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416, Fall 2001
Homework 6, due Monday, November 19.

Consider an economy in which household preferences have the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \beta = .99, \text{and } u(c_t, n_t) = \log(c_t) + \psi \log(1 - n_t).$$

The household budget constraint is:

$$c_t + k_{t+1} - (1 - \delta)k_t = r_t k_t + w_t n_t + \pi_t, \ \delta = .028,$$

where r_t, w_t, π_t denote the rental rate on capital, the wage rate, and profits, respectively. Firms operate the following technology:

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha}, \ \alpha = 0.36.$$

Here,

$$A_t = Y_t^{\gamma},$$

where Y_t denotes the economy-wide level of output, in per capita terms. Firms are competitive, and maximize profits, which are zero in equilibrium.

1. Derive the household static and dynamic Euler equations. Derive the firm Euler equation.

2. In the household Euler equations, substitute out the rental rate on capital and the wage rate for labor, using the firm first order conditions. You should have a static and dynamic Euler equation. You should focus on symmetric equilibria, in which the economy-wide average stock of capital and level of employment coincide with the firm's stock of capital and employment. After making this identification, the static and dynamic Euler equations should have the following arguments:

$$\begin{aligned} \text{`static'} &: \quad v_h(n_t, k_t, k_{t+1}) = 0 \\ \text{`dynamic'} &: \quad v_k(k_t, k_{t+1}, k_{t+2}, n_t, n_{t+1}) = 0. \end{aligned}$$

   Linearize these equations about the steady state values of employment and the firm's stock of capital. In computing the steady states, fix steady state employment at 1/3 and choose the value of ψ that rationalizes that. Initially, fix $\gamma = 0.01$.

3. Substitute out for labor in the dynamic Euler equation, using the static Euler equation. This will give you one dynamic Euler equation in k_t, k_{t+1}, k_{t+2}.

4. Set the Euler equation up as a first order difference equation:

$$aY_{t+1} + bY_t = 0, t = 0, 1, 2, ... \tag{1}$$

   Show the values of a, and b. What are the eigenvalues of $\Pi = -a^{-1}b$? Find a value of γ such that, for values larger than that, both eigenvalues of Π are less than unity in absolute value. Display the left eigenvectors of Π associated with each of these two eigenvalues. Hint: to find out how to compute eigenvalues and eigenvectors in MATLAB, type 'help eig', in MATLAB. Display the policy rules corresponding to each eigenvector, i.e., the representations, $\tilde{k}_{t+1} = d\tilde{k}_t$, $\tilde{n}_t = q\tilde{k}_t$, where a tilde denotes deviation from steady state.

5. Solve the model again, but this time do it *without* substituting out for labor in the dynamic Euler equation. In this question, set the externality parameter, γ to zero. Set up the state space system, (1). Note that the exact procedure applied before cannot be applied now because a is not invertible. There are two ways to proceed. The brute force way is to do the substitutions you did in question (5). That has the effect of eliminating the singularity in (1) and converting the system into one in which a is invertible after all.

   But, this is an awkward approach in more complicated environments. An alternative strategy that works quite generally is based on the QZ decomposition. It also removes the singularity in a system, reduces its size, and produces a new system where the a matrix is invertible. This approach is described in detail in the section, 'the non-invertible a case' in Christiano, Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients.

   The idea is to find the QZ decomposition of a and b, i.e., orthonormal matrices Q and Z with the properties

$$QaZ = H_0, \; QbZ = H_1,$$

   where H_0 and H_1 are upper triangular matrices. All these matrices are 6×6. The matrix H_0 is structured so that the l zeros on its diagonal are located in the lower right part of H_0 (you should determine what the value of l is in the model of the question).[1] Denote the upper $(6-l)\times(6-l)$ block of H_0 by G_0. This matrix must be non-singular. Let the corresponding upper left $(6-l)\times(6-l)$ block in H_1 be denoted G_1. I assume that the diagonal terms in the lower right $l\times l$ block of H_1 are non-zero. Also, it is useful to partition Z' as follows:

$$Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

where L_1 is $(6-l)\times 6$ and L_2 is $l\times 6$.

Inserting ZZ' $(=I)$ before Y_{t+1} and Y_t in (1), defining $\gamma_t \equiv Z'Y_t$, and pre-multiplying (1) by Q, (1) becomes:

$$H_0\gamma_{t+1} + H_1\gamma_t = 0,\ t = 0, 1, \ldots\ . \tag{2}$$

Partition γ_t as follows:

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}, \tag{3}$$

where γ_t^1 is $(6-l)\times 1$ and γ_t^2 is $l\times 1$. It is easy to verify that (2) implies $\gamma_t^2 = 0$, $t\geq 0$, i.e.,

$$L_2 Y_t = 0,\ t = 0, 1, \ldots\ . \tag{4}$$

With (4) imposed, the last l equations in (2) are redundant, so (2) can be written

$$G_0\gamma_{t+1}^1 + G_1\gamma_t^1 = 0,\ t = 0, 1, \ldots\ . \tag{5}$$

This system looks just like (1), except that its dimension is less than 6. Also, the analog of the a matrix, G_0, is now nonsingular.

[1] The QZ decomposition can be implemented on a and b with MATLAB routine QZ. A problem with MATLAB's routine is that it does not order H_0 so that the zeros on the diagonal are located along the bottom right part of the diagonal. This can be done by following MATLAB's QZ program with programs written by Chris Sims to do the reordering. The programs are QZDIV.M and QZSWITCH.M. The latter is called by QZDIV.M. Here is the way I implement the decomposition:

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[H0,H1,q,z,v]=qz(a,b);
stake1 = 1e+08;
stake2 = 1e-05;
[H0,H1,q,z] = qzdiv(stake1,H0,H1,q,z);
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The set of solutions to the new system can be expressed as $\gamma_t^1 = (-G_0^{-1}G_1)^t\gamma_0^1$, $t \geq 0$, or,

$$P^{-1}\gamma_t^1 = \Lambda^t P^{-1}\gamma_0^1, \tag{6}$$

where $P\Lambda P^{-1} = -G_0^{-1}G_1$ is the eigenvector, eigenvalue decomposition of $-G_0^{-1}G_1$. The γ_t^1 that solve (6) converge to zero asymptotically if, and only if, $\tilde{p}\gamma_0^1 = 0$, where $\tilde{p}$ is composed of the rows of P^{-1} corresponding to diagonal terms in Λ that exceed 1 in absolute value. Taking into account the definition of γ_t^1, this condition is:

$$\tilde{p}L_1Y_0 = 0. \tag{7}$$

The number of free elements in Y_0 is 2 (i.e., n_0 and k_1). Equation (4) for $t = 0$ represents l (actually, l had better be unity in this example) restrictions on Y_0, so that to pin Y_0 down uniquely, $2-l$ more restrictions are needed. Thus, uniqueness of convergent paths requires that there be $2 - l$ explosive eigenvalues in Λ, i.e., that $\tilde{p}L_1$ contain $2 - l$ rows. Then, define

$$D = \begin{bmatrix} \tilde{p}L_1 \\ L_2 \end{bmatrix}. \tag{8}$$

Use the condition, $DY_t = 0$, to express the minimal state solutions in the form $\tilde{k}_{t+1} = d\tilde{k}_t$, $\tilde{n}_t = q\tilde{k}_t$. How many are there? How do they compare with what you got before, when you substituted out for hours worked?