

\[
E_{t-1} P_{k',t} u_{c,t} = \beta E_{t-1} u_{c,t+1} [R_{t+1} + (1-\delta)Q_{1,t+1}P_{k',t+1}],
\]  

(1.11)
where the $E_{t-1}$ reflects the assumption that this decision is made before the current period monetary shock. Also, 
\[ r^{k}_{t+1} = \frac{R^{k}_{t+1}}{P^{k}_{t+1}}. \]

We can define the rate of return on equity as 
\[ r^{e}_{t+1} = \frac{r^{k}_{t+1} + (1 - \delta) Q_{1,t+1} P_{k',t+1}}{P_{k',t}}. \]

Note there is a dividend term, $r^{k}_{t+1}/P_{k',t}$, and a capital gain term in this expression. The Euler equation is simply: 
\[ E_{t-1} \frac{\beta u_{c,t+1}}{u_{c,t}} r^{e}_{t+1} = 1. \]

By recursively iterating on this formula, we can get the usual present value formula for stock prices:
\[ P_{k',t} = E_{t} \sum_{j=1}^{\infty} \left( \prod_{i=1}^{j} p_{c,t+i} [Q_{1,t+i}(1 - \delta)] \right) r^{k}_{t+j}, \]
\[ P_{c,t+1} = \frac{\beta u_{c,t+1}}{u_{c,t}} \]  
(1.12)

Expressions (1.10) and (1.12) are two alternative ways of characterizing stock prices, where the latter is the most standard one. The former, (1.10), although less standard, is more convenient for computation purposes and for understanding how stock prices respond to shocks in the model. (All this material was taken from a Christiano-Fisher paper, and assumes no leverage.)

We now turn to the cost-of-change formulation of technology:
\[ k^{h'} = (1 - \delta) k^{h} + F(I, L_{-1}), \]
where 
\[ F(I, L_{-1}) = \left( 1 - S \left( \frac{I}{L_{-1}} \right) \right) I \]
\[ S(x) = g_{3} \left\{ \exp \left[ g_{1} (x - 1) \right] + \frac{g_{1}}{g_{2}} \exp \left[ -g_{2} (x - 1) \right] - 2 \right\}. \]

Let’s study $S$ a little. The derivative is:
\[ S'(x) = g_{1} g_{3} \left\{ \exp \left[ g_{1} (x - 1) \right] - \exp \left[ -g_{2} (x - 1) \right] \right\}, \]
so that \( S(1) = S'(1) = 0 \), and the steady state is invariant to \( g_1, g_2, g_3 \). With this parameterization one can separately control the slope when \( x \) is positive and when it is negative.

To understand the first order conditions for this problem, consider the following simplified Lagrangian:

\[
\max \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t - h_t) + \mu_t \left[ -P_t (I_t + c_t) + R_t^k k_t + A_t \right] + \lambda_t \left[ (1 - \delta)k_t + F(I_t, I_{t-1}) - k_{t+1} \right] \right\},
\]

where \( A_t \) is exogenous. Also,

\[ P_t \mu_t = u_{c,t}. \]

The first order conditions are, for \( I_t \):

\[ u_{c,t} = \lambda_t F_{1,t} + \beta \lambda_{t+1} F_{2,t+1} \]

and, for \( k_{t+1} \):

\[ \lambda_t = \beta \left[ \frac{u_{c,t+1}}{P_{t+1}} R_{t+1}^k + \frac{\lambda_{t+1}(1 - \delta)}{u_{c,t+1}} \right], \]

or,

\[ \frac{u_{c,t}}{u_{c,t}} \lambda_t = \beta u_{c,t+1} \left[ r_{t+1}^k + \frac{\lambda_{t+1}(1 - \delta)}{u_{c,t+1}} \right]. \]

Let

\[ P_{k', t} = \frac{\lambda_t}{u_{c,t}}. \]

This is the household’s (shadow) price of a unit of installed capital. Note that in the case of no adjustment costs, \( F_1 = 1, F_2 = 0 \), then \( P_{k', t} = 1 \). Then, the Euler equations reduce to:

\[ u_{c,t} = \beta u_{c,t+1} \left[ r_{t+1}^k + P_{k', t}(1 - \delta) \right]. \]

This last expression requires no explanation. Also

\[ u_{c,t} = u_{c,t} P_{k', t} F_{1,t} + \beta u_{c,t+1} P_{k', t+1} F_{2,t+1}. \]

>From this equation, it is obvious that \( P_{k', t} = 1 \) in steady state, when \( F_{1,t} = 1, F_{2,t+1} = 0 \). This equation is easy to understand when \( F_2 = 0 \). In this case, it says that \( P_{k', t} \), which is the marginal cost (in consumption units) of \( k_{t+1} \), has to equal the ratio of the price of
investment goods (i.e., 1), divided by the marginal product of investment goods, \( F_1 \). In our setting marginal cost involves future considerations, and that’s why this equation is dynamic. Here’s a little more intuition about this equation. Imagine you have a consumption good in hand. On the one hand, you could eat it and ‘earn’ \( u_{c,t} \) in utility. On the other hand, you could transform it, one-for-one, into an investment good and use that to produce \( F_{1,t} \) extra \( k' \). The goods value of this is \( u_{c,t} P_{k',t} F_{1,t} \). Another consequence of the additional investment good is that you have an effect on next period’s cost, \( F_{2,t+1} \), of producing investment goods. That’s what the other term in the above expression captures.

Here is yet another approach to capital. Here, we suppose that the capital services that the household rents out is \( u_t k_t \), where \( k_t \) is the stock of physical capital, and \( u_t \) is the utilization rate of that stock. Now, we suppose that with a higher utilization of capital requires an additional expenditure of resources, according to the function, \( \bar{\delta}(u_t) \), where \( \bar{\delta} > 0 \). The simplified version of the household problem now is:

\[
\max \sum_{t=0}^{\infty} \beta^t \{ u(c_t - h_t) + \mu_t \left[ -P_t \left( I_t + c_t + \bar{\delta}(u_t)k_t \right) + R^k_t u_t k_t + A_t \right] \\
+ \lambda_t \left[ (1 - \delta)k_t + F(I_t, I_{t-1}) - \bar{k}_{t+1} \right] \},
\]

where the bar over \( k_t \) is to distinguish this capital, the physical capital, from the capital that goes into firms’ production functions, which is

\( k_t = u_t \bar{k}_t \).

Firms know nothing of the distinction between capital utilization and physical capital. Those are variables that are of concern to the households only. The capital accumulation technology is:

\( \bar{k}_{t+1} = (1 - \delta) \bar{k}_t + F(I_t, I_{t-1}) \).

The first order conditions for this problem are like the ones above, except that the condition for physical capital, \( \bar{k}_{t+1} \), now becomes

\[
E_{t-1} u_{c,t} \frac{\lambda_t}{u_{c,t}} = \beta E_{t-1} u_{c,t+1} \left[ u_{t+1} r_{t+1}^k - \bar{\delta}(u_{t+1}) + \frac{\lambda_{t+1}}{u_{c,t+1}} (1 - \delta) \right],
\]

where the conditional expectation operator indicates the information that is available at the time that the investment decision is known. As before, we can call \( \lambda_t / u_{c,t} \) the price of capital, so that

\[
E_{t-1} \left\{ u_{c,t} P_{k',t} - \beta u_{c,t+1} \left[ u_{t+1} r_{t+1}^k - \bar{\delta}(u_{t+1}) + P_{k',t+1} (1 - \delta) \right] \right\} = 0.
\]
There is, of course, now an extra first order condition pertaining to capital utilization. Differentiating with respect to \(u_t\), we obtain:

\[
E_{t-1} \mu_t \left[ R_t^k \bar{k}_t - P_t \bar{\delta}'(u_t) \bar{k}_t \right] = 0.
\]

Again, the assumption is that the capacity utilization decision is taken before the realization of the current period uncertainty. Then,

\[
E_{t-1} u_{c,t} \left[ r_t^k - \bar{\delta}'(u_t) \right] = 0,
\]

after dividing by \(\bar{k}_t\), a variable that is known at time \(t - 1\). This says that the marginal benefit of raising utilization a little, \(r_t^k\), must match the marginal cost. Note that if \(\bar{\delta}' = 0\), then basically they would like to set \(u_t = \infty\). The ‘correct’ way to handle this is to specify some upper bound on \(u_t\) and then define the household’s problem as a constrained problem. But, this makes things too complicated for us. So, instead we should just eliminate this equation altogether when \(\delta\) is a constant, and include it when \(\delta\) is a non-trivial function.

2. Firms

There are three types of firms. One hires labor from the households and transforms it into a homogeneous input good, denoted \(X\). This was discussed in the previous section. The other type of firm buys \(X\) and rents capital, and produces an intermediate good which it sells to a final goods producer. There is a continuum of these intermediate goods producers, each of which is a monopoly supply of its own good and is competitive in the markets for inputs.

2.1. Final Good Firms

Production function of representative final good firm:

\[
Y_t = \left[ \int_0^1 Y_{it} \frac{1}{\gamma} \, di \right] \lambda_f, \quad 1 \leq \lambda_f < \infty
\]  

(2.1)

Fonc:

\[
\left( \frac{P_t}{P_{it}} \right)^{\frac{\lambda_f}{\gamma - 1}} = \frac{Y_{it}}{Y_t},
\]

(2.2)

Price of final Goods:

\[
P_t = \left[ \int_0^1 P_{it}^{\frac{1}{1-\gamma}} \, di \right]^{(1-\lambda_f)}.
\]

(2.3)
2.2. Intermediate Good Firms

Production function of $i^{th}$, $i \in (0, 1)$, intermediate good firm:

\[ Y_{it} = K_{it}^{\alpha} X_{it}^{1-\alpha}, \quad 0 < \alpha < 1. \quad (2.4) \]

2.2.1. Non-Price Euler Equations

Suppose the $i^{th}$ firm’s price, $P_{it}$, is somehow given (see next subsection for how this is determined). Then the firm must produce $Y_{it}$ in (2.2). Firm solves:

\[
\min_{K_i, X_i} WRX_i + R^k K_i + \zeta [Y_i - K_i^{\alpha} X_i^{1-\alpha}],
\]

where $\zeta$ is the Lagrange multiplier. Solving:

\[
\zeta = \left( \frac{1}{1-\alpha} \right)^{(1-\alpha)} \left( \frac{1}{\alpha} \right)^{\alpha} (R^k)^{\alpha}(WR)^{1-\alpha},
\]

\[
WR = \zeta F_X,
\]

\[
R^k = \zeta F_K,
\]

were $\zeta$ is marginal cost, $MC$. Let $s$ denote real marginal cost (the inverse of the markup):

\[
s \equiv \frac{MC}{P} = \frac{\zeta}{P}.
\]

Then, we get the usual foncs:

\[
\frac{W}{P} = \frac{sF_X}{R},
\]

\[
\frac{R^k}{P} = sF_K.
\]

The first of these is the ‘X-input demand’ equation. In a graph with $W/P$ on the vertical axis and $X$ on the horizontal, demand is downward-sloping. A fall in $R$ or a rise in $s$ shifts labor demand to the right. Note:

\[
R^k = WR \frac{\alpha}{1-\alpha} \frac{X_i}{K_i} = WR \frac{\alpha}{1-\alpha} \frac{X}{K},
\]

\[
Y_i = \left( \frac{X}{K} \right)^\alpha X_i,
\]

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all \( i \in (0, 1) \), where
\[
X = \int_0^1 X_i d_i, \quad K = \int_0^1 K_i d_i.
\]
Substituting out for \( R^k \) in the marginal cost expression,
\[
s_t = \frac{1}{1 - \alpha} \left( \frac{X_t}{K_t} \right)^\alpha \frac{W_t}{P_t} R_t.
\]

2.2.2. Price Euler Equation

Randomly, \( 1 - \xi_p \) firms get to set price, and \( \xi_p \) must set \( P_t = \tilde{\pi}_t P_{t-1} \). The ones that get to change their price, \( \tilde{P}_t \), do so to solve:
\[
E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j u_{t+j} Y_{t+j} P_{t+j} \left( \frac{\tilde{P}_{t+j}}{\tilde{\pi}_{t+j}} \right)^{\frac{\lambda_j}{1 - \gamma_j}} \left[ \left( \tilde{P}_{t+j} \tilde{\pi}_{t+j} \right)^{1 - \frac{\lambda_j}{1 - \gamma_j}} - MC_{t+j} \left( \tilde{P}_{t+j} \tilde{\pi}_{t+j} \right)^{-\frac{\lambda_j}{1 - \gamma_j}} \right] = 0.
\]  
(2.5)

Fonc:
\[
E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j u_{t+j} Y_{t+j} P_{t+j} \frac{\tilde{P}_{t+j}}{\tilde{\pi}_{t+j}} \frac{\tilde{P}_{t+j}^{1 - \gamma_j} - \gamma_j MC_{t+j}}{\lambda_j - 1} \tilde{\pi}_{t+j}^j = 0.
\]  
(2.6)

or, after rearranging:
\[
E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \left( u_{t+j} P_{t+j} \right) Y_{t+j} \left( \frac{P_{t+j}}{\tilde{P}_{t+j}} \right)^{\frac{\lambda_j}{1 - \gamma_j}} \left[ \frac{\tilde{P}_{t+j} \tilde{\pi}_{t+j}}{P_{t+j}} - \gamma_j MC_{t+j} \right] \tilde{\pi}_{t+j}^j = 0.
\]

Define:
\[
\tilde{\pi}_t = \frac{\tilde{P}_t}{P_t}, \quad s_t = \frac{MC_t}{P_t} = \frac{1}{1 - \alpha} \left( \frac{L_t}{K_t} \right)^\alpha w_t R_t, \quad \pi_{t+1} = \frac{P_{t+1}}{P_t}, \quad \tilde{v}_t = u_t P_t,
\]
\[
X_{t,j} = \begin{cases} 
\prod_{k=1}^{j} \frac{\pi_{t+k}}{\pi_{t+1}} & j > 0 \\
1 & j = 0
\end{cases}
\]

After multiplying by \( P_t^{\lambda_j/(\gamma_j - 1)} \) and rearranging:
\[
E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \tilde{v}_{t+j} Y_{t+j} \left( X_{t,j} \right)^{\frac{-\lambda_j}{\gamma_j - 1}} [\tilde{\pi}_t X_{t,j} - \gamma_j s_{t+j}] = 0.
\]  
(2.7)
2.3. Final Good Price

\[ P_t = \left[ (1 - \xi_p) \hat{P}_t^{1-\lambda_f} + \xi_p \left( \hat{\pi}_t \right)^{1-\lambda_f} \right]^{1-\lambda_f}, \]

or

\[ 1 = \left[ (1 - \xi_p) \hat{P}_t^{1-\lambda_f} + \xi_p \left( \frac{\hat{\pi}_t}{\pi_t} \right)^{1-\lambda_f} \right]^{1-\lambda_f} \]

3. Monetary Policy

We will consider two types of policies. In one, monetary policy is an exogenous stochastic process for money growth,

\[ \mu_t = \frac{\hat{M}_{t+1}}{M_t}. \]

Here is the time series representation we will assume for this:

\[ \mu_t = \phi \mu_{t-1} + \varepsilon_t + \omega_1 \varepsilon_{t-1} + \omega_2 \varepsilon_{t-2}. \]  \hfill (3.1)

We will also consider policies like this:

\[ \hat{R}_t = \rho \hat{R}_{t-1} + (1 - \rho) \hat{R}_t^* \]

where

\[ \hat{R}_t^* = a_0 + a_\pi E_t \hat{\pi}_{t+1} + a_y \hat{y}_t, \]  \hfill (3.2)

where \( y_t = \frac{Y_t - Y^*}{Y^*} \) denotes the percent deviation of output from steady state (‘trend’).

4. Loan Market Clearing

Financial intermediaries receive \( M_t - Q_t \) from households and a transfer, \( (1 - \mu_t)M_t \) from the monetary authority. They lend the full amount out to the intermediate good firms which use the funds to pay for \( X \). Loan market clearing requires

\[ W_t X_t = \mu_t M_t - Q_t. \]  \hfill (4.1)
5. Resource Constraint

We want to develop an expression that relates aggregate consumption and investment to aggregate employment and the aggregate stock of capital. It turns out that we get close to such an expression. An extra term having to do with the distribution of prices and wages, enters too.

Let \( Y^* \) denote the unweighted average of output in each sector:\(^4\)

\[
Y^* = \int_0^1 Y(f)df = \int_0^1 F(K(f), X(f))df = \int_0^1 F\left(\frac{K}{X}, 1\right)X(f)df,
\]

where

\[
K = \int_0^1 K(f)df, \quad X = \int_0^1 X(f)df,
\]

and \( X \) is the total output of the labor contractor (the new line here is that the labor contractor hires the labor effort of individual households to produce a homogeneous good used by the intermediate good producers). Now, each intermediate good firm confronts the same factor prices, so they hire capital and labor in the same proportions. That’s why the capital/labor ratio for each individual firm is identical, and identical to the aggregate. Then,

\[
Y^* = F\left(\frac{K}{X}, 1\right)\int_0^1 X(f)df = F\left(\frac{K}{X}, 1\right)X = F(K, X).
\]

So, the unweighted sum of output of the intermediate goods producers can be represented as a function of aggregate capital and the aggregate output of the labor contractor. There are two reasons why this expression falls short. First, the unweighted sum of output, \( Y^* \), does not correspond to anything in this model. Second, we want to relate output to the

---

\(^4\)Here, we’re assuming more than we need. For example, if the production function were linear homogeneous in \( K \) and \( X - X^0 \), where \( X^0 \) is a fixed cost, then all firms would use the same ratio of \( K \) to \( X \). To see this, note that they all face the same rental rate on capital, \( r^k \) and the same wage rate, \( W \):

\[ F_K = r^k, \quad F_X = W. \]

Then, the ratio,

\[
\frac{r^k}{W} = \frac{F_K}{F_X} = f\left(\frac{K}{X - X^0}\right),
\]

so everyone has the same \( K/X \).
unweighted sum of labor, not $X$, because that is how the labor input is measured in the data.

Note,

$$Y^* = \int_0^1 Y(f)df = \int_0^1 \left[ \frac{P}{P(f)} \right]^{\frac{\lambda_f}{1-\lambda_f}} Ydf$$

$$= YP^{\frac{\lambda_f}{1-\lambda_f}} (P^*)^{\frac{\lambda_f}{1-\lambda_f}},$$

where $Y$ is aggregate output of the final good sector, as defined above, and $P^*$ is the indicated weighted average of the individual prices, where the weights differ from what they are in $P$. So,

$$Y^* = \left( \frac{P}{P^*} \right)^{\frac{\lambda_f}{1-\lambda_f}} Y.$$

Now, actual output is $Y$, and this is what is available to be divided into consumption and investment. So, we write the resource constraint as follows:

$$\delta(u)\bar{k} + C + I \leq \left( \frac{P^*}{P} \right)^{\frac{\lambda_f}{1-\lambda_f}} F(K, X),$$

where $I$ denotes investment.

We also want a measure of ‘employment’ for our economy (this usage of $L$ is inconsistent with the usage of $L$ way up above...but we intend to go with this new usage):

$$L = \int_0^1 L(h)dh$$

Now

$$L = \int_0^1 \left[ \frac{W(h)}{W} \right]^{\frac{\lambda_w}{1-\lambda_w}} Xdh$$

$$= WX^{\frac{\lambda_w}{1-\lambda_w}} (W^*)^{\frac{\lambda_w}{1-\lambda_w}}$$

$$= X \left( \frac{W}{W^*} \right)^{\frac{\lambda_w}{1-\lambda_w}},$$

where

$$W^* = \left[ \int_0^1 W(h)^{\frac{\lambda_w}{1-\lambda_w}} dh \right]^{\frac{1-\lambda_w}{\lambda_w}}.$$
Then, the resource constraint can be written as a function of aggregate capital and labor as follows:

\[
\delta(u)k + C + I \leq \left( \frac{P^*}{P} \right)^{\frac{\lambda_f}{\lambda_f - 1}} F \left[ K, \left( \frac{W^*}{W} \right)^{\frac{\lambda_w}{\lambda_w - 1}} L \right].
\]

This is our aggregate resource constraint.

The analysis is still slightly incomplete, however, because we have to discuss the determination of \(P^*\) and \(W^*\).

The same logic that delivers expressions for \(P^*\) and \(W^*\) delivers expressions for these things too:

\[
W_t^* = \left[ (1 - \xi_w) \left( \frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} + \xi_w \left( \frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} \right]^{\frac{1 - \lambda_w}{\lambda_w}}.
\]

Divide both sides by \(W_t\):

\[
\frac{W_t^*}{W_t} = \left[ (1 - \xi_w) \left( \frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} + \xi_w \left( \frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} \right]^{\frac{1 - \lambda_w}{\lambda_w}}
\]

or,

\[
w_t^* = \left[ (1 - \xi_w) \left( \frac{\tilde{w}_t}{w_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} + \xi_w \left( \frac{\tilde{w}_t}{w_t} \right)^{\frac{\lambda_w}{1 - \lambda_w}} \right]^{\frac{1 - \lambda_w}{\lambda_w}},
\]

where

\[
w_t^* = \frac{W_t^*}{W_t}, \quad w_t = \frac{W_t}{P_t}.
\]

6. Solving the Model

We now turn to solving the model. We do so by first linearizing the model’s Euler equations and other constraints around steady state. The first section computes the steady state. The second subsection presents the linearization formulas. The third section shows how we solved the model.

6.1. Steady State

These are the nine steady state variables we seek: \(w, r^k, k, L, R, \pi, q, c, m\). (In steady state, \(X = L\)). The investment equation is:

\[
1 = \beta \left[ r^k + (1 - \delta) \right], \quad \text{(6.1)}
\]
where $r^k = (MC/P) \times F_k$ implies:

$$r^k = \frac{\alpha}{1 - \alpha} wR\frac{L}{K}$$  (6.2)

Note that in (6.1), we have set $\delta(u) = 0$ in steady state. As a result, $\bar{k} = k$ ($k$ and $K$ mean the same thing in these notes) in steady state. The $M'$ equation is:

$$1 = \beta \frac{R}{\pi}$$  (6.3)

The money equation is:

$$v'(q) = u_c(R - 1).$$  (6.4)

The labor decision is:

$$w \frac{u_c}{\lambda_w} + f_L = 0.$$  (6.5)

The marginal cost equation is:

$$1 = \lambda_f \left( \frac{1}{1 - \alpha} \right)^{(1-\alpha)} \left( \frac{1}{\alpha} \right)^{\alpha} (r^k)^{\alpha} (wR)^{1-\alpha}.$$  (6.6)

The money market clearing condition is:

$$wL = \mu m - q.$$  (6.7)

The resource constraint is:

$$c + \delta K = K^\alpha L^{1-\alpha}.$$  (6.8)

Also, constancy of $m$ implies:

$$\mu = \pi.$$  (6.9)

We solve for the nine variables like this. First, compute $r^k$ from (6.1). Then, use (6.6) to obtain $wR$. Next, use (6.3) to compute $R$ and then get $w$ from $w = wR/R$. Then, use (6.2) to get $L/K$. Now we use (6.5) and (6.8) to solve for $c$ and $L$. By (6.8), we get:

$$c = \left[ \left( \frac{K}{L} \right)^{\alpha} - \delta \frac{K}{L} \right] L.$$  (6.10)

Let the utility function be

$$U = \frac{(c - H)^{1-\sigma_c}}{1 - \sigma_c}, \quad H = \frac{bc}{1 - \chi}, \quad b < 1 - \chi,$$

$$f(L) = \frac{-L^{1+\sigma_L}}{1 + \sigma_L}, \quad v(q) = \frac{q^{1-\sigma_q}}{1 - \sigma_q}. $$  (6.11)
so that, from (1.1):

\[ u_c = \left(1 - \frac{b}{1 - \chi}\right)^{-\sigma_e} \cdot c^{-\sigma_e}, \]

in the case of external habit. In the case of internal habit,

\[ u_c = \left(1 - \frac{b}{1 - \chi}\right)^{-\sigma_e} \cdot c^{-\sigma_e} \left[1 - \frac{b\beta}{1 - \beta\chi}\right]. \]

Writing out (6.5):

\[ w \frac{u_c}{\lambda_w} = L^{\sigma_e} \psi_L. \quad (6.12) \]

Combining (6.10) and (6.12), we obtain, in the case of external habit:

\[ w \left(1 - \frac{b}{1 - \chi}\right)^{-\sigma_e} \left[\left(\frac{K}{L}\right)^\alpha - \delta\frac{K}{L}\right]^{-\sigma_e} \frac{\lambda_w}{\lambda_w} = L^{(\sigma_L + \sigma_e)} \psi_L, \]

and

\[ w \left(1 - \frac{b}{1 - \chi}\right)^{-\sigma_e} \left[\left(\frac{K}{L}\right)^\alpha - \delta\frac{K}{L}\right]^{-\sigma_e} \left[1 - \frac{b\beta}{1 - \beta\chi}\right] = L^{(\sigma_L + \sigma_e)} \psi_L, \]

in the case of internal habit. Since \( w \) and \( K/L \) have already been determined, this can trivially be solved for \( L \). With \( L \) and \( K/L \), we compute \( K \). We get \( c \) from (6.10) and \( q \) from (6.4). Finally, we get \( m \) from (6.7).

Note that nothing is said here about capital utilization or the utilization cost function. It just doesn’t appear anywhere in the steady state equations described above, given our assumption that this cost is zero in steady state. We assume the steady state derivative of the function, \( \delta'(u) \), is equal to \( r^k \):

\[ r^k = \delta'(u), \]

so that \( \delta'(u) \) is determined recursively, after all the other steady state variables are determined in the manner described above. Finally, we assume:

\[ u = 1, \]

so that, in steady state,

\[ \bar{k}_t = k_t. \]

These assumptions will play a role in the linearized system.
The utilization cost function will play an important role in the dynamics of the linearized system. There, we will need its curvature:

\[ \sigma_{i} = \frac{\tilde{\delta}''}{\tilde{\delta}}. \]

So, adding capital utilization adds just this one parameter to the system, \( \sigma_{i} \).

6.2. Linearizing the Equations

We will now linearize the various equations.

6.2.1. Linearizing the Price Equation

The Euler equation of a firm that changes its price in period \( t \) is given in (2.7). Letting

\[ A_{t,j} = u_{c,t+j} X_{t,j}^{\frac{\pi}{1 - \beta}}, \]

we can write this as follows:

\[
A_{t,0} \left[ \tilde{p}_{t} X_{t,0} - \lambda_{f} s_{t} \right] + \beta \xi_{p} A_{t,1} \left[ \tilde{p}_{t} X_{t,1} - \lambda_{f} s_{t+1} \right] \\
+ (\beta \xi_{p})^{2} A_{t,2} \left[ \tilde{p}_{t} X_{t,2} - \lambda_{f} s_{t+2} \right] + (\beta \xi_{p})^{3} A_{t,3} \left[ \tilde{p}_{t} X_{t,3} - \lambda_{f} s_{t+3} \right] + ...
\]

where

\[ X_{t,j} = \frac{\frac{\pi^{j}}{\pi_{t+1} \cdots \pi_{t+j}}}, \]

under scheme #1 and

\[ X_{t,j} = \frac{\pi^{t}}{\pi_{t+j}}, \]

under scheme #2.

Under scheme #1, differentiate with respect to \( \pi_{t+j}, j > 0 \):

\[- (\beta \xi_{p})^{j} A_{t,j} \tilde{p}_{t} \frac{X_{t,j}}{\pi_{t+j}} - (\beta \xi_{p})^{j+1} A_{t,j+1} \tilde{p}_{t} \frac{X_{t,j+1}}{\pi_{t+j+1}} - ...
\]

or, in steady state (\( \tilde{p} = 1 \)):

\[- (\beta \xi_{p})^{j} \frac{1}{\pi} \frac{u_{c} Y}{1 - \beta \xi_{p}}. \]

Under scheme #2, differentiate with respect to \( \pi_{t+j}, j > 0 \):

\[- (\beta \xi_{p})^{j} A_{t,j} \tilde{p}_{t} \frac{\pi^{t}}{\pi_{t+j}}. \]
or, in steady state:

\[- (\beta \xi_p)^j \hat{p} \frac{u_c Y}{\hat{\pi}}. \]

Under scheme #2, \( \pi_t \) matters too. Differentiate with respect to \( \pi_t \):

\[ \frac{\beta \xi_p u_c Y \hat{p}}{1 - \beta \xi_p} \frac{1}{\hat{\pi}}. \]

Differentiate this with respect to \( \hat{p}_t \) and evaluate the result in steady state

\[ \frac{u_c Y}{1 - \beta \xi_p}, \]

where we have used the fact that in steady state, \( A_{t,j} = u_c Y \). Differentiate with respect to \( s_{t+j}, j \geq 0 \) and evaluate the result in steady state:

\[- (\beta \xi_p)^j u_c Y. \]

Using this information to take the Taylor series expansion of (2.7) about steady state, we get:

\[ 0 = \frac{u_c Y}{1 - \beta \xi_p} \hat{p}_t - \lambda_f u_c Y s \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j} \]

\[ + \frac{\beta \xi_p u_c Y \hat{p}}{1 - \beta \xi_p} (1 - \varphi) \hat{\pi}_t - \frac{u_c Y \hat{p}}{1 - \varphi \beta \xi_p} \sum_{j=1}^{\infty} (\beta \xi_p)^j \hat{\pi}_{t+j}, \]

or, \( s\lambda_f = 1 \),

\[ \hat{\pi}_t = (1 - \beta \xi_p) \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j} - \beta \xi_p (1 - \varphi) \hat{\pi}_t + \frac{1 - \beta \xi_p}{1 - \varphi \beta \xi_p} \sum_{t=1}^{\infty} (\beta \xi_p)^t \hat{\pi}_{t+l} \]

The intuition for the difference between scheme #1 and #2 is discussed in the Q section.

From the formula for the aggregate price level, and using the Calvo trick in the case of scheme #1:

\[ P_t = \left[ (1 - \xi_p) \left( \hat{P}_t \right)^{1 - \lambda_f} + \xi_p (\hat{\pi} P_{t-1})^{1 - \lambda_f} \right]^{1 - \lambda_f}, \]

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or, after dividing by \( P_t \):

\[
1 = \left[ (1 - \xi_p) \left( \tilde{p}_t \right)^{\frac{1}{\lambda_f}} + \xi_p \left( \frac{\bar{\pi}}{\pi_t} \right)^{\frac{1}{\lambda_f}} \right]^{1-\lambda_f}.
\]

Linearizing

\[
\tilde{p}_t = \frac{\xi_p}{1 - \xi_p} \bar{\pi}_t.
\]

Now consider scheme \#2:

\[
P_t = \left[ (1 - \xi_p) \left( \tilde{p}_t \right)^{\frac{1}{\lambda_f}} + \xi_p (\pi_{t-1} P_{t-1})^{\frac{1}{\lambda_f}} \right]^{1-\lambda_f},
\]

or, after dividing by \( P_t \):

\[
1 = \left[ (1 - \xi_p) \left( \tilde{p}_t \right)^{\frac{1}{\lambda_f}} + \xi_p \left( \frac{\pi_{t-1}}{\pi_t} \right)^{\frac{1}{\lambda_f}} \right]^{1-\lambda_f}.
\]

Then,

\[
\tilde{p}_t = \frac{\xi_p}{1 - \xi_p} \left[ \bar{\pi}_t - (1 - \varphi) \bar{\pi}_{t-1} \right].
\]

Substituting,

\[
\bar{\pi}_t - (1 - \varphi) \bar{\pi}_{t-1} = \frac{(1 - \xi_p) (1 - \beta \xi_p)}{\xi_p} \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j}
- \beta (1 - \xi_p) (1 - \varphi) \bar{\pi}_t
+ \frac{1 - \xi_p}{\xi_p} \frac{1 - \beta \xi_p}{1 - \varphi \beta \xi_p} \sum_{t=1}^{\infty} (\beta \xi_p)^t \bar{\pi}_{t+t}
\]

or,

\[
\frac{1 + \beta (1 - \xi_p) (1 - \varphi)}{(1 - \xi_p) (1 - \beta \xi_p)} \sum_{j=0}^{\infty} (\beta \xi_p)^j \hat{s}_{t+j}
+ (1 - \varphi) \bar{\pi}_{t-1}
+ \frac{1 - \xi_p}{\xi_p} \frac{1 - \beta \xi_p}{1 - \varphi \beta \xi_p} \sum_{t=1}^{\infty} (\beta \xi_p)^t \bar{\pi}_{t+t}
\]

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Then, quasi-forward differencing

\[
\frac{[1 + \beta (1 - \xi_p) (1 - \varphi)] (\hat{\pi}_t - \beta \xi_p \hat{\pi}_{t+1})}{(1 - \xi_p) (1 - \beta \xi_p) \hat{s}_t} + (1 - \varphi) (\hat{\pi}_{t-1} - \beta \xi_p \hat{\pi}_t) + \frac{1 - \xi_p}{\xi_p} \frac{1 - \beta \xi_p}{1 - \varphi \beta \xi_p} \beta \xi_p \hat{\pi}_{t+1}
\]

or, after combining:

\[
0 = - \left[ \beta \xi_p (1 + \beta (1 - \xi_p) (1 - \varphi)) + \frac{1 - \xi_p}{\xi_p} \frac{1 - \beta \xi_p}{1 - \varphi \beta \xi_p} \beta \xi_p \right] \hat{\pi}_{t+1}
+ [1 + \beta (1 - \varphi)] \hat{\pi}_t - (1 - \varphi) \hat{\pi}_{t-1} - \frac{(1 - \xi_p) (1 - \beta \xi_p)}{\xi_p} \hat{s}_t
\]

Expanding the expression for \( r^k \):

\[
r^k \hat{r}_t = r^k \hat{w}_t + r^k \hat{R}_t + r^k \hat{L}_t - r^k \hat{K}_t,
\]

so that, 

\[
\hat{r}_t = \hat{w}_t + \hat{R}_t + \hat{L}_t - \hat{K}_t \tag{6.13}
\]

**6.2.2. Linearizing the Q Equation**

We consider scheme \#1 first. Rewriting (1.4):

\[
E_t \sum_{j=0}^{\infty} (\beta \xi_q)^j \{ u' (q_t X_{t,j}) - u_{c,t+j} [R_{t+j} - 1] \} X_{t,j} = 0,
\]

where

\[
X_{t,j} = \frac{\pi^j}{\pi_{t+1} \cdots \pi_{t+j}},
\]

under scheme \#1 and

\[
X_{t,j} = \frac{\pi_t}{\pi_{t+j}},
\]

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under scheme #2.

Writing this out:

\[
0 = v'(\tilde{q}_t) - u_{c,t}(R_t - 1) + \beta \xi_q \{ v'(\tilde{q}_t X_{t,1}) - u_{c,t+1}[R_{t+1} - 1]\} X_{t,1}
+ (\beta \xi_q)^2 \{ v'(\tilde{q}_t X_{t,2}) - u_{c,t+2}[R_{t+2} - 1]\} X_{t,2} + ...
\]

Differentiating with respect to \(\pi_{t+j}, j > 0\), for scheme #1:

\[
- (\beta \xi_q)^j v''_{t+j} \tilde{q}_{t+j} \frac{1}{\pi_{t+j}} - (\beta \xi_q)^{j+1} v''_{t+j+1} \tilde{q}_{t+j+1} \frac{1}{\pi_{t+j+1}} - ...
\]

or, in steady state:

\[
- (\beta \xi_q)^j v'' \tilde{q} \frac{1}{\bar{\pi}}.
\]

Differentiating with respect to \(\pi_{t+j}, j > 0\), for scheme #2:

\[
- (\beta \xi_q)^j v''_{t+j} \tilde{q}_{t+j} \frac{1}{\pi_{t+j}}.
\]

In steady state:

\[
- (\beta \xi_q)^j v'' \tilde{q} \frac{1}{\bar{\pi}}.
\]

Differentiating with respect to \(\pi_t\):

\[
\frac{\beta \xi_q v'' \tilde{q}}{1 - \beta \xi_q}.
\]

Differentiate with respect to \(\tilde{q}_t\):

\[
v''_t + \beta \xi_q v''_{t+1} + (\beta \xi_q)^2 v''_{t+2} + ...
\]

or, in steady state:

\[
\frac{v''}{1 - \beta \xi_q}.
\]

Differentiating with respect to \(R_{t+j}, j \geq 0\), and evaluating in steady state:

\[
- (\beta \xi_q)^j u_c.
\]

Differentiating with respect to \(u_{c,t+j}\) and evaluating that in steady state:

\[
- (\beta \xi_q)^j (R - 1).
\]
Putting all this together:

\[
0 = \frac{v'' \tilde{q}}{1 - \beta \xi_q} \hat{q}_t + \frac{(1 - \varphi) \beta \xi_q v'' \tilde{q}}{1 - \beta \xi_q} \hat{\pi}_t - \frac{v'' \tilde{q}}{1 - \varphi \beta \xi_q} \sum_{j=1}^{\infty} (\beta \xi_q)^j \tilde{\pi}_{t+j} - u_c (R - 1) \sum_{j=0}^{\infty} \left( \frac{R}{R - 1} \hat{R}_{t+j} + \hat{\tilde{u}}_{c,t+j} \right),
\]

where \( \varphi = 1 \) under scheme #1 and \( \varphi = 0 \) under scheme #2.

Rewriting

\[
\hat{q}_t = \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \sum_{j=1}^{\infty} (\beta \xi_q)^j \tilde{\pi}_{t+j} - (1 - \varphi) \beta \xi_q \hat{\pi}_t + (1 - \beta \xi_q) \frac{u_c}{v'' \tilde{q}} (R - 1) \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[ \frac{R}{R - 1} \hat{R}_{t+j} + \tilde{u}_{c,t+j} \right].
\]

Now, note that \( u_c/v' = 1/(R - 1) \). Then, \( u_c/(v'' \tilde{q}) = (u_c/v') v'/v'' \tilde{q} = [1/(R - 1)] v'/v'' \tilde{q} \).

Also,

\[
\sigma_q = -\frac{v'' \tilde{q}}{v'},
\]

so that

\[
\frac{u_c}{v'' \tilde{q}} = \frac{1}{R - 1} \frac{v'}{v'' \tilde{q}} = -\frac{1}{(R - 1) \sigma_q}.
\]

Substituting this into the expression for \( \hat{q}_t \), we get

\[
\hat{q}_t = \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \sum_{j=1}^{\infty} (\beta \xi_q)^j \tilde{\pi}_{t+j} - (1 - \varphi) \beta \xi_q \hat{\pi}_t + \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[ \frac{R}{R - 1} \hat{R}_{t+j} + \hat{\tilde{u}}_{c,t+j} \right],
\]

or

\[
\hat{q}_t = \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \sum_{j=1}^{\infty} (\beta \xi_q)^j \tilde{\pi}_{t+j} - (1 - \varphi) \beta \xi_q \hat{\pi}_t - \frac{1 - \beta \xi_q}{\sigma_q} \sum_{j=0}^{\infty} (\beta \xi_q)^j \left[ \frac{R}{R - 1} \hat{R}_{t+j} + \hat{\tilde{u}}_{c,t+j} \right].
\]
Note that the weight on future inflation is unity under scheme #1 and it is fairly small under scheme #2. This makes a lot of sense. Under scheme #2 the q decision is indexed to inflation, and so there is no need for front-loading in this case.

Then, multiplying by $(1 - \beta \xi_q L^{-1})$, where $L^j x_t \equiv x_{t-j}$, for integer values of $j$:

\[
\hat{q}_t = \beta \xi_q \hat{q}_{t+1} + \beta \xi_q \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \hat{\pi}_{t+1} - (1 - \varphi) \beta \xi_q (\hat{\pi}_t - \beta \xi_q \hat{\pi}_{t+1}) - \frac{1 - \beta \xi_q}{\sigma_q} \left( \hat{u}_{c,t} + \frac{R}{R - 1} \hat{R}_t \right)
\]

The aggregate Q is:

\[
Q_t = (1 - \xi_q) \hat{Q}_t + \xi_q \hat{\pi} Q_{t-1},
\]

in scheme #1 and

\[
Q_t = (1 - \xi_q) \hat{Q}_t + \xi_q \pi_{t-1} Q_{t-1},
\]

in scheme #2.

Working with scheme #1

\[
q_t = (1 - \xi_q) \hat{q}_t + \xi_q \hat{\pi}_t.
\]

Linearizing:

\[
\hat{q}_t = (1 - \xi_q) \hat{q}_t + \xi_q \hat{q}_{t-1} - \xi_q \hat{\pi}_t.
\]

Working on scheme #2, divide by $P_t$:

\[
q_t = (1 - \xi_q) \hat{q}_t + \xi_q \frac{\pi_{t-1}}{\pi_t} q_{t-1}.
\]

Linearizing:

\[
\hat{q}_t = (1 - \xi_q) \hat{q}_t + \xi_q \hat{q}_{t-1} - \xi_q (\hat{\pi}_t - \hat{\pi}_{t-1}).
\]

Then,

\[
(1 - \xi_q) \hat{q}_t = \hat{q}_t - [\xi_q \hat{q}_{t-1} - \xi_q (\hat{\pi}_t - (1 - \varphi) \hat{\pi}_{t-1})].
\]

Then, multiplying the $\hat{q}$ equation by $(1 - \xi_q)$:

\[
\hat{q}_t - [\xi_q \hat{q}_{t-1} - \xi_q (\hat{\pi}_t - (1 - \varphi) \hat{\pi}_{t-1})] = \beta \xi_q (\hat{q}_{t+1} - [\xi_q \hat{q}_t - \xi_q (\hat{\pi}_{t+1} - (1 - \varphi) \hat{\pi}_t)] - (1 - \xi_q) (1 - \varphi) \beta \xi_q (\hat{\pi}_t - \beta \xi_q \hat{\pi}_{t+1}) + (1 - \xi_q) \beta \xi_q \frac{1 - \beta \xi_q}{1 - \varphi \beta \xi_q} \hat{\pi}_{t+1} - (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left( \hat{u}_{c,t} + \frac{R}{R - 1} \hat{R}_t \right)
\]
Collecting terms:

\[ 0 = (1 + \beta \xi^2_q) \hat{q}_t - \beta \xi_q \hat{q}_{t+1} - \xi_q \hat{q}_{t-1} \\
+ \left( -\beta \xi^2_q - \frac{(1 - \xi_q)\beta \xi_q}{1 - \varphi \beta \xi_q} \right) \hat{t}_{t+1} \\
+ \left( \xi_q + \beta \xi^2_q (1 - \varphi) + (1 - \xi_q) (1 - \varphi) \beta \xi_q \right) \hat{t}_t \\
- \xi_q (1 - \varphi) \hat{t}_{t-1} + (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left( \hat{u}_{c,t} + \frac{R}{R - 1} \hat{R}_t \right). \]

6.2.3. Linearizing the Wage Equation

The household’s Euler equation associated with the wage rate is (1.7). Letting \( \tilde{w}_t = \tilde{W}_t/W_t \) and \( u_t = W_t/P_t \), we rewrite (1.7). First, note that:

\[
\tilde{L}_{t+j} = \left( \frac{\tilde{W}_{t+j}}{W_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \\
= \left( \frac{\tilde{W}_t}{w_{t+j} P_t} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} = \left( \frac{\tilde{w}_t W_t}{w_{t+j} P_t} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \\
= \left( \frac{\tilde{w}_t w_t}{w_{t+j}} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j}
\]

So, (1.7) can be written as:

\[ 0 = E_t \sum_{j=0}^{\infty} (\beta \xi^2 w)^j \tilde{L}_{t+j} \left[ \tilde{w}_t w_t X_{t,j} \frac{\lambda_w}{1-\lambda_w} u_{c,t+j} + f_L \left( \left( \frac{\tilde{w}_t w_t}{w_{t+j}} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+j} \right) \right]. \]

Writing this out, term by term:

\[
0 = \tilde{L}_t \left[ \tilde{w}_t w_t \frac{\lambda_w}{1-\lambda_w} u_{c,t} + f_L \left( \left( \frac{\tilde{w}_t w_t}{w_t} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_t \right) \right] \\
+ \beta \xi w \tilde{L}_{t+1} \left[ \tilde{w}_t w_t X_{t,1} \frac{\lambda_w}{1-\lambda_w} u_{c,t+1} + f_L \left( \left( \frac{\tilde{w}_t w_t}{w_{t+1}} X_{t,1} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+1} \right) \right] \\
+ (\beta \xi^2 w)^2 \tilde{L}_{t+2} \left[ \tilde{w}_t w_t X_{t,2} \frac{\lambda_w}{1-\lambda_w} u_{c,t+2} + f_L \left( \left( \frac{\tilde{w}_t w_t}{w_{t+2}} X_{t,2} \right)^{\frac{\lambda_w}{1-\lambda_w}} L_{t+2} \right) \right] + \ldots
\]
In steady state, $\bar{w} = 1, \pi_t = \bar{\pi}, \quad w \frac{\alpha}{\lambda_w} + f_L = 0$. The latter implies that there is no need to worry about differentiating the term outside the square brackets in the wage-setting problem.

Consider scheme #1 first. Differentiate with respect to $\pi_{t+j}, j > 0$ and evaluate that in steady state:

$$-\left(\frac{\beta \xi_w}{1 - \beta \xi_w}\right)^j L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{\bar{\pi}}.$$

Now consider scheme #2. Differentiate with respect to $\pi_{t+j}, j > 0$:

$$-\left(\frac{\beta \xi_w}{1 - \beta \xi_w}\right)^j L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{\bar{\pi}}.$$

For $j = 0$,

$$\frac{\beta \xi_w}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{\bar{\pi}}.$$

Differentiate with respect to $\bar{w}_t$, and evaluate the result in steady state:

$$\frac{1}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right].$$

Differentiate with respect to $w_t$, and evaluate that in steady state:

$$L \frac{u_c}{\lambda_w} + \frac{\beta \xi_w}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L \right]$$

$$= \frac{1}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \frac{1}{w} - f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L^2.$$

Differentiate with respect to $w_{t+j}, j > 0$, and evaluate in steady state:

$$-\left(\frac{\beta \xi_w}{1 - \beta \xi_w}\right)^j L f_{LL} \frac{\lambda_w}{1 - \lambda_w} \frac{1}{w} L.$$

Do the same for $L_{t+j}$ (aggregate employment) and $u_{c,t+j}$:

$$(\beta \xi_w)^j L f_{LL}, \quad (\beta \xi_w)^j L w \frac{1}{\lambda_w}.$$

Collecting terms

$$0 = \frac{1}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] (\bar{w}_t + \bar{w}_t) - L f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \sum_{j=0}^{\infty} (\beta \xi_w)^j \bar{w}_{t+j}.$$
\[- \frac{L}{1 - \varphi \beta \xi_w} \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \sum_{j=1}^{\infty} (\beta \xi_w)^j \hat{\pi}_{t+j} \]
\[+ \frac{(1 - \varphi \beta \xi_w)}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \hat{\pi}_t \]
\[+ L^2 f_{LL} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{L}_{t+j} + \frac{L w u_c}{\lambda_w} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{u}_{ct+j} \]

After dividing by \( L \), this expression is written:

\[0 = \frac{1}{1 - \beta \xi_w} \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \left[ \hat{w}_t + \hat{u}_t \right] - \frac{\left[ w \frac{u_c}{\lambda_w} + \frac{f_{LL} \lambda_w}{1 - \lambda_w} L \right]}{1 - \varphi \beta \xi_w} \sum_{t=1}^{\infty} (\beta \xi_w)^t \hat{\pi}_{t+t} \]
\[+ w \frac{u_c}{\lambda_w} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{u}_{ct+j} - f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \sum_{t=0}^{\infty} (\beta \xi_w)^t \hat{w}_{t+t} \]
\[+ f_{LL} L \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{L}_{t+j} + \frac{(1 - \varphi \beta \xi_w)}{1 - \beta \xi_w} L \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] \hat{\pi}_t \]

Denoting \( \left[ w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L \right] = - f_L + f_{LL} \frac{\lambda_w}{1 - \lambda_w} L = \tilde{\sigma}_L \), \( w \frac{u_c}{\lambda_w} = - f_L \) we obtain:

\[0 = \frac{1}{1 - \beta \xi_w} \tilde{\sigma}_L \left[ \hat{w}_t + \hat{u}_t \right] - \frac{\tilde{\sigma}_L}{1 - \varphi \beta \xi_w} \sum_{t=1}^{\infty} (\beta \xi_w)^t \hat{\pi}_{t+t} - f_L \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{u}_{ct+j} \]
\[- f_{LL} L \frac{\lambda_w}{1 - \lambda_w} \sum_{t=0}^{\infty} (\beta \xi_w)^t \hat{w}_{t+t} + f_{LL} L \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{L}_{t+j} \]
\[+ \frac{(1 - \varphi \beta \xi_w)}{1 - \beta \xi_w} \tilde{\sigma}_L \hat{\pi}_t \]

or,

\[\frac{1}{1 - \beta \xi_w} \tilde{\sigma}_L \left[ \hat{w}_t + \hat{u}_t \right] = \frac{\beta \xi_w}{1 - \beta \xi_w} \tilde{\sigma}_L \left[ \hat{w}_{t+1} + \hat{u}_{t+1} \right] + \frac{\tilde{\sigma}_L \beta \xi_w}{1 - \varphi \beta \xi_w} \hat{\pi}_{t+1} + f_L \hat{u}_{ct} \]
\[+ f_{LL} L \frac{\lambda_w}{1 - \lambda_w} \hat{w}_t - f_{LL} L \hat{L}_t = \frac{(1 - \varphi \beta \xi_w)}{1 - \beta \xi_w} \tilde{\sigma}_L \left[ \hat{\pi}_t - \beta \xi_w \hat{\pi}_{t+1} \right] \]
Define
\[ \sigma_L = \frac{f_{LL,L}}{f_L}. \]

Note, that \( \tilde{\sigma}_L = -f_L + f_{LL} \frac{\lambda_w}{1-\lambda_w} = f_L \left[ \frac{\lambda_w}{1-\lambda_w} \frac{f_{LL,L}}{f_L} - 1 \right] = f_L \left[ \frac{\lambda_w}{1-\lambda_w} \sigma_L - 1 \right], \) so dividing expression above by \( f_L \)

\[
\begin{align*}
\frac{1}{1 - \beta \xi_w} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] [\hat{\tilde{w}}_t + \hat{\tilde{w}}_t] \\
\beta \xi_w \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] [\hat{\tilde{w}}_{t+1} + \hat{\tilde{w}}_{t+1}] \\
+ \frac{1}{1 - \varphi_w \beta \xi_w} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \beta \xi_w \hat{\tau}_{t+1} + \hat{\tau}_{c,t} \\
+ \sigma_L \frac{\lambda_w}{1 - \lambda_w} \hat{\tau}_t - \sigma_L \hat{\tilde{L}}_t \\
- \frac{(1 - \varphi_w) \beta \xi_w}{1 - \beta \xi_w} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] [\hat{\tau}_t - \beta \xi_w \hat{\tau}_{t+1}] 
\end{align*}
\]

The aggregate wage looks like this:
\[ W_t = \left[ \int_0^1 (W_t(i))^{\frac{1}{\xi_w}} d\xi \right]^{1-\lambda_w}. \]

There are two types of firms: \( 1 - \xi_w \) have the new wage, \( \bar{W}_t, \) this time, because they got tapped. The \( \xi_w \) others are stuck with the wage they had last period. Because the ones that are stuck are randomly selected, the distribution of stuck wages is exactly the same as the overall distribution of wages last time. The difference is that there is a smaller overall measure. Then, under scheme \#1:

\[ W_t = \left[ (1 - \xi_w) \left( \bar{W}_t \right)^{\frac{1}{\lambda_w}} + \xi_w \left( \hat{\tau} W_{t-1} \right)^{\frac{1}{\lambda_w}} \right]^{1-\lambda_w} \]

Dividing this by \( P_t, \) we get:

\[ w_t = \left[ (1 - \xi_w) \left( \tilde{w}_t w_t \right)^{\frac{1}{\lambda_w}} + \xi_w \left( \frac{\hat{\tau}}{\hat{\tau}_t} \right) w_{t-1} \right]^{1-\lambda_w}. \]
Now, differentiate (ignoring the constant terms, which cancel):

\[
w \dot{w}_t = \left(1 - \lambda_w \right) w^{\frac{\lambda_w}{1 - \lambda_w}} \left[ \frac{1 - \xi_w}{1 - \lambda_w} w^{\frac{\lambda_w}{1 - \lambda_w} - 1} \left( \dot{w}_t + \ddot{w}_t \right) + \frac{\xi_w}{1 - \lambda_w} (\dot{w}_{t-1} - \dot{\pi}_t) \right] w^{\frac{1}{1 - \lambda_w}} \]

Now, simplify:

\[
\dot{w}_t = \left(1 - \xi_w \right) \left( \dot{w}_t + \ddot{w}_t \right) + \xi_w \left( \dot{w}_{t-1} - \dot{\pi}_t \right)
\]

or,

\[
\left(1 - \xi_w \right) \ddot{w}_t = \xi_w \dot{w}_t - \xi_w \left( \dot{w}_{t-1} - \dot{\pi}_t \right)
\]

Under scheme #2:

\[
W_t = \left[ \left(1 - \xi_w \right) \left( \ddot{W}_t \right)^{\frac{1}{1 - \lambda_w}} + \xi_w \left( \ddot{\pi}_{t-1} W_{t-1} \right)^{\frac{1}{1 - \lambda_w}} \right]^{1 - \lambda_w}
\]

Dividing this by \( P_t \), we get:

\[
w_t = \left[ \left(1 - \xi_w \right) \left( \ddot{w}_t w_t \right)^{\frac{1}{1 - \lambda_w}} + \xi_w \left( \ddot{\pi}_{t-1} w_{t-1} \right)^{\frac{1}{1 - \lambda_w}} \right]^{1 - \lambda_w}
\]

Now, differentiate (ignoring the constant terms, which cancel):

\[
w \dddot{w}_t = \left(1 - \lambda_w \right) w^{\frac{\lambda_w}{1 - \lambda_w}} \left[ \frac{1 - \xi_w}{1 - \lambda_w} w^{\frac{\lambda_w}{1 - \lambda_w} - 1} \left( \dot{w}_t + \ddot{w}_t \right) + \frac{\xi_w}{1 - \lambda_w} (\dot{w}_{t-1} - (\dot{\pi}_t - \dot{\pi}_{t-1})) \right] w^{\frac{1}{1 - \lambda_w}} \]

Now, simplify:

\[
\ddot{w}_t = \left(1 - \xi_w \right) \left( \dot{w}_t + \ddot{w}_t \right) + \xi_w \left( \dot{w}_{t-1} - (\dot{\pi}_t - \dot{\pi}_{t-1}) \right)
\]

or,

\[
\left(1 - \xi_w \right) \ddot{w}_t = \xi_w \dot{w}_t - \xi_w \left( \dot{w}_{t-1} - (\dot{\pi}_t - \dot{\pi}_{t-1}) \right)
\]

Thus,

\[
\left(1 - \xi_w \right) \ddot{w}_t = \xi_w \dot{w}_t - \xi_w \left( \dot{w}_{t-1} - (\dot{\pi}_t - (1 - \varphi) \dot{\pi}_{t-1}) \right).
\]

Combining this with the wage equation:

\[
\frac{1}{1 - \beta \xi_w} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \frac{1}{1 - \xi_w} \left[ \dot{w}_t - \xi_w \left( \dot{w}_{t-1} - (\dot{\pi}_t - (1 - \varphi) \dot{\pi}_{t-1}) \right) \right]
\]

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\[
\beta_{\xi_w} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \frac{1}{(1 - \xi_w)} [\hat{w}_{t+1} - \xi_w (\hat{w}_t - (\hat{\pi}_{t+1} - (1 - \varphi) \hat{\pi}_t))] \\
+ \frac{1}{1 - \varphi_w \beta_{\xi_w}} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \beta_{\xi_w} \hat{\pi}_{t+1} + \hat{u}_{c,t} \\
+ \sigma_L \frac{\lambda_w}{1 - \lambda_w} \hat{w}_t - \sigma_L \hat{L}_t \\
- \frac{(1 - \varphi_w) \beta_{\xi_w}}{1 - \beta_{\xi_w}} \left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] [\hat{\pi}_t - \beta_{\xi_w} \hat{\pi}_{t+1}]
\]

It is convenient to multiply this expression by \(1 - \lambda_w\):

\[
\frac{1}{1 - \beta_{\xi_w}} \left[ \lambda_w \sigma_L - (1 - \lambda_w) \right] \frac{1}{(1 - \xi_w)} [\hat{w}_t - \xi_w (\hat{w}_{t-1} - (\hat{\pi}_t - (1 - \varphi) \hat{\pi}_{t-1}))] \\
= \frac{\beta_{\xi_w}}{1 - \beta_{\xi_w}} \left[ \lambda_w \sigma_L - (1 - \lambda_w) \right] \frac{1}{(1 - \xi_w)} [\hat{w}_{t+1} - \xi_w (\hat{w}_t - (\hat{\pi}_{t+1} - (1 - \varphi) \hat{\pi}_t))] \\
+ \frac{1}{1 - \varphi_w \beta_{\xi_w}} [\lambda_w \sigma_L - (1 - \lambda_w)] \beta_{\xi_w} \hat{\pi}_{t+1} + (1 - \lambda_w) \hat{u}_{c,t} \\
+ \sigma_L \lambda_w \hat{w}_t - \sigma_L (1 - \lambda_w) \hat{L}_t \\
- \frac{(1 - \varphi_w) \beta_{\xi_w}}{1 - \beta_{\xi_w}} [\lambda_w \sigma_L - (1 - \lambda_w)] [\hat{\pi}_t - \beta_{\xi_w} \hat{\pi}_{t+1}]
\]

Letting \(b_w = [\lambda_w \sigma_L - (1 - \lambda_w)] / [(1 - \xi_w) (1 - \beta_{\xi_w})]\), we obtain

\[
b_w [\hat{w}_t - \xi_w (\hat{w}_{t-1} - (\hat{\pi}_t - (1 - \varphi) \hat{\pi}_{t-1}))] \\
= \beta_{\xi_w} b_w [\hat{w}_{t+1} - \xi_w (\hat{w}_t - (\hat{\pi}_{t+1} - (1 - \varphi) \hat{\pi}_t))] \\
+ b_w \frac{(1 - \beta_{\xi_w}) (1 - \xi_w) \beta_{\xi_w}}{1 - \varphi_w \beta_{\xi_w}} [\hat{w}_{t+1} - \xi_w \hat{w}_t + (1 - \lambda_w) \hat{u}_{c,t} \\
+ \sigma_L \lambda_w \hat{w}_t - \sigma_L (1 - \lambda_w) \hat{L}_t \\
- \xi_w \left(1 - \xi_w\right) \beta_{\xi_w} [\hat{\pi}_t - \beta_{\xi_w} \hat{\pi}_{t+1}]
\]

Note, that when we set \(\xi_w = 0\) this expression, becomes:

\[
\left[ \frac{\lambda_w}{1 - \lambda_w} \sigma_L - 1 \right] \hat{w}_t = \hat{u}_{c,t} + \sigma_L \frac{\lambda_w}{1 - \lambda_w} \hat{w}_t - \sigma_L \hat{L}_t
\]

or

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\[ \dot{u}_{c,t} - \sigma_L \dot{L}_t + \ddot{u}_t = 0, \]

which is just the static Euler equation we would have started out with if we had set \( \xi_w \) at the very beginning. Note that \( \lambda_{w} \) is not present here. Its only influence on the system is on the steady state.

6.2.4. Fisher Equation

The Euler equation for \( M \) is given in (1.3). It can be rewritten

\[ u_{c,t} = \beta u_{c,t+1} \frac{\dot{R}_{t+1}}{\pi_{t+1}}. \]

Linearizing:

\[ \dot{u}_{c,t} = \dot{u}_{c,t+1} + \dot{R}_{t+1} - \pi_{t+1}. \]

Note that this is a slightly weird equation. It’s got tomorrow’s interest rate in it. That reflects that consumption is a credit good.

6.2.5. Marginal Utility of Consumption

As noted above, we consider two cases: internal and external habit. Consider external habit first. With period utility defined as in (6.11),

\[ \dot{u}_{c,t} = \dot{U}_{c,t} = -\ddot{\sigma}_c^e \left( \dot{c}_t - \frac{b}{1 - \chi} \dot{\hat{H}}_t \right), \]

where

\[ \ddot{\sigma}_c^e = -\frac{u_{cc} c}{u_c} = \sigma_c \frac{1 - \chi}{1 - \chi - b}. \]

Also, \( H \dot{\hat{H}}_t = \chi H \dot{\hat{H}}_{t-1} + b \dot{c}_{t-1} \), so that

\[ \dot{\hat{H}}_t = \chi \dot{\hat{H}}_{t-1} + (1 - \chi) \dot{c}_{t-1}. \]

Now consider internal habit. Let \( U_t \) denote the present discounted value of utility from consumption:

\[ U_t = \sum_{t=0}^{\infty} \beta^t U(c_{t+j} - H_{t+j}). \]
The derivative of $u_t$ with respect to $c_t$, $u_{c,t}$, is

$$u_{c,t} = U_c(c_t - H_t) - b\beta[U_c(c_{t+1} - H_{t+1}) + \beta\chi U_c(c_{t+2} - H_{t+2}) + \beta^2\chi^2 U_c(c_{t+3} - H_{t+3})... ]$$

In steady state:

$$u_c = U_c - b\beta[U_c + \beta\chi U_c + \beta^2\chi^2 U_c...]$$

$$u_c = U_c[1 - \frac{b\beta}{1 - \beta\chi}]$$

Linearizing:

$$u_{c,t} - u_c = U_{cc}c\frac{(c_t - c)}{c} - H\frac{(H_t - H)}{H}$$

$$- b\beta U_{cc}\left\{ [c\frac{(c_{t+1} - c)}{c} - H\frac{(H_{t+1} - H)}{H}]$$

$$+ \beta\chi[c\frac{(c_{t+2} - c)}{c} - H\frac{(H_{t+2} - H)}{H}]$$

$$+ \beta^2\chi^2[c\frac{(c_{t+3} - c)}{c} - H\frac{(H_{t+3} - H)}{H}]... \right\}$$

$$c - H = c[1 - \frac{b}{1 - \chi}]$$

$$\hat{u}_{c,t} = \frac{u_{c,t} - u_c}{u_c} = U_{cc}c\frac{(c_t - c)}{c} - H\frac{(H_t - H)}{H}$$

$$- b\beta[\hat{c}_{t+1} - \frac{b}{1 - \chi}\hat{H}_{t+1}]$$

$$+ \beta\chi[\hat{c}_{t+2} - \frac{b}{1 - \chi}\hat{H}_{t+2}]$$

$$+ \beta^2\chi^2[\hat{c}_{t+3} - \frac{b}{1 - \chi}\hat{H}_{t+3}]... \right\}$$

Let

$$-\hat{\sigma}_c\frac{U_{cc}c}{u_c} = \frac{U_{cc}cU_c}{U_cu_c} = \left[ -\sigma_c\frac{1 - \chi}{1 - \chi - b} \right] \left[ \frac{1 - \beta\chi}{1 - \beta\chi - \beta b} \right].$$

Then,

$$\hat{u}_{c,t} = \beta\chi\hat{u}_{c,t+1} - \hat{\sigma}_c\frac{b}{1 - \chi}\hat{H}_t + (b + \chi)\beta\hat{\sigma}_c\frac{b}{1 - \chi}\hat{H}_{t+1}$$

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6.2.6. Resource Constraint

Log-linearizing:

\[
\hat{\hat{w}}^*_t = \frac{1 - \lambda_w}{\lambda_w} \left\{ \lambda_w \frac{(1 - \xi_w)\hat{\hat{w}}_t + \xi_w}{1 - \lambda_w} \left[ -\hat{\pi}_t - \hat{\hat{w}}_t + \hat{\hat{w}}_{t-1} + \hat{\hat{w}}^*_{t-1} \right] \right\},
\]

or

\[
\hat{\hat{w}}^*_t = (1 - \xi_w)\hat{\hat{w}}_t + \xi_w \left[ -\hat{\pi}_t - \hat{\hat{w}}_t + \hat{\hat{w}}_{t-1} + \hat{\hat{w}}^*_{t-1} \right]
\]

From before

\[
(1 - \xi_w)\hat{\hat{w}}_t = \xi_w\hat{\hat{w}}_t - \xi_w (\hat{\hat{w}}_{t-1} - \hat{\pi}_t)
\]

Substituting this into the \( \hat{\hat{w}}^*_t \) equation, we get:

\[
\hat{\hat{w}}^*_t = \xi_w\hat{\hat{w}}^*_{t-1}
\]

Now we’ll figure out \( \hat{P}^*/\hat{P} \). The following expression holds for \( \hat{P}^* \) :

\[
\hat{P}^*_t = \left[ (1 - \xi_f) \left( \hat{P}_t \right)^{\frac{\lambda_f}{1 - \lambda_f}} + \xi_f \left( \hat{\pi} P^*_t \right)^{\frac{\lambda_f}{1 - \lambda_f}} \right]^{\frac{1 - \lambda_f}{\lambda_f}}.
\]

Dividing by \( \hat{P}_t \):

\[
\hat{p}^*_t = \left[ (1 - \xi_f) \left( \hat{p}_t \right)^{\frac{\lambda_f}{1 - \lambda_f}} + \xi_f \left( \frac{\hat{\pi}}{\hat{p}^*_t} \right)^{\frac{\lambda_f}{1 - \lambda_f}} \right]^{\frac{1 - \lambda_f}{\lambda_f}}.
\]

Taking the log-linear approximation:

\[
\hat{\hat{p}}^*_t = \frac{1 - \lambda_f}{\lambda_f} \left\{ \frac{\lambda_f}{1 - \lambda_f} (1 - \xi_f)\hat{\hat{p}}_t + \frac{\lambda_f}{1 - \lambda_f} \xi_f \left[ -\hat{\pi}_t + \hat{\hat{p}}^*_{t-1} \right] \right\}
\]

or,

\[
\hat{\hat{p}}^*_t = (1 - \xi_f)\hat{\hat{p}}_t + \xi_f \left[ -\hat{\pi}_t + \hat{\hat{p}}^*_{t-1} \right]
\]

Substituting out for \( \hat{\hat{p}}_t \) in terms of \( \hat{\pi}_t \), we get

\[
\hat{\hat{p}}^*_t = \xi_f \hat{\hat{p}}^*_{t-1}
\]

The message here is that \( \hat{\hat{p}}^*_t = 0 \) if you start from steady state. This means that the variations in \( \hat{P}^*/\hat{P} \) and \( W^*/W \) will be nil in the linear approximation.
We now linearize the resource constraint:

$$\bar{\delta}(u_t) \tilde{k}_t + c_t + I_t \leq \left( \frac{P_t^*}{P_t} \right)^{\frac{\lambda_f}{\sigma_f - 1}} F(K_t, \left( \frac{W_t^*}{W_t} \right)^{\frac{\lambda_w}{\sigma_w - 1}} L_t).$$

Log-linearizing this, and ignoring $P^* / P$ and $W^* / W$:

$$\bar{\delta}'(u) K \hat{u}_t + c \hat{c}_t + I \hat{I}_t = F_K K \hat{K}_t + F_L L \hat{L}_t.$$

Divide by $Y$:

$$\bar{\delta}'(u) \frac{K}{Y} \hat{u}_t + \frac{c}{Y} \hat{c}_t + I \frac{\hat{I}_t}{Y} = \frac{F_K K}{Y} K \hat{K}_t + \frac{F_L L}{Y} \hat{L}_t.$$

Let

$$s_c = \frac{c}{Y}, \quad s_l = \frac{I}{Y}, \quad \alpha_k = \frac{F_K K}{Y}, \quad \alpha_L = \frac{F_L L}{Y},$$

so that

$$\left[ \frac{1}{\beta} - (1 - \delta) \right] \frac{K}{Y} \frac{Y}{c} \hat{u}_t + \frac{c}{Y} \hat{c}_t + \frac{s_l}{s_c} \hat{I}_t = \frac{\alpha_k}{s_c} K \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t,$$

or

$$\left[ \frac{1}{\beta} - (1 - \delta) \right] \frac{s_k}{s_c} \hat{u}_t + \frac{c}{s_c} \hat{c}_t + \frac{s_l}{s_c} \hat{I}_t = \frac{\alpha_k}{s_c} K \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t,$$

where the steady state relation, $r^k = 1/\beta - (1 - \delta)$ has been used, and $s_k = K/Y$.

Log-linearizing the capital accumulation technology under the Lucas-Prescott formulation:

$$K_{t+1} = Q((1 - \delta)K_t, I_t),$$

we obtain:

$$K \hat{K}_{t+1} = Q_1(1 - \delta)K \hat{K}_t + Q_2 I \hat{I}_t,$$

or, taking into account $Q_1 = Q_2 = 1$,

$$\hat{K}_{t+1} = (1 - \delta) \hat{K}_t + \delta \hat{I}_t, \quad (6.15)$$

since $I/K = \delta$. We can use this to substitute out for $\hat{I}_t$ in the resource constraint, to get

$$\left[ \frac{1}{\beta} - (1 - \delta) \right] \frac{s_k}{s_c} \hat{u}_t + \frac{c}{s_c} \hat{c}_t + \frac{s_k}{s_c} \delta \hat{I}_t = \frac{\alpha_k}{s_c} K \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t, \quad s_k = K/Y, \quad (6.16)$$

where we have made use of the fact, $s_l/(\delta s_c) = s_k / s_c$. 

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Under the cost-of-change formulation with capital utilization:

\[ K\hat{K}_{t+1} = (1 - \delta) K\hat{K}_t + F_1 I\hat{I}_t + F_2 I\hat{I}_{t-1} \]

\[ = (1 - \delta) K\hat{K}_t + I\hat{I}_t, \]

since \( F_1 = 1, F_2 = 0 \) in steady state. So, the steady state investment equation is:

\[ \hat{K}_{t+1} = (1 - \delta) \hat{K}_t + \delta \hat{I}_t. \]

Also,

\[ \hat{K}_t = \hat{u}_t + \hat{K}_t. \]

One could substitute out for \( \hat{K}_t \) in the capital accumulation technology. However, this could result in trouble, since \( \hat{K}_{t+1} \) is known at time \( t \), but \( \hat{u}_{t+1} \) (and, hence, \( \hat{K}_{t+1} \)) are not.

**6.2.7. Investment Euler Equation**

We first address the Lucas-Prescott formulation. We now linearize the investment Euler equation, (1.11). Doing so, leaving out no steps:

\[ P_{k'} u_c \hat{P}_{k',t} + P_{k'} u_c \hat{u}_{c,t} \]

\[ = \beta u_c \left[ r^k + 1 - \delta \right] \hat{u}_{c,t+1} + \beta u_c r^k \hat{r}_{t+1} \]

\[ + \beta u_c (1 - \delta) P_{k'} Q_1 P_{k',t+1} Q_{1,t+1}, \]

where \( P_{k',t+1} Q_{1,t+1} \) denotes the percent change in the product, \( P_{k',t+1} Q_{1,t+1} \). Divide by \( u_c \) and take into account \( Q_1 = P_{k'} = 1 = \beta \left[ r^k + 1 - \delta \right] : \)

\[ \hat{P}_{k',t} + \hat{u}_{c,t} = \hat{u}_{c,t+1} + (1 - \beta (1 - \delta)) \hat{r}_{t+1} + \beta (1 - \delta) P_{k',t+1} Q_{1,t+1}. \]

Now,

\[ P_{k',t+1} Q_{1,t+1} = \frac{Q_{1,t+1}}{Q_{2,t+1}} = \frac{a_1}{a_2} \left( \frac{I_{t+1}}{(1 - \delta) K_{t+1}} \right)^{1-\psi} = \left( \frac{I_{t+1}}{\delta K_{t+1}} \right)^{1-\psi}, \]

so that

\[ \hat{P}_{k',t} Q_{1,t} = (1 - \psi) \left( \hat{I}_t - \hat{K}_t \right) \]

\[ = \frac{1 - \psi}{\delta} \left( \hat{K}_{t+1} - \hat{K}_t \right), \]
using (6.15). Also,

\[
P_{k'} P_{k',t} = \frac{\psi - 1}{\psi} \left[ a_1 \left( \frac{(1 - \delta)K}{I} \right)^\psi + a_2 \right] \frac{\psi - 1}{\psi} a_1 \left( \frac{(1 - \delta)K}{I} \right)^\psi \frac{(1 - \delta)K}{I} \left( \hat{K}_t - \hat{I}_t \right).
\]

Taking into account \( P_{k'} = 1, I = \delta K \), and the formula for \( P_{k'} \) in steady state,

\[
\hat{P}_{k',t} = \frac{\psi - 1}{\psi} \left[ a_1 \left( \frac{1 - \delta}{\delta} \right)^\psi + a_2 \right] a_1 \left( \frac{1 - \delta}{\delta} \right)^{\psi - 1} \frac{(1 - \delta)}{\delta} \left( \hat{K}_t - \hat{I}_t \right),
\]

or, after taking into account the definition of \( a_1 \) and \( a_2 \),

\[
\hat{P}_{k',t} = (1 - \psi)(1 - \delta) \left( \hat{I}_t - \hat{K}_t \right) = \frac{(1 - \psi)(1 - \delta)}{\delta} \left( \hat{K}_{t+1} - \hat{K}_t \right).
\]

Here again, we see that a shock that drives up investment drives up stock prices too, as long as \( \psi < 1 \).

Substituting out for \( \hat{P}_{k',t+1} Q_{1,t+1} \) and \( \hat{P}_{k',t} \) in our Euler equation, we get

\[
\frac{(1 - \psi)(1 - \delta)}{\delta} \left( \hat{K}_{t+1} - \hat{K}_t \right) + \hat{u}_{c,t}
\]

\[
= \hat{u}_{c,t+1} + (1 - \beta(1 - \delta)) \hat{r}_{k,t+1} + \beta(1 - \delta) \frac{1 - \psi}{\delta} \left( \hat{K}_{t+2} - \hat{K}_{t+1} \right).
\]

Previously, we derived an expression for \( \hat{r}_{k,t+1} \):

\[
\hat{r}_{k,t+1} = \hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}.
\]

Substituting this into the linearized intertemporal Euler equation:

\[
\frac{(1 - \psi)(1 - \delta)}{\delta} \left( \hat{K}_{t+1} - \hat{K}_t \right) + \hat{u}_{c,t}
\]

\[
= \hat{u}_{c,t+1} + (1 - \beta(1 - \delta)) \left( \hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1} \right) + \beta (1 - \delta) \frac{1 - \psi}{\delta} \left( \hat{K}_{t+2} - \hat{K}_{t+1} \right).
\]

Now, we go after the cost-of-change formulation. Linearizing the intertemporal Euler equation:

\[
u_c \hat{u}_{c,t} + u_c \hat{P}_{k',t}
\]

\[
= \beta u_c \left[ r^k \hat{r}_{k,t+1} + r^k \hat{u}_{t+1} + \hat{P}_{k',t+1}(1 - \delta) - \delta \hat{u}_{t+1} \right]
\]

\[
+ \beta \left[ r^k + (1 - \delta) \right] u_c \hat{u}_{c,t+1}
\]

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or, after dividing by $u_c$ :

$$
\hat{u}_{c,t} + \hat{P}_{k',t} = \beta \left[ r^k \hat{r}_{t+1} + (r^k - \bar{r}) \hat{u}_{t+1} + \hat{P}_{k',t+1}(1 - \delta) \right] \\
+ \beta \left[ r^k + (1 - \delta) \right] \hat{u}_{c,t+1}
$$

and, after taking into account the steady state condition, $r^k = \bar{r}$,

$$
\hat{P}_{k',t} + \hat{u}_{c,t} = \beta \left[ r^k + 1 - \delta \right] \hat{u}_{c,t+1} + \beta r^k \hat{r}_{t+1} \\
+ \beta(1 - \delta) \hat{P}_{k',t+1}.
$$

After substituting out for $\hat{r}_{t+1}$ and using $\beta \left[ r^k + 1 - \delta \right] = 1$ :

$$
\hat{P}_{k',t} + \hat{u}_{c,t} = \hat{u}_{c,t+1} + (1 - \beta(1 - \delta)) \left[ \hat{w}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1} \right] \\
+ \beta(1 - \delta) \hat{P}_{k',t+1}.
$$

Now consider the equation that defines $P_{k',t}$. For this, we need the derivatives of $F_1$ and $F_2$, in steady state:

$$
F(I, I_{-1}) = [1 - S(I/I_{-1})] I,
$$

so

$$
F_1(I, I_{-1}) = -S'(I/I_{-1}) \times (I/I_{-1}) + 1 - S(I/I_{-1}).
$$

Then,

$$
F_{11}(I, I_{-1}) = -S''(I/I_{-1})(I/I_{-1})^2 - 2S'(I/I_{-1})/I_{-1} \\
F_{12}(I, I_{-1}) = S''(I/I_{-1})(I/I_{-1}) + S'(I/I_{-1}) \times (I/I_{-1}) + S'(I/I_{-1}) S'(I/I_{-1}) \times (I/I_{-1})^2,
$$

so that, when $I/I_{-1} = 1$,

$$
F_{11} = -S''/I \\
F_{12} = S''/I
$$

Now, for $F_2$ :

$$
F_2(I, I_{-1}) = S'(I/I_{-1}) \times (I/I_{-1})^2.
$$
\[ F_{22}(I, I_{-1}) = -S''(I/I_{-1})(I/I_{-1}^2)(I/I_{-1})^2 - 2S'(I/I_{-1})(I/I_{-1}^2) \]

In steady state:
\[ F_{22} = -S''/I. \]

Here,
\[ S'' = g_1g_3[g_1 + g_2]. \]

\[ u_c \hat{u}_{c,t} = u_c \hat{u}_{c,t} + u_c \hat{P}_{k',t} + u_c \left[ F_{11} \hat{I}_t + F_{12} \hat{I}_{t-1} \right] + \beta u_c \left[ F_{21} \hat{I}_{t+1} + F_{22} \hat{I}_t \right] \]

or,
\[ 0 = u_c \hat{P}_{k',t} + S''u_c \left[ -\hat{I}_t + \hat{I}_{t-1} \right] + \beta S''u_c \left[ \hat{I}_{t+1} - \hat{I}_t \right] \]

Finally,
\[ \hat{P}_{k',t} = S'' \{ \hat{I}_t - \hat{I}_{t-1} - \beta [\hat{I}_{t+1} - \hat{I}_t] \}. \]

According to this, the marginal cost of installed capital corresponds to the change in the slope of investment: the rise from yesterday to today minus the rise from today to tomorrow. Marginal costs of extra investment are reduced if that slope is very smooth. It’s sort of intriguing that marginal cost before was a function of the level of investment and now it’s a function of the second derivative, even though the cost function was only changed by increasing the degree of differentiation in \( I \) by one.

The Euler equation for capital utilization is:
\[ u_c [r^k \hat{r}_t - \delta' (ut)] = 0, \]
so that
\[ u_c [r^k \hat{r}_t - \delta'' \hat{u}_t] + u_c \hat{u}_{c,t} [r^k - \delta'] = 0, \]
or, since \( r^k = \delta' \) and \( u_c > 0, \)
\[ r^k \hat{r}_t - \delta'' \hat{u}_t = 0. \]
so that,
\[ E_{t-1} [\hat{r}_t - \sigma \hat{u}_t] = 0, \]
after taking into account $\tilde{\delta} > 0$, $\tilde{\delta} = \delta^k$, and $\sigma_{\tilde{\delta}} = \delta''/\tilde{\delta}$.

We can combine this with the equation determining the difference between the physical stock of capital and its services:

$$\hat{K}_t = \hat{u}_t + \hat{K}_t.$$ 

That is,

$$\sigma_{\tilde{\delta}} \left( \hat{K}_t - \hat{K}_t \right) = \sigma_{\tilde{\delta}} \hat{u}_t,$$

or, applying the conditional expectation operator:

$$E_{t-1} \left\{ \hat{K}_t - \hat{K}_t - \frac{1}{\sigma_{\tilde{\delta}}} \hat{r}_t^k \right\} = 0.$$

Finally, substituting out for $\hat{r}_t^k$ from (6.13), we obtain:

$$E_{t-1} \left\{ \hat{K}_t - \hat{K}_t - \frac{1}{\sigma_{\tilde{\delta}}} \left( \hat{w}_t + \hat{R}_t + \hat{L}_t - \hat{K}_t \right) \right\} = 0.$$

6.2.8. Loan Market Clearing

Dividing the loan market clearing condition, (4.1), by $P_t$, we obtain

$$w_t X_t = \mu t m_t - q_t.$$

Here, $m_t = M_t/P_t$ evolves as follows:

$$m_{t+1} = \frac{\mu t}{\pi_{t+1}} m_t.$$

Linearizing these expressions:

$$w X \left( \hat{w}_t + \hat{X}_t \right) = \mu m (\hat{\mu}_t + \hat{m}_t) - q_{\tilde{\pi}_t}$$

and

$$\hat{m}_{t+1} = \hat{\mu}_t + \hat{m}_t - \hat{\pi}_{t+1}.$$
6.2.9. Collecting the Linearized Equations

Price equation

\[
0 = - \left[ \beta \xi_p \left( 1 + \beta (1 - \xi_p) (1 - \varphi_p) \right) + \frac{1 - \xi_p}{\xi_p} \frac{1 - \beta \xi_p}{1 - \varphi_p \beta \xi_p} \right] \tilde{\pi}_{t+1}

+ \left[ 1 + \beta (1 - \varphi_p) \right] \hat{\pi}_t - (1 - \varphi_p) \hat{\pi}_{t-1}

- \frac{(1 - \xi_p)(1 - \beta \xi_p)}{\xi_p} \left[ \alpha \left( \hat{X}_t - \hat{K}_t \right) + \hat{\omega}_t - \hat{\pi}_t + \hat{R}_t \right]
\]

where \( \hat{s}_t = \alpha \left( \hat{X}_t - \hat{K}_t \right) + \hat{\omega}_t + \hat{R}_t \) and \( \hat{\omega}_t = \hat{\omega}_t - \hat{\pi}_t \). Collecting,

\[
(1) \ 0 = - \left[ \xi_p \left( 1 + \beta (1 - \xi_p) (1 - \varphi_p) \right) + (1 - \xi_p) \frac{1 - \beta \xi_p}{1 - \varphi_p \beta \xi_p} \right] \beta \xi_p \hat{\pi}_{t+1}

+ \left[ \xi_p + \xi_p \beta (1 - \varphi_p) + (1 - \xi_p)(1 - \beta \xi_p) \right] \hat{\pi}_t - \xi_p (1 - \varphi_p) \hat{\pi}_{t-1}

- (1 - \xi_p)(1 - \beta \xi_p) \left[ \alpha \left( \hat{X}_t - \hat{K}_t \right) + \hat{\omega}_t + \hat{R}_t \right]
\]

\[
(1') \ 0 = \beta \xi_p \hat{\pi}_{t+1}

- \left[ \xi_p + \xi_p \beta (1 - \varphi) + (1 - \xi_p)(1 - \beta \xi_p) \right] \hat{\pi}_t + \xi_p (1 - \varphi) \hat{\pi}_{t-1}

+ (1 - \xi_p)(1 - \beta \xi_p) \left[ \alpha \left( \hat{X}_t - \hat{K}_t \right) + \hat{\omega}_t + \hat{R}_t \right]
\]

where \( \hat{s}_t = \alpha \left( \hat{X}_t - \hat{K}_t \right) + \hat{\omega}_t + \hat{R}_t \) and \( \hat{\omega}_t = \hat{\omega}_t - \hat{\pi}_t \), has been used.

This is the \( Q \) equation, after adopting the change of variable, \( \hat{\varphi}_t = \hat{\pi}_t - \hat{\pi}_t \):

\[
0 = \left( 1 + \beta \varepsilon^2_q \right) \left( \hat{\varphi}_t - \hat{\pi}_t \right) - \beta \xi_q \left( \hat{\varphi}_{t+1} - \hat{\pi}_{t+1} \right) - \xi_q \left( \hat{\varphi}_{t-1} - \hat{\pi}_{t-1} \right)

+ \left( - \beta \varepsilon^2_q - \left( 1 - \xi_q \right) \beta \xi_q \frac{1 - \beta \xi_q}{1 - \varphi_q \beta \xi_q} \right) \hat{\pi}_{t+1}

+ \left( \xi_q + \beta \varepsilon^2_q (1 - \varphi_q) + (1 - \xi_q)(1 - \varphi_q) \beta \xi_q \right) \hat{\pi}_t

- \xi_q (1 - \varphi_q) \hat{\pi}_{t-1} + (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left( \hat{u}_{c,t} + \frac{R}{R - 1} \hat{R}_t \right).
\]

Collecting terms:

\[
(2) \ 0 = \left( 1 + \beta \varepsilon^2_q \right) \hat{\varphi}_t - \beta \xi_q \hat{\varphi}_{t+1} - \xi_q \hat{\varphi}_{t-1}
\]
\[
\begin{align*}
+ \left( -\beta \xi_q^2 - \left( 1 - \xi_q \right) \beta \xi_q \frac{1 - \beta \xi_q}{1 - \varphi_q \beta \xi_q} \right) + \beta \xi_q - (1 - \xi_q) (1 - \varphi_q) (\beta \xi_q)^2 \right) \pi_t + 1 \\
+ \left( \xi_q + \beta \xi_q^2 (1 - \varphi_q) - (1 + \beta \xi_q^2) + (1 - \xi_q) (1 - \varphi_q) \beta \xi_q \right) \pi_t \\
+ \xi_q \pi_{t-1} + (1 - \xi_q) \frac{1 - \beta \xi_q}{\sigma_q} \left( \hat{u}_{c,t} + \frac{R}{R - 1} \hat{R}_t \right).
\end{align*}
\]

This change of variable for \( q \) was adopted because we want to think of \( \hat{q}_t \) as the choice variable, not \( q_t \). The latter is \( Q_t \) divided by \( P_t \), but we are thinking of \( Q_t \) as chosen before \( P_t \) is observed. So, a given choice of \( \hat{q}_t \) corresponds to a random choice of \( Q_t \), conditional on the realization of \( P_t \).

This is the wage equation:

\[
\begin{align*}
b_w \left[ \hat{w}_t - \xi_w \left( \hat{w}_{t-1} - (\pi_t - (1 - \varphi_w) \pi_{t-1}) \right) \right] \\
= \beta \xi_w b_w \left[ \hat{w}_{t+1} - \xi_w \left( \hat{w}_t - (\pi_{t+1} - (1 - \varphi_w) \pi_t) \right) \right] \\
+ b_w \left( 1 - \beta \xi_w \right) (1 - \xi_w) \beta \xi_w \\
+ \sigma_L \lambda_w \hat{w}_t - \sigma_L (1 - \lambda_w) \hat{L}_t \\
- b_w (1 - \xi_w) (1 - \varphi_w) \beta \xi_w [\pi_t - \beta \xi_w \pi_{t+1}]
\end{align*}
\]

or, after collecting terms:

\[
\begin{align*}
0 &= b_w \xi_w \hat{w}_{t-1} - \left[ b_w (1 + \beta \xi_w^2) - \sigma_L \lambda_w \right] \hat{w}_t + \beta \xi_w \hat{w}_t \hat{w}_{t+1} \\
+ b_w \xi_w (1 - \varphi_w) \pi_t - b_w \xi_w \left[ 1 + (1 - \varphi_w) \beta \right] \pi_t \\
+ b_w \beta \xi_w \left[ \xi_w + \frac{(1 - \beta \xi_w) (1 - \xi_w)}{1 - \varphi_w \beta \xi_w} + (1 - \xi_w) (1 - \varphi_w) \beta \xi_w \right] \pi_{t+1} \\
+ (1 - \lambda_w) \hat{u}_{c,t} - \sigma_L (1 - \lambda_w) \hat{L}_t.
\end{align*}
\]

Then,

\[
E_t \left\{ \eta_0 \hat{w}_{t-1} + \eta_1 \hat{w}_t + \eta_2 \hat{w}_{t+1} + \eta_3 \pi_{t-1} + \eta_4 \pi_t + \eta_5 \pi_{t+1} + \eta_6 \hat{L}_t + \eta_7 \hat{u}_{c,t} \right\} = 0
\]

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where
\[
\eta = \begin{pmatrix}
    b_w \xi_w \\
    -b_w [1 + \beta \xi_w^2] + \sigma_L \lambda_w \\
    \beta \xi_w b_w \\
    b_w \xi_w (1 - \varphi_w) \\
    -\xi_w b_w [1 + (1 - \varphi_w) \beta] \\
    \eta_0 \\
    \eta_1 \\
    \eta_2 \\
    \eta_3 \\
    \eta_4 \\
    \eta_5 \\
    \eta_6 \\
\end{pmatrix} = \begin{pmatrix}
    \eta_0 \\
    \eta_1 \\
    \eta_2 \\
    \eta_3 \\
    \eta_4 \\
    \eta_5 \\
\end{pmatrix}.
\]

Now, we’d actually like to think of \( \hat{w}_t \) instead of \( \hat{w}_t \), where \( \hat{w}_t = W_t / P_{t-1} \). This will allow us to consider situations in which \( W_t \) is chosen before the current realization of \( P_t \). Now, \( \hat{w}_t = \hat{w}_t - \hat{\pi}_t \).

\[
E_t \left\{ \eta_0 (\hat{w}_{t-1} - \hat{\pi}_{t-1}) + \eta_1 (\hat{w}_t - \hat{\pi}_t) + \eta_2 (\hat{w}_{t+1} - \hat{\pi}_{t+1}) + \eta_3 \hat{\pi}_{t-1} + \eta_4 \hat{\pi}_t + \eta_5 \hat{\pi}_{t+1} + \eta_6 \hat{L}_t + \eta_6 \hat{u}_{c, t} \right\} = 0,
\]

so that the wage equation reduces to:

(3) \( E_t \left\{ \eta_0 \hat{w}_{t-1} + \eta_1 \hat{w}_t + \eta_2 \hat{w}_{t+1} + (\eta_3 - \eta_0) \hat{\pi}_{t-1} + (\eta_3 - \eta_1) \hat{\pi}_t + (\eta_3 - \eta_2) \hat{\pi}_{t+1} + \eta_5 \hat{L}_t + \eta_6 \hat{u}_{c, t} \right\} = 0, \)

The Fisher equation:

(4) \( E_t \left\{ \hat{u}_{c, t+1} + \hat{R}_{t+1} - \hat{\pi}_{t+1} - \hat{u}_{c, t} \right\} = 0. \)

The investment euler equation in the Lucas-Prescott case:

(5) \( 0 = E_{t-1} \left\{ \hat{u}_{c, t+1} + (1 - \beta (1 - \delta)) \left( \hat{w}_{t+1} - \hat{\pi}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1} \right) \right. \)

\[ + \beta (1 - \delta) \frac{1 - \psi}{\delta} \left( \hat{K}_{t+2} - \hat{K}_{t+1} \right) - \left( 1 - \psi \right) \frac{1 - \delta}{\delta} \left( \hat{K}_{t+1} - \hat{K}_t \right) - \hat{u}_{c, t} \right\}. \)

In the cost-of-change case, (5) is replaced by (5'), (11) and (12):

(5') \( 0 = E_{t-1} \left\{ -\hat{P}_{\kappa, t} - \hat{u}_{c, t} + \hat{u}_{c, t+1} + (1 - \beta (1 - \delta)) \left[ \hat{w}_{t+1} - \hat{\pi}_{t+1} + \hat{R}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1} \right] \right. \)

\[ + \beta (1 - \delta) \hat{P}_{\kappa, t+1}, \right\}, \)

(11) \( 0 = E_{t-1} \left\{ -\hat{P}_{\kappa, t} + S'' \left[ \hat{I}_t - \hat{I}_{t-1} - \beta \left( \hat{I}_{t+1} - \hat{I}_t \right) \right] \right\}. \)

(12) \( 0 = -\hat{K}_{t+1} + (1 - \delta) \hat{K}_t + \delta \hat{I}_t. \)

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When there are no capital adjustment costs, then there is no distinction between \( \hat{K}_t \) and \( \tilde{K}_t \). If there are capital adjustment costs, then there is a distinction. In this case the stock of capital, \( \hat{K}_t \), is (as before) a state variable at time \( t \), while the services from capital, \( \tilde{K}_t \), is a date \( t \) choice variable. We assume that the choice is made at the beginning of time \( t \), before the current period uncertainty is realized. The associated first order condition is:

\[
E_{t-1} \left\{ \hat{K}_t - \tilde{K}_t - \frac{1}{\sigma_{\xi}} \left( \hat{w}_t - \hat{\pi}_t + \hat{R}_t + \hat{L}_t - \tilde{K}_t \right) \right\} = 0.
\]

(Note here that if \( \sigma_{\xi} = \infty \), then \( \hat{K}_t - \tilde{K}_t = 0 \) for all possible realizations of shocks.) The resource constraint:

\[
\left( \frac{1}{\beta} - (1 - \delta) \right) \frac{s_k}{s_c} \left( \hat{K}_t - \hat{\tilde{K}}_t \right) + \hat{c}_t + \frac{s_k}{s_c} \left[ \hat{K}_{t+1} - (1 - \delta) \hat{K}_t \right] - \left[ \frac{\alpha_k}{s_c} \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t \right] = 0
\]

Clearing in the loan market

\[
\left( \frac{1}{\beta} - (1 - \delta) \right) \frac{s_k}{s_c} \left( \hat{K}_t - \hat{\tilde{K}}_t \right) + \hat{c}_t + \frac{s_k}{s_c} \left[ \hat{K}_{t+1} - (1 - \delta) \hat{K}_t \right] - \left[ \frac{\alpha_k}{s_c} \hat{K}_t + \frac{\alpha_L}{s_c} \hat{L}_t \right] = 0
\]

Definitional equation for money growth:

\[
\hat{\mu}_{t-1} + \hat{m}_{t-1} - \hat{\pi}_t - \hat{\pi}_t = 0.
\]

Habit evolution equation:

\[
\hat{H}_t - \chi \hat{H}_{t-1} - (1 - \chi) \hat{c}_{t-1} = 0.
\]

Finally, there is the marginal utility of consumption. In the case of internal habit:

\[
E_{t} \left\{ -\beta \chi \hat{u}_{c,t+1} + \beta \hat{\sigma}_{e}^2 \hat{c}_t - \frac{b}{1 - \chi} \hat{H}_t \right\} - \left( b + \chi \right) \beta \hat{\sigma}_{e}^2 \hat{c}_{t+1} - \frac{b}{1 - \chi} \hat{H}_{t+1} + \hat{u}_{c,t} = 0
\]

and in the case of external habit:

\[
\hat{u}_{c,t} + \hat{\sigma}_{e}^2 \left( \hat{c}_t - \frac{b}{1 - \chi} \hat{H}_t \right) = 0.
\]

The only uncertainty in this model comes from monetary policy, which we represent as (3.1). It is useful to write this in first order vector form:

\[
\theta_t = \rho \theta_{t-1} + e_t,
\]

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where

\[
\theta_t = \begin{pmatrix} \mu_t \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \mu_{t-1} \end{pmatrix}, \quad \rho = \begin{bmatrix} \phi & \omega_1 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad e_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \end{pmatrix}.
\]

Equations (1)-(9), together with the law of motion for \(\theta_t\), together with a convergence condition, characterize equilibria in a neighborhood of the model's steady state.

6.2.10. Solving the Equations

Let \(z_t\) denote a vector of the model variables determined at time \(t\). The Euler equations can be written in the following format:

\[
\mathcal{E}_t \{ \alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 \theta_{t+1} + \beta_1 \theta_t \} = 0,
\]

where \(\alpha_i\) are 8 \times 8 matrices, \(\beta_i\) is 8 \times 2, \(i = 0, 1\). Also, \(\mathcal{E}_t\) is the expectation operator, which reflects the timing assumptions on the expectations in (1)-(8). A solution is an 8 \times 8 matrix \(A\) and an 8 \times n matrix \(B\), where \(z_t = A z_{t-1} + B s_t\), which satisfies the Euler equations and the information constraints. Also, \(s_t = \theta_t\) when the information set for all expectations includes all date \(t\) shocks and and \(s_t = (\theta_t, \theta_{t-1})'\) when the information set is lagged for one or more equations. We have \(n = 4\) in the first case and \(n = 8\) in the second.

6.2.11. Benchmark case

In the benchmark case, money has the exogenous representation. Also, in the benchmark case we adopt the Lucas-Prescott formulation of adjustment costs. We discuss this case first, and then discuss the changes that are necessary to accommodate the cost-of-change specification. Finally, we discuss the further changes that are necessary to accommodate the
cost-of-change specification with variable capital utilization. Let $z_t$ be defined as follows

$$
\begin{align*}
z_t = & \begin{pmatrix}
\hat{\pi}_t \\
\hat{q}_t \\
\hat{w}_t \\
\hat{c}_t \\
\hat{K}_{t+1} \\
\hat{m}_t \\
\hat{L}_t \\
\hat{R}_t \\
\hat{H}_t \\
\hat{u}_{ct}
\end{pmatrix}
\end{align*}
$$

In the Euler equation,

$$
\alpha_0(1 : 4) = \begin{bmatrix}
\xi_p \beta & 0 & 0 & 0 \\
0 & \beta \xi_q & 0 & 0 \\
\eta_1 - \eta_2 & 0 & \eta_2 & 0 \\
-1 & 0 & 0 & 0 \\
-1 + \beta(1 - \delta) & 0 & 1 - \beta(1 - \delta) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_h(b + \chi)\beta \delta^i_c
\end{bmatrix}
$$
where $I_h = 1$ when habit is internal and $I_h = 0$ when habit is external. Also

$$
\alpha_0(5:10) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

$$
\alpha_0 = [\alpha_0(1:4), \alpha_0(5:9)]
$$

Also, (there were errors in the 7,4 and 8,4 components of this matrix)

$$
\alpha_1(1:4) = \begin{bmatrix}
\hat{\pi}_t & \hat{q}_t & \hat{w}_t & \hat{c}_t & \hat{K}_{t+1} & \hat{m}_t & \hat{L}_t & \hat{R}_t & \hat{H}_t & \hat{u}_{c,t} \\
-a_p - \xi_p \beta (1 - \varphi_p) - \xi_p & 0 & a_p & 0 \\
1 + \xi_q (\beta \xi_q - 1) - \beta \xi_q (1 - \varphi_q) & -(\beta \xi_q^2 + 1) & 0 & 0 \\
\eta_3 - \eta_1 & 0 & \eta_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
q + wL & -q & -wL & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_h \hat{\sigma}_c + (1 - I_h) \hat{\sigma}_c^e \\
\end{bmatrix}
$$

where $a_t = (1 - \xi_t)(1 - \beta \xi_t)$ (the 7,2 element in the following matrix looked wrong)

$$
\hat{\pi}_t \quad \hat{q}_t \quad \hat{w}_t \quad \hat{c}_t \quad \hat{K}_{t+1} \quad \hat{m}_t \quad \hat{L}_t \quad \hat{R}_t \quad \hat{H}_t \quad \hat{u}_{c,t}
$$

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$$\alpha_1(5:7) = \begin{bmatrix} 0 & 0 & \alpha a_p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_5 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1+\beta(1-\psi)(1-\delta)}{\delta} - (1-\beta(1-\delta)) & 0 & 0 & -\frac{\alpha a_p}{\gamma} \\ \frac{1}{\gamma} & 0 & \mu m & -wL \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_1(8:10) = \begin{bmatrix} \frac{\alpha a_p}{\sigma_q} & 0 & 0 \\ -\frac{(1-\xi)(1-\beta\xi)}{\xi(1-R)} & 0 & -\frac{(1-\xi)(1-\beta\xi)}{\xi(1-R)} \\ 0 & 0 & \eta_5 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\sigma b}{\chi} & -\frac{\sigma b}{\chi} \left(1 - I_h\right) \end{bmatrix}$$

$$\alpha_1 = [\alpha_1(1:4), \alpha_1(5:9)]$$

Finally, (the 1,1 element here looked like there was an $a$ where an $\alpha$ should have been)

$$\alpha_2 = \begin{bmatrix} \xi_p(1-\varphi_p) & 0 & 0 & 0 \\ -\xi_q \varphi_q & \xi_q & 0 & 0 \\ \eta_5 - \eta_0 & 0 & \eta_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
\[ \beta_0 = 0 \begin{pmatrix} 0_{10 	imes 4} \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

We now discuss the cost-of-change case. Now, \( z \) is defined as follows:

\[ z_t = \begin{pmatrix} \hat{\pi}_t \\ \hat{q}_t \\ \hat{w}_t \\ \hat{c}_t \\ \hat{K}_{t+1} \\ \hat{\bar{m}}_t \\ \hat{L}_t \\ \hat{R}_t \\ \hat{\bar{H}}_t \\ \hat{u}_{ct} \\ \hat{P}_{k,t} \\ \hat{I}_t \end{pmatrix}. \]

We now discuss the changes that need to be made to the \( \alpha \)’s. First, the row and column dimensions need to be raised by 2. Apart from the fifth row, the first ten rows must have zeros in the right two columns. Also, the fifth row has to be changed, and the last two rows need to be added.

The fifth row is obtained by adjusting what we have as follows. First, the last column in each of the \( \alpha \)’s of the fifth row is zero. Also:

\[ \begin{align*}
\alpha_0 & : \quad (5,11) \text{ element has } \beta(1 - \delta); \quad (5,5) \text{ element has } 0 \\
\alpha_1 & : \quad (5,11) \text{ element has } -1; \quad (5,5) \text{ element has } -(1 - \beta(1 - \delta)) \\
\alpha_2 & : \quad (5,5) \text{ element has } 0.
\end{align*} \]

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must look like this. The last two rows of the \( \alpha \)'s must be zero, except:

\[
\begin{align*}
\alpha_0 : & \quad (11,12) \text{ element has } -\beta S'' \\
\alpha_1 : & \quad (11,11) \sim -1; (11,12) \sim (1+\beta)S''; (12,5) \sim -1; (12,12) \sim \delta \\
\alpha_2 : & \quad (11,12) \sim -S''; (12,5) \sim 1 - \delta.
\end{align*}
\]

We now turn to cost-of-change with variable capital utilization. Now, \( z \) is defined as follows:

\[
z_t = \begin{pmatrix}
\hat{n}_t \\
\hat{q}_t \\
\hat{w}_t \\
\hat{c}_t \\
\hat{K}_{t+1} \\
\hat{m}_t \\
\hat{L}_t \\
\hat{R}_t \\
\hat{H}_t \\
\hat{u}_ct \\
\hat{P}_{\hat{K},t} \\
\hat{I}_t \\
\hat{K}_t
\end{pmatrix}
\]

Note that we have made two changes to the structure of \( z_t \). First, \( \hat{K}_{t+1} \) is replaced by \( \hat{K}_{t+1} \). The latter is a choice variable at time \( t \). The former is not. Also, \( \hat{K}_t \) was added at the bottom of the vector. The necessary modification to the \( \alpha \)'s is simple. Among the first 12 equations, only the first and fifth need to be adjusted. The \( K \)'s in those equations have to be changed to \( \hat{K} \)'s. Regarding the first equation:

\[
(1,5) \text{ element of } \alpha_2 \text{ should be zero} \\
(1,13) \text{ element of } \alpha_1 \text{ should be } -\alpha(1 - \xi_p)(1 - \beta \xi_p).
\]

In the fifth equation, the following changes need to be made:

\[
(5,5) \text{ element of } \alpha_1 \text{ should be zero} \\
(5,13) \text{ element of } \alpha_0 \text{ should be } -(1 - \beta(1 - \delta))
\]

Some changes need to be made to the resource constraint, which is the 6th equation:

\[
(6,13) \text{ element of } \alpha_1 \text{ should be } \left[ \frac{1}{\beta} - (1 - \delta) \right] s_k - \frac{\alpha_k}{s_c} \\
\]

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(6,13) element of \( \alpha_0 \) should be 0

(6,5) element of \( \alpha_2 \) should be replaced by \(- \left[ \frac{1}{\beta} - (1 - \delta) \right] \frac{s_k}{s_c} \)

(6,5) element of \( \alpha_1 \) should be zero

(6,12) element of \( \alpha_1 \) should be \( \delta s_k / s_c \).

The entries in the 13th column of the first 12 rows of \( \alpha \)'s (except rows 1, 5 and 6) should be zero. The 13th equation is:

\[
(13) \quad \hat{K}_t - \hat{K}_t - \frac{1}{\sigma_{\hat{z}}} \left( \hat{w}_t - \hat{\pi}_t + \hat{R}_t + \hat{I}_t - \hat{K}_t \right) = 0.
\]

The bottom row of \( \alpha_0 \) is all zeros. Now consider the bottom row of \( \alpha_1 \):

\[
\begin{align*}
(13,13) & \quad -1 + (1/\sigma_{\hat{z}}) \\
(13,3) & \quad -1/\sigma_{\hat{z}} \\
(13,1) & \quad -1/\sigma_{\hat{z}} \\
(13,8) & \quad -(1/\sigma_{\hat{z}}) \\
(13,7) & \quad -(1/\sigma_{\hat{z}}) \\
(13,11) & \quad 0
\end{align*}
\]

The bottom row of \( \alpha_2 \) is all zero except

\[
(13,5) \quad -1.
\]

7. Some Other Variables

We are interested in some other variables, in addition to those above. For example, there is the rate of return on capital:

\[
\hat{R}_t^k = \frac{r_{t+1}^k + P_{k,t+1}(1 - \delta)}{P_{k,t}}.
\]

We are after \( \hat{R}_t^k \):

\[
R^k \hat{R}_t^k = r_{t+1}^k + (1 - \delta) \hat{P}_{k,t+1} - R^k \hat{P}_{k,t},
\]

or,

\[
\hat{R}_t^k = \frac{r_{t+1}^k + (1 - \delta) \hat{P}_{k,t+1} - R^k \hat{P}_{k,t}}{R^k \hat{P}_{k,t+1}}.
\]
or,
\[ \hat{R}_t^k = [1 - \beta (1 - \delta)] \hat{r}_{t+1}^k + \beta (1 - \delta) \hat{P}_{k,t+1} - \hat{P}_{k,t}. \]

We also want nominal profits. Real profits are:
\[ prof_t = y_t - R_t w_t L_t, \]
so that
\[ prof \times \hat{prof}_t = y \hat{y}_t - wLR[\hat{R}_t + \hat{w}_t + \hat{L}_t], \]
or,
\[ \hat{prof}_t = \frac{y}{\text{prof}} \hat{y}_t - \frac{wLR}{\text{prof}}[\hat{R}_t + \hat{w}_t + \hat{L}_t]. \]

But, we want nominal profits:
\[ P_t \times \hat{prof}_t = \hat{P}_t + \hat{prof}_t \]
\[ = \sum_{j=1}^{t} \hat{\pi}_j + \hat{prof}_t, \]
for \( t > 0 \) and \( P_t \times \hat{prof}_t = \hat{prof}_t \) for \( t = 0 \).

In the simulations, quantities should be reported as \( \hat{x}_t \times 100 \), so that they are in percent deviations from steady state. Rates of return should be reported as \( \hat{x}_t x \times 400 \), so they are in annualized percentage point deviations from steady state (APR).

### 8. Menu Costs

We are interested in menu costs, the fixed cost that an agent would be willing to pay to receive a Calvo tap. We first consider the firms and their prices. We then turn to households.

We have in mind a situation in which the economy is in a steady state. In period 0, prices are set and the money shock is revealed afterward. Since the money shock is unexpected, all prices in period 0 remain at their steady state values, whether firms are allowed to reoptimize or not. Nothing unexpected ever happens again to the money supply after period 0. In period 1, the period after the shock, \( 1 - \xi_p \) firms are ‘tapped’ to reoptimize their price and \( \xi_p \) firms are not. We want to know how much a firm that is not tapped in period 1 would pay to be tapped. Consider the table below, which contains the ratio of the price chosen by the indicated type of firm, divided by the aggregate price level. Note that in period 0, everyone chooses \( P^s \) and has a relative price of unity. The price chosen by the tapped firm in period
1 is denoted by \( \tilde{P} \), and the price of the untapped firm in that period is \( P^s \pi_0 \), etc. Note how being tapped in period 1 implies a proportional shift in price at each date. For example, the period 2 price level of a ‘never tapped’ person is the price level of the tapped person, times \( \tilde{P}/(\pi_0 P^s) \). The same factor applies to all the other dates.

<table>
<thead>
<tr>
<th>Tapped in Period 1 Only</th>
<th>Never Tapped</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 ) ( P^s_{t} ) ( P^s_{t} = \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \tilde{P} \frac{1}{\pi_0} ) ( \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \pi_0 )</td>
<td></td>
</tr>
<tr>
<td>( t = 1 ) ( P^s_{t} ) ( P^s_{t} = \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \tilde{P} \frac{1}{\pi_0} ) ( \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \pi_0 )</td>
<td></td>
</tr>
<tr>
<td>( t = 2 ) ( P^s_{t} ) ( P^s_{t} = \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \tilde{P} \frac{1}{\pi_0} ) ( \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \pi_0 )</td>
<td></td>
</tr>
<tr>
<td>( t = 3 ) ( P^s_{t} ) ( P^s_{t} = \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \tilde{P} \frac{1}{\pi_0} ) ( \tilde{P} ) ( \frac{P^s_{t}}{P^s_{t-1}} = \pi_0 )</td>
<td></td>
</tr>
</tbody>
</table>

Let the vector of relative prices of the tapped person be \( p \) and the corresponding vector of the untapped firm be \( \tilde{p} \). Let \( \vartheta(p) \) denote the profits of the never tapped firm and \( \vartheta(\tilde{p}) \) denote the corresponding object for the firm tapped in period 1. We are interested in the difference, \( \vartheta = \vartheta(\tilde{p}) - \vartheta(p) \). Note that in this difference, date \( t = 0 \) is irrelevant. Note too that in evaluating the difference, we do not concern ourselves situations in which a firm is tapped in \( t \geq 2 \). This can be understood by studying the history tree in the attached figure (see tapped.pdf). The solid lines are histories where firm profits differ. In all other histories firm profits are identical and so they can be ignored in \( \vartheta(\tilde{p}) - \vartheta(p) \). Here, \( p = [p_0, p_1, ...] \), where \( p_t = \pi_0/\pi_t \) for \( t \geq 0 \). Also,

\[
\tilde{p} = [\tilde{p}_0, \tilde{p}_1, ...] = \frac{\tilde{P} \frac{1}{\pi_0}}{P^s_{\pi_0}} p = \frac{\tilde{P} \frac{1}{\pi_0}}{P^s_{\pi_0}} p,
\]

where \( \pi \) is steady state inflation. The objective function is:

\[
\vartheta(p) = \sum_{t=1}^{\infty} (\beta \xi_p)^t u_t Y_t P_t^{\gamma_{t-1}} \left[ \left( P^s_{\pi_0} \pi_1 \pi_2 \cdots \pi_{t-1} \right)^{1-\gamma_{t-1}} - MC_t \times \left( P^s_{\pi_0} \pi_1 \pi_2 \cdots \pi_{t-1} \right)^{\gamma_{t-1}} \right]
\]

\[
= \sum_{t=1}^{\infty} (\beta \xi_p)^t u_t Y_t P_t^{\gamma_{t-1}} \left[ \left( P^s \right)^{1-\gamma_{t-1}} - MC_t \times \left( P^s \right)^{\gamma_{t-1}} \right], \quad P_t^s = P^s_{\pi_0} \pi_1 \pi_2 \cdots \pi_{t-1},
\]

\[
\vartheta(\tilde{p}) = \sum_{t=1}^{\infty} (\beta \xi_p)^t u_t Y_t P_t^{\gamma_{t-1}} \left[ \left( \tilde{P}_t \pi_1 \pi_2 \cdots \pi_{t-1} \right)^{1-\gamma_{t-1}} - MC_t \times \left( \tilde{P}_t \pi_1 \pi_2 \cdots \pi_{t-1} \right)^{\gamma_{t-1}} \right]
\]

\[
= \sum_{t=1}^{\infty} (\beta \xi_p)^t u_t Y_t P_t^{\gamma_{t-1}} \left[ \left( \frac{\tilde{P}_{\pi_0} P_t}{P^s_{\pi_0}} \right)^{1-\gamma_{t-1}} - MC_t \left( \frac{\tilde{P}_{\pi_0} P_t}{P^s_{\pi_0}} \right)^{\gamma_{t-1}} \right],
\]

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where it is understood that $\pi_1 \pi_2 \cdots \pi_{t-1} = 1$ for $t = 1$ in $\vartheta(\bar{p})$. Also, we have dropped $t = 0$ because that is constant across the tapped and untapped firms in our experiment. Then,

$$
\vartheta(\bar{p}) - \vartheta(p) = \sum_{t=1}^{\infty} (\beta \xi_p)^t \nu_t Y_t P_t^{\frac{\lambda_f}{r}} \left[ \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{\frac{1}{r}} - MC_t \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{-\frac{\lambda_f}{r}} \right] - \sum_{t=1}^{\infty} (\beta \xi_p)^t \nu_t Y_t P_t^{\frac{\lambda_f}{r}} \left[ (P_t^s)^{\frac{1}{r}} - MC_t (P_t^s)^{-\frac{\lambda_f}{r}} \right]
$$

or

$$
\vartheta(\bar{p}) - \vartheta(p) = \sum_{t=1}^{\infty} (\beta \xi_p)^t \nu_t Y_t P_t^{\frac{\lambda_f}{r}} \left\{ (P_t^s)^{\frac{1}{r}} - MC_t (P_t^s)^{-\frac{\lambda_f}{r}} \right\} - \sum_{t=1}^{\infty} (\beta \xi_p)^t \nu_t Y_t P_t^{\frac{\lambda_f}{r}} \left[ \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{\frac{1}{r}} - 1 \right] - MC_t (P_t^s)^{-\frac{\lambda_f}{r}} \left[ \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{-\frac{\lambda_f}{r}} - 1 \right]
$$

Rearranging, (taking into account, $\nu_t P_t = u_{c,t}$)

$$
\vartheta = \vartheta(\bar{p}) - \vartheta(p) = \sum_{t=1}^{\infty} (\beta \xi_p)^t \nu_{c,t} Y_t \left( \frac{P_t^s}{P_t} \right)^{\frac{1}{r}} \left[ \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{\frac{1}{r}} - 1 \right] - s_t \left( \frac{P_t^s}{P_t} \right)^{\frac{1}{r}} \left[ \left( \frac{\bar{P}_t}{P^s \pi_0} \right)^{-\frac{1}{r}} - 1 \right]
$$

where $s_t = MC_t / P_t$. Note that:

$$
\frac{P_t^s}{P_t} = \frac{P^s \pi_0 \pi_1 \pi_2 \cdots \pi_{t-1}}{P^s \pi_1 \pi_2 \pi_3 \cdots \pi_t} = \frac{\pi_0}{\pi_t},
$$
$$
\frac{\bar{P}_t}{P^s \pi_0} = \frac{\bar{p}_t}{P^s \pi_0} = \frac{\bar{P}_1}{P^s \pi_0} = \frac{\bar{P}_1}{P^s \pi_0} = \frac{p_1 \pi_1}{\pi_0}
$$

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in the notes above. Thus,

\[ \vartheta = \sum_{t=1}^{\infty} (\beta \xi_t)^t u_{c,t} Y_t \left\{ \left( \frac{\pi_0}{\pi_t} \right)^{1-\frac{\lambda_f}{\lambda_f-1}} \left[ \left( \frac{\tilde{p}_t}{\pi_0} \right)^{1-\frac{\lambda_f}{\lambda_f-1}} - 1 \right] \right\} - s_t \left( \frac{\pi_0}{\pi_t} \right)^{\frac{\lambda_f}{\lambda_f-1}} \left[ \left( \frac{\tilde{p}_t}{\pi_0} \right)^{\frac{\lambda_f}{\lambda_f-1}} - 1 \right]. \]

We now turn to the households’ choice of the wage rate. We have in mind the same situation considered above. Namely, the economy starts in steady state. Then, a money shock occurs. Of course, the household has already made its wage decision. So, in the current period, everyone chooses the same wage rate. But, in the next period, some people receive a Calvo tap, and others do not. The situation is as it is in the table above, but with the wage rate replacec by \( W \):

<table>
<thead>
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<tbody>
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</tr>
<tr>
<td>( t = 1 ) ( \frac{W^*}{W} = \frac{\tilde{w}}{W} )</td>
<td>( \frac{W^<em>}{W^</em> \pi_t} = \frac{\pi_t}{\pi_0} )</td>
</tr>
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</tr>
<tr>
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<td>( \frac{W^<em>}{W^</em> \pi_t} = \frac{\pi_t}{\pi_0} )</td>
</tr>
</tbody>
</table>

9. Estimating the Parameters

We have the following 20 parameters. For preferences:

\[ \chi, b, \beta, \psi_q, \sigma_q, \psi_L, \sigma_L, \sigma_c. \]

Technology:

\[ \alpha, \lambda_w, \lambda_f, \xi_q, \xi_p, \xi_w, \delta, \psi. \]

Money:

\[ \phi, \omega_1, \omega_2, \omega_3. \]

We should just pin down \( \beta \) to 0.99, and \( \sigma_L = \sigma_c = 1 \). We can estimate \( \alpha \) and \( \delta \) the way Craig and Marty did. The parameter \( \psi_L \) will come off some mean associated with the static Euler equation. Parameters to get from impulse response functions include \( \chi, b, \psi \) (consumption, investment) \( \psi_q, \sigma_q, \xi_q \) (fed funds response) and \( \xi_p, \xi_w \) (impulse response of inflation and the real wage). The law of motion for money will be the VAR itself.
Parameters estimated on the VAR could be based on functions of the impulse responses. We can get $\xi_p$ from data on marginal cost and inflation. Here are the two equations:

$$\hat{\pi}_t = \frac{1}{1 + \beta} \hat{\pi}_{t-1} + \frac{\beta}{1 + \beta} E_t \hat{\pi}_{t+1} + \frac{(1 - \beta \xi_p)(1 - \xi_p)}{(1 + \beta) \xi_p} \hat{s}_t,$$

under scheme #2. Scheme #1 gives:

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1 - \beta \xi_p)(1 - \xi_p)}{\xi_p} \hat{s}_t.$$

Define

$$\tilde{\pi}^1_t = \hat{\pi}_t - \frac{1}{1 + \beta} \hat{\pi}_{t-1} - \frac{\beta}{1 + \beta} \hat{\pi}_{t+1},$$

$$\tilde{\pi}^2_t = \hat{\pi}_t - \beta \hat{\pi}_{t+1}.$$

We set $\beta = 0.99$. We measured $\hat{s}_t$ as the (log of) the ratio of compensation of employees (GCOMP) plus nonfarm proprietor’s income (GPROBJ) times the gross, quarterly, funds rate (FYFF) to GDP. (Capitalized mnemonics are from Citibase.) We measured $\hat{\pi}_t$ as the log-first difference of the GDP deflator (GD). We regressed $\tilde{\pi}^i_t$ on $\hat{s}_t$ for $i = 1, 2$, with data covering the period 1960Q1 to 1998Q1. When $i = 1$, the coefficient is 0.0053. This implies a value of $\xi_p = 0.87$, assuming $\beta = 0.99$. For $i = 2$, we obtain a coefficient of 0.0022 implying $\xi_p = 0.96$. The standard errors on these coefficients are extremely high. A scatterplot looks like a shotgun pattern. There is little or no information in this regression.

The wage equation is, when $\varphi_w = 1$:

$$0 = \hat{w}_{t-1} - \frac{b_w (1 + \beta \xi_w^2) - \sigma_L \lambda_w}{b_w \xi_w} \hat{w}_t + \beta \hat{w}_{t+1}$$

$$-\pi_t + \beta \pi_{t+1} + \frac{1 - \lambda_w}{b_w \xi_w} \hat{u}_{c,t} - \frac{1 - \lambda_w}{b_w \xi_w} \hat{f}_t.$$

When $\varphi_w = 0$

$$0 = \hat{w}_{t-1} - \frac{b_w (1 + \beta \xi_w^2) - \sigma_L \lambda_w}{b_w \xi_w} \hat{w}_t + \beta E_t \hat{w}_{t+1}$$

$$+ \beta(E_t \pi_{t+1} - \pi_t) - (\pi_t - \pi_{t-1}) + \frac{1 - \lambda_w}{b_w \xi_w} \hat{u}_{c,t} - \frac{1 - \lambda_w}{b_w \xi_w} \hat{f}_t.$$

where $b_w = [\lambda_w \sigma_L - (1 - \lambda_w)] / [(1 - \xi_w) (1 - \beta \xi_w)]$. Note the ‘accelerationist’ inflation term in the last expression.