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Economics 416
Fall, 2001
Due: November 5.

Homework 4: Sunspot Equilibria in a Limited Participation Monetary Model

This homework explores sunspots. The best way to understand them is to grow some yourself. That's what this homework asks you to do. For some background on the relevant concepts, see section 3.2.1 in the Christiano-Rostagno paper and the class handout of Tuesday, October 23.

Section 1 below lays out a simple monetary model in which the monetary policy authority implements a Taylor rule. We ignore the various interesting issues that arise when there is a government budget constraint by simply assuming the fiscal authority has no outstanding debt and finances its expenditures with lump sum taxes. The monetary authority affects the size of the money stock by executing lump sum transfers (or, taxes, if negative) of cash.

Section 2 explores the global properties of equilibrium in the model. The first subsection sets the stage, by studying the deterministic equilibria of the model. The following subsection constructs sunspot equilibria. Technically, there are two messages here. First, indeterminacy of a particular deterministic equilibrium is sufficient for the existence of a sunspot equilibrium. This illustrates the point of a classic (unpublished!) 1986 manuscript by Woodford. Second, indeterminacy is not necessary for a sunspot equilibrium to exist. This illustrates a point made recently by Carlstrom and Fuerst (see their 'Forward Looking versus Backward Looking Taylor Rules' on the course website). Substantively, the homework involves exploring a new model of sunspots, one which may turn out to be useful for interpreting data.

1. The Model

1.1. Households and Firms

Preferences:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}). \tag{1.1}$$

where

$$u(c,l) = \frac{\left[c - \frac{\psi_0}{1+\psi} l^{1+\psi}\right]^{1-\sigma}}{1-\sigma},\ c - \frac{\psi_0}{1+\psi} l^{1+\psi} \geq 0.$$

We will refer to the non-negativity condition above as the 'non-negativity constraint on utility'. Cash constraint:

$$P_t c_t \leq W_t l_t + M_t - T_t - D_t, \tag{1.2}$$

where D_t denotes household deposits, M_t denotes the household's beginning of period money holdings, P_t is the price of consumption goods, c_t, W_t is the wage rate and l_t is work effort. Also, T_t denotes lump sum taxes paid to the government. Later, we will specify that fiscal policy sets T_t so that government spending, $P_t g$, is financed exactly in each period.

The household's cash evolution equation is:

$$M_{t+1} = (1+R_t)(X_t + D_t) + M_t - D_t + W_t l_t - P_t c_t - T_t + \text{Profits}_t.$$

The household's static labor Euler equation is:

$$\frac{W_t}{P_t} = \psi_0 l_t^{\psi}. \tag{1.3}$$

The intertemporal Euler equation associated with the D_t decision is:

$$E_{t-1} \frac{u_{c,t}}{P_t} = E_{t-1}\left(1+R_t\right) \frac{\beta u_{c,t+1}}{P_{t+1}}. \tag{1.4}$$

The lagged information set here reflects that we are working with a limited participation economy: households make their date t deposit decision, D_t, before the realization of date t random variables. (As we'll see, the only such variables are sunspot variables.)

The firm Euler equation is (firms have to borrow to pay their wage bill and there is monopoly power, accounting for the presence of λ):

$$\frac{W_t}{P_t} = \frac{\lambda}{1+R_t},\ 0 < \lambda \leq 1. \tag{1.5}$$

Combining the household and firm intratemporal Euler equations:

$$1 + R_t = \frac{\lambda}{\psi_0 l_t^{\psi}}. \tag{1.6}$$

Substitute this into the intertemporal Euler equation:

$$E_{t-1}\frac{u_{c,t}}{P_t} = E_{t-1}\frac{\lambda}{\psi_0 l_t^{\psi}}\frac{\beta u_{c,t+1}}{P_{t+1}}.$$

We can also rewrite the marginal utility of consumption, using the resource constraint:

$$l_t = c_t + g,$$

to obtain:

$$E_{t-1}\frac{\left[l_t - g - \frac{\psi_0}{1+\psi}l_t^{1+\psi}\right]^{-\sigma}}{P_t} = E_{t-1}\frac{\lambda}{\psi_0 l_t^{\psi}}\frac{\beta\left[l_{t+1} - g - \frac{\psi_0}{1+\psi}l_{t+1}^{1+\psi}\right]^{-\sigma}}{P_{t+1}}. \tag{1.7}$$

Optimization requires that (1.2) hold as a strict equality if $R_t > 0$. The weak inequality holds in the event $R_t = 0$.

1.2. Taylor Rule

We adopt the following specification of the Taylor rule:

$$R_t = \max\left\{0, R^* + \alpha\left(E_t\pi_{t+1} - \pi^*\right)\right\},$$

or, inverting this:

$$E_t\pi_{t+1} = \begin{cases} \pi^* + \frac{1}{\alpha}\left(R_t - R^*\right) & R_t > 0 \\ [0, \pi^* - \frac{1}{\alpha}R^*] & R_t = 0 \end{cases}, \quad \pi_t = \frac{P_t}{P_{t-1}}, \tag{1.8}$$

where π^* is the 'target inflation rate' implicit in the Taylor rule, and $R^* = \pi^*/\beta$. We refer to a non-stochastic steady state equilibrium in which the inflation rate and interest rate are π^* and R^*, respectively as the *inflation target equilibrium*.

2. Equilibria

Before considering sunspot equilibria, it is convenient to first study the set of deterministic equilibria.

2.1. Deterministic Equilibria

In the absence of shocks, (1.7) reduces to:

$$\left[l_t - g - \tfrac{\psi_0}{1+\psi} l_t^{1+\psi}\right]^{-\sigma} \frac{\psi_0 l_t^{\psi}}{\lambda\beta}$$

$$= \begin{cases} \dfrac{\left[l_{t+1} - g - \frac{\psi_0}{1+\psi} l_{t+1}^{1+\psi}\right]^{-\sigma}}{\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}} - 1 - R\right)} & l_{t+1} < \left[\frac{\lambda}{\psi_0}\right]^{\frac{1}{\psi}} \\ \dfrac{\left[l_{t+1} - g - \frac{\psi_0}{1+\psi} l_{t+1}^{1+\psi}\right]^{-\sigma}}{\pi_{t+1}}, \ \pi_{t+1} \in [0, \pi - \frac{1}{\alpha} R] & l_{t+1} = \left[\frac{\lambda}{\psi_0}\right]^{\frac{1}{\psi}} \end{cases},$$

or,

$$A(l_t) = B(l_{t+1}), \tag{2.1}$$

in obvious notation.

In the first subsection that follows, we discuss the domain, D, and range of the mappings, A and B. In the second subsection, we state and prove a partial characterization result, that solutions to (2.1) correspond to equilibria. In the final subsection, we illustrate the characterization result with some examples.

2.1.1. The Domain and Range of A and B

There are two factors that impose an upper bound on D. Consider the case, $g = 0$. First, nonnegativity of utility requires:

$$l \leq \left[\frac{1+\psi}{\psi_0}\right]^{\frac{1}{\psi}}.$$

Second, the condition, $R_{t+1} > 0$, requires

$$l < \left[\frac{\lambda}{\psi_0}\right]^{\frac{1}{\psi}}.$$

So, the domain of A and B is:

$$D = \left\{ l : 0 \leq l \leq \min\left[\left(\frac{1+\psi}{\psi_0}\right)^{\frac{1}{\psi}}, \left(\frac{\lambda}{\psi_0}\right)^{\frac{1}{\psi}}\right]\right\}$$

Now consider the case, $g > 0$. The bounds that define D cannot be expressed analytically. Consider first the restrictions imposed by non-negative utility. Consider the following expression:

$$l - g = \frac{\psi_0}{1+\psi} l^{1+\psi}.$$

The function on the left is parallel to the 45 degree line, shifted down by g. The function on the right is monotonically increasing, with slope $\psi_0 l^{\psi}$. This slope is zero at $l = 0$ and is increasing as l increases. This suggests that, if there is a value of l that satisfies the above expression, two values of l will do it. Denote the lower and upper values by $\bar{l}^l$ and $\tilde{l}$, respectively. It is easy to see that as $g \to 0$, $\bar{l}^l \to 0$, consistent with our discussion of the $g = 0$ case. A practical algorithm for finding $\bar{l}^l$ and $\tilde{l}$ works like this. First, we need to verify that these numbers exist in the first place, i.e., that we're not going in with a value of g that is too large. To determine this, simply find the value of l that maximizes $l - \frac{\psi_0}{1+\psi} l^{1+\psi}$. This is just $l = (1/\psi_0)^{1/\psi}$. Then, the largest value of g for which the interval, $[\bar{l}^l, \tilde{l}]$, is non-empty is

$$\bar{g} = (1/\psi_0)^{1/\psi} - \frac{\psi_0}{1+\psi} \left[(1/\psi_0)^{1/\psi} \right]^{1+\psi}.$$

When $g = \bar{g}$, then $[\bar{l}^l, \tilde{l}]$ is composed of a single point. As long as $g < \bar{g}$, $[\bar{l}^l, \tilde{l}]$ is a non-trivial interval, i.e., $\bar{l}^l < \tilde{l}$. We can find the boundaries of this interval by putting a grid on l. The lower bound is zero and the upper bound is the unique positive value of l that sets utility to zero when $g = 0$ (this is easy to verify in a picture.) This is just $[(1+\psi)/\psi_0]^{1/\psi}$. So, we put a fine grid on $\{0, [(1+\psi)/\psi_0]^{1/\psi}\}$ and find the boundaries of $[\bar{l}^l, \tilde{l}]$ by identifying the two grid intervals were $l - g - \frac{\psi_0}{1+\psi} l^{1+\psi}$ switches sign. The exact l's that set this function to zero can then be found by applying a standard numerical algorithm.[1] So, think of the two zeros, $\bar{l}^l$ and $\tilde{l}$, as having been found.

The restrictions imposed by $R_t \geq 0$ are the same as before. Thus, we conclude:

$$D = \left\{ l : \bar{l}^l \leq l \leq \bar{l}^u \right\}, \ \bar{l}^u \equiv \min \left[\tilde{l}, \left(\frac{\lambda}{\psi_0} \right)^{\frac{1}{\psi}} \right].$$

[1]For example, fzero in MATLAB will find the zero of a function and limit its search to a user-supplied interval.

It is interesting to identify the set of model parameter values where $\bar{l}^u < \tilde{l}$. We can determine this as follows. Substitute $l = [\lambda/\psi_0]^{1/\psi}$ into the argument of the utility function:

$$\begin{aligned} & (\lambda/\psi_0)^{1/\psi} - \tfrac{\psi_0}{1+\psi}\left[(\lambda/\psi_0)^{1/\psi}\right]^{1+\psi} - g \\ = & \left(\frac{\lambda}{\psi_0}\right)^{\frac{1}{\psi}}\left[1 - \tfrac{\psi_0}{1+\psi}\frac{\lambda}{\psi_0}\right] - g \\ = & \left(\frac{\lambda}{\psi_0}\right)^{\frac{1}{\psi}}\left[1 - \tfrac{\lambda}{1+\psi}\right] - g. \end{aligned}$$

It is useful to draw a picture of $l - g$ and $\frac{\psi_0}{1+\psi}l^{1+\psi}$. It is easily verified that if the above expression is negative, then $\tilde{l} < (\lambda/\psi_0)^{1/\psi}$, so that $\bar{l}^u = \tilde{l}$. If the expression is positive, then $\tilde{l} > (\lambda/\psi_0)^{1/\psi}$ and $\bar{l}^u = (\lambda/\psi_0)^{1/\psi}$.

So, the domain for A and B is D. It is easily verified that A is a function over this domain, mapping D into the positive real line. For $l \to \bar{l}^l, \tilde{l}$, $A \to \infty$, as long as $g > 0$. In the middle, A is finite and positive. We have found that the graph of the function, A, exhibits a 'U' shape over the domain, $[\bar{l}^l, \tilde{l}]$. Over the domain, D, this 'U' shape is truncated on the right if $\bar{l}^u < \tilde{l}$. This is the case when the upper bound on D is defined by the non-negativity constraint on R_t. This is an important case, because then B is set-valued for l on the upper boundary of D.

Now consider the $B(l)$ function:

$$B(l) = \begin{cases} \dfrac{1}{\left[l-g-\frac{\psi_0}{1+\psi}l^{1+\psi}\right]^{\sigma}\left[\pi+\frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l^{\psi}}-1-R\right)\right]} & l < (\lambda/\psi_0)^{1/\psi} \\ \dfrac{\left[l-g-\frac{\psi_0}{1+\psi}l^{1+\psi}\right]^{-\sigma}}{\pi_{t+1}}, \ \pi_{t+1} \in [0, \pi - \frac{1}{\alpha}R] & l = (\lambda/\psi_0)^{1/\psi} \end{cases}.$$

Evidently, for $l < (\lambda/\psi_0)^{1/\psi}$, this is an ordinary function. For $l = (\lambda/\psi_0)^{1/\psi}$, $B(l)$ is set-valued. The upper bound is associated with the lower bound on π_{t+1}, and corresponds to $B = \infty$. The lower bound is associated with $\pi_{t+1} = \pi - \frac{1}{\alpha}R$, and coincides with

$$\lim_{l \to (\lambda/\psi_0)^{1/\psi}} B(l).$$

So, the graph of B is a line for $l < (\lambda/\psi_0)^{1/\psi}$. As $l \to (\lambda/\psi_0)^{1/\psi}$, that line becomes the lower bound for the set, $B\left[(\lambda/\psi_0)^{1/\psi}\right]$, which is not bounded above.

2.1.2. A Partial Characterization Result for Equilibria

We now discuss the construction of an equilibrium for this economy. Suppose we have a sequence, $l_0, l_1, ...$, which satisfies (2.1) and $l_t \in D$ for $t = 0, 1, 2, ...$. We can compute a sequence, $\pi_1, \pi_2, ...$ using (1.8). From this, we have

$$P_t = \pi_t P_{t-1},\ t = 1, 2, ...\ ,$$

with P_0 being an arbitrary value for the period 0 price level given initial condition.[2] This candidate equilibrium is an actual equilibrium if we can find a set of values for the remaining equilibrium variables, which satisfy all the equilibrium conditions.

We can solve for R_t from (1.6) and W_t/P_t from (1.5). Note that by the fact that $l_t \in D$,we have $R_t \geq 0$, and it is trivial to verify that $W_t/P_t \geq 0$. Given the P_t's, we have the W_t's. We obtain c_t from the resource constraint, $c_t = l_t - g$. We are guaranteed that $c_t \geq 0$ from the fact, $l_t \in D$. We still need deposits, D_t, and money growth, X_t.

We can construct an equilibrium in which the household's cash constraint and the money market clearing condition are satisfied as strict equalities, even in periods when $R_t = 0$. Thus

$$D_t = W_t l_t - T_t + M_t - P_t c_t,$$

where $T_t = P_t g$. Multiplying the firm Euler equation, (1.5), by l_t and P_t, we obtain:

$$W_t l_t = \frac{\lambda P_t l_t}{1 + R_t}.$$

Substitute this and the resource constraint into the expression for deposits, to obtain:

$$\begin{aligned} D_t &= \frac{\lambda P_t l_t}{1 + R_t} - P_t l_t + M_t \\ &= P_t l_t \left[\frac{\lambda}{1 + R_t} - 1 \right] + M_t. \end{aligned}$$

[2] That only π_1 and not π_0 is pinned down is the reason this model exhibits 'nominal indeterminacy'.

Since $\lambda \leq 1$, it follows that $\lambda/(1+R_t) \leq 1$, so that $D_t \leq M_t$. If $\lambda = 1$, then $D_t \to M_t$ as $R_t \to 0$.[3]

With D_t, $t = 0, 1, 2, ...$ in hand, we can compute X_t from

$$X_t = W_t l_t - D_t.$$

This can then be used to compute a sequence of money stocks:

$$M_{t+1} = X_t + M_t,$$

with M_0 given.

We have established:

Proposition 2.1. *If $l_0, l_1, ...,$ satisfy (2.1) and $l_t \in D$ for $t = 0, 1, 2, ...$, then these l_t's correspond to an equilibrium.*

2.1.3. Questions

A. Consider a version of the model with $\alpha = 1.5$, $\beta = 0.99264$, $\sigma = 1$, $\psi_0 = 1$, $\psi = 2$, $\pi^* = 1.0043$, $\lambda = 1$, $g = 0.08$.

1. Compute l^*, the level of employment in the target inflation steady state. Show that that equilibrium is determinate?[4] Support your conclusion with a graph of A and B in a narrow range about l^*.

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[3] The intuition here is simple. The households set $D_t < M_t$ to the extent that they need cash in addition to the wage bill to finance consumption. Suppose profits, taxes and the interest rate were zero. In this case the wage bill equals total output (in general, total output equals profits plus earnings of the financial intermediary, plus the wage bill). But, total output equals household consumption plus government consumption. So, in this case, the wage bill exceeds what households need to finance consumption. In this case, they would set $D_t > M_t$. But, we levy taxes in the amount of government spending, and so if R_t and profits were still zero, then we'd have $D_t = M_t$. But, if $R_t > 0$, then some of output goes to the financial intermediary, so that the wage bill net of taxes is less than total consumption. In this case, $D_t < M_t$, consistent with the formula above. Finally, if profits were positive because $\lambda < 1$, once again wages after taxes would be less than total consumption and there is another reason to set $D_t < M_t$.

[4] For this, you will have to totally differentiate the expression, $A(l_t) = B(l_{t+1})$ with respect to l_t and l_{t+1} and study dl_{t+1}/dl_t.

2. Graph A and B over the range, $l < l^*$. Are there sequences with $l_0 < l^*$ that satisfy the conditions of the above proposition? Now graph A and B for $l > l^*$. Are there equilibria with $l_0 > l^*$. Is there a steady state equilibrium with $R = 0$? Is that equilibrium determinate, or indeterminate?

B. Now consider a version of the model with a higher labor supply elasticity, say with $\psi = 0.05$. Note that now the inflation target steady state is indeterminate. What is the intuition underlying the result that a higher labor supply elasticity results in indeterminacy?

2.2. Stochastic Equilibria

The household's intertemporal Euler equation in the stochastic version of the economy is:

$$E_{t-1}\left[\frac{u_{c,t}}{P_t} - \frac{\lambda}{\psi_0 l_t^{\psi}} \frac{\beta u_{c,t+1}}{P_{t+1}}\right] = 0,$$

$t = 0, 1, 2, ...$. Substituting:

$$E_{t-1}\left\{\frac{\left[l_t - g - \frac{\psi_0}{1+\psi} l_t^{1+\psi}\right]^{-\sigma}}{P_t} - \frac{\lambda}{\psi_0 l_t^{\psi}} \frac{\beta \left[l_{t+1} - g - \frac{\psi_0}{1+\psi} l_{t+1}^{1+\psi}\right]^{-\sigma}}{P_{t+1}}\right\} = 0.$$

Substitute the forward-looking Taylor rule in here:

$$E_{t-1}\left\{\frac{\left[l_t - g - \frac{\psi_0}{1+\psi} l_t^{1+\psi}\right]^{-\sigma}}{P_t} - \frac{\lambda}{\psi_0 l_t^{\psi}} \frac{\beta \left[l_{t+1} - g - \frac{\psi_0}{1+\psi} l_{t+1}^{1+\psi}\right]^{-\sigma}}{P_t \left[\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}} - 1 - R\right) + \nu_{t+1}\right]}\right\} = 0,$$

where $\nu_{t+1} = \pi_{t+1} - E_t \pi_{t+1}$. Since

$$D_t = P_t l_t \left[\psi_0 l_t^{\psi} - 1\right] + M_t,$$

it follows that

$$P_t = \frac{D_t - M_t}{l_t \left[\psi_0 l_t^{\psi} - 1\right]}.$$

Then,

$$E_{t-1}\{\frac{l_t\left[\psi_0 l_t^{\psi}-1\right]\left[l_t-g-\frac{\psi_0}{1+\psi}l_t^{1+\psi}\right]^{-\sigma}}{D_t-M_t}$$
$$-\frac{\lambda}{\psi_0 l_t^{\psi}}\frac{l_t\left[\psi_0 l_t^{\psi}-1\right]\beta\left[l_{t+1}-g-\frac{\psi_0}{1+\psi}l_{t+1}^{1+\psi}\right]^{-\sigma}}{[D_t-M_t]\left[\pi+\frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}}-1-R\right)+\nu_{t+1}\right]}\} = 0.$$

Since $D_t - M_t$ is in the E_{t-1} information set, $D_t - M_t$ can be cancelled from both sides:

$$E_{t-1}\{l_t\left[\psi_0 l_t^{\psi}-1\right]\left[l_t-g-\frac{\psi_0}{1+\psi}l_t^{1+\psi}\right]^{-\sigma}$$
$$-\frac{\lambda}{\psi_0 l_t^{\psi}}\frac{l_t\left[\psi_0 l_t^{\psi}-1\right]\beta\left[l_{t+1}-g-\frac{\psi_0}{1+\psi}l_{t+1}^{1+\psi}\right]^{-\sigma}}{\left[\pi+\frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}}-1-R\right)+\nu_{t+1}\right]}\} = 0.$$

Let,

$$\begin{aligned}
A(l_t) &= l_t\left[\psi_0 l_t^{\psi}-1\right]\left[l_t-g-\frac{\psi_0}{1+\psi}l_t^{1+\psi}\right]^{-\sigma} \\
a(l_t) &= \frac{\lambda\beta l_t\left[\psi_0 l_t^{\psi}-1\right]}{\psi_0 l_t^{\psi}} \\
B(l_t, l_{t+1}, \nu_{t+1}) &= \frac{\left[l_{t+1}-g-\frac{\psi_0}{1+\psi}l_{t+1}^{1+\psi}\right]^{-\sigma}}{\left[\pi+\frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}}-1-R\right)+\nu_{t+1}\right]}
\end{aligned}$$

Then,

$$E_{t-1}\left\{A(l_t) - a(l_t)B(l_t, l_{t+1}, \nu_{t+1})\right\} = 0$$

There are additional conditions. We shall limit ourselves to stochastic equilibria in which $R_t > 0$, so that the cash constraint and money market clearing condition must hold as a strict equality. The household's cash constraint is: $P_t c_t = W_t l_t - T_t + M_t - D_t$. After substituting out the household's labor Euler equation and the fact, $T_t = P_t g$:

$$D_t = P_t l_t\left[\psi_0 l_t^{\psi}-1\right] + M_t. \tag{2.2}$$

Substituting the money market clearing condition, $W_t l_t = D_t + X_t$, into the household's cash constraint we obtain $P_t c_t = M_t + X_t - T_t$, or,

$$P_t l_t = M_t + X_t,$$

after using $c_t + g = l_t$. Divide the previous two expressions by M_t :

$$\begin{aligned} d_t &= p_t l_t \left[\psi_0 l_t^{\psi} - 1\right] + 1, && (2.3) \\ p_t l_t &= 1 + x_t, && (2.4) \end{aligned}$$

where

$$d_t = \frac{D_t}{P_t},\ x_t = \frac{X_t}{M_t}.$$

From the (inverse of the) Taylor rule,

$$P_{t+1} = P_t \left[\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}} - 1 - R\right) + \nu_{t+1}\right],$$

or, after dividing by M_t :

$$p_{t+1}(1 + x_t) = p_t \left[\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}} - 1 - R\right) + \nu_{t+1}\right].$$

Substitute the cash constraint in here, and cancel p_t from both sides:

$$p_{t+1} = \frac{\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^{\psi}} - 1 - R\right) + \nu_{t+1}}{l_t}$$

Also, taking the ratio of the money market clearing condition, $W_t l_t = X_t + D_t$, and the cash constraint on households, $P_t l_t = M_t + X_t$, one obtains:

$$\psi_0 l_t^{\psi} = \Gamma_t = \frac{x_t + d_t}{1 + x_t}. \tag{2.5}$$

Not surprisingly, of (2.3), (2.4) and (2.5), only two are independent. For example, substituting out for x_t in (2.5) from (2.4), one obtains (2.3).

We now summarize the basic equations that characterize the equilibrium:

$$A(l_t) - a(l_t)B(l_t, l_{t+1}, \nu_{t+1}) = \xi_0 \eta_{t+1} + \xi_1 \eta_t \tag{2.6}$$

$$\psi_0 l_t^\psi = \frac{x_t + d_t}{1 + x_t} \tag{2.7}$$

$$d_t = p_t l_t \left[\psi_0 l_t^\psi - 1\right] + 1 \tag{2.8}$$

$$p_t l_t = 1 + x_t, \tag{2.9}$$

$$p_{t+1} = \frac{\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^\psi} - 1 - R\right) + \nu_{t+1}}{l_t} \tag{2.10}$$

with the following additional restrictions:

$$E_t \omega_{t+1} = E_t \eta_{t+1} = E_t \nu_{t+1} = 0$$

d_t is not a function of date t shocks,

$$R_t = \frac{\lambda}{\psi_0} l_t^{-\psi} - 1 \geq 0,$$

and of equations (2.7)-(2.9), either (2.7) or (2.9) should be dropped, depending on what is convenient. We find an equilibrium if we compute a set, $\{l_t, d_t, p_t, x_t\}$ that satisfy the four independent equations and the additional restrictions (the random variables, η_t and ω_t are 'sunspots').

Now, x_t is actually a 'nuisance' variable from our perspective, and so we can drop it, and an equation too. So, let this be our system of equations:

$$A(l_t) - a(l_t)B(l_t, l_{t+1}, \nu_{t+1}) = \xi_0 \eta_{t+1} + \xi_1 \eta_t \tag{2.11}$$

$$d_t = p_t l_t \left[\psi_0 l_t^\psi - 1\right] + 1 \tag{2.12}$$

$$p_{t+1} = \frac{\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_t^\psi} - 1 - R\right) + \nu_{t+1}}{l_t} \tag{2.13}$$

We can substitute out for p_t in (2.12) from (2.13):

$$A(l_t) - a(l_t)B(l_t, l_{t+1}, \nu_{t+1}) = \xi_0 \eta_{t+1} + \xi_1 \eta_t \tag{2.14}$$

$$d_t = \frac{\pi + \frac{1}{\alpha}\left(\frac{\lambda}{\psi_0 l_{t-1}^\psi} - 1 - R\right) + \nu_t}{l_{t-1}} l_t \left[\psi_0 l_t^\psi - 1\right] + 1 \tag{2.15}$$

This is two equations in the two unknowns, $\{l_t, d_t\}$. Let's linearize this and see where we get. From (2.11),

$$\begin{aligned} \alpha_0 \hat{l}_{t+1} + \alpha_1 \hat{l}_t &= \xi_0 \eta_{t+1} + \xi_1 \eta_t + \beta_0 \nu_{t+1} && (2.16) \\ \hat{d}_t &= a_0 \hat{l}_t + a_1 \hat{l}_{t-1} + a_2 \nu_t && (2.17) \end{aligned}$$

where ξ_0, ξ_1 are free parameters, and $\alpha_0, \alpha_1, a_0, a_1, a_2$ are known parameters that are functions of the model parameters. Consider the following candidate solution:

$$\begin{aligned} \hat{l}_t &= \Lambda_l \hat{l}_{t-1} + \gamma_0 \eta_t + \gamma_1 \nu_t + B_0 \eta_{t-1} + B_1 \nu_{t-1} \\ \hat{d}_t &= \Lambda_d \hat{l}_{t-1} + B_2 \eta_{t-1} + B_3 \nu_{t-1}, \end{aligned}$$

where $\gamma_0, \gamma_1, B_0, B_1, B_2$ are coefficients to be determined. To pin the value of these down, substitute these into (2.16)-(2.17). Begin with (2.16):

$$\alpha_0 \left[\Lambda_l \hat{l}_t + \gamma_0 \eta_{t+1} + \gamma_1 \nu_{t+1} + B_0 \eta_t + B_1 \nu_t \right] + \alpha_1 \hat{l}_t = \xi_0 \eta_{t+1} + \xi_1 \eta_t + \beta_0 \nu_{t+1},$$

so that

$$\Lambda_l = -\frac{\alpha_1}{\alpha_0},\ \gamma_0 = \xi_0/\alpha_0,\ \gamma_1 = \beta_0/\alpha_0,\ B_0 = \xi_1/\alpha_0,\ B_1 = 0.$$

Now substitute into (2.17):

$$\Lambda_d \hat{l}_{t-1} + B_2 \eta_{t-1} + B_3 \nu_{t-1} = a_0 \left[\Lambda_l \hat{l}_{t-1} + \gamma_0 \eta_t + \gamma_1 \nu_t + B_0 \eta_{t-1} + B_1 \nu_{t-1} \right] + a_1 \hat{l}_{t-1} + a_2 \nu_t$$

This requires:

$$\Lambda_d = a_0 \Lambda_l + a_1,\ B_2 = a_0 \xi_1/\alpha_0,\ B_3 = a_0 B_1 = 0,\ \gamma_0 = 0,\ \gamma_1 = -a_2/a_0.$$

For the solution to be consistent, we require $\xi_0 = 0$. Problem is, we require γ_1 to be β_0/α_0 and $-a_2/a_0$ at the same time. Impossible! So, there does not exist a sunspot equilibrium of the type sought.

But, here is a candidate sunspot process that does work. Let $\nu_t = \varphi \eta_t$, where φ is free. Then, our system becomes

$$\begin{aligned} \alpha_0 \hat{l}_{t+1} + \alpha_1 \hat{l}_t &= \tilde{\xi}_0 \eta_{t+1} + \xi_1 \eta_t && (2.18) \\ \hat{d}_t &= a_0 \hat{l}_t + a_1 \hat{l}_{t-1} + \tilde{a}_2 \eta_t && (2.19) \end{aligned}$$

where $\tilde{\xi}_0 = \xi_0 + \varphi\beta_0$, $\tilde{a}_2 = a_2\varphi$. In effect, $\tilde{\xi}_0$, ξ_1, $\tilde{a}_2$ are free parameters here, with α_0, α_1, a_0, a_1 being functions of the model parameters. Consider the following candidate solution:

$$\hat{l}_t = \Lambda_l \hat{l}_{t-1} + \gamma_0 \eta_t + B_0 \eta_{t-1} \tag{2.20}$$

$$\hat{d}_t = \Lambda_d \hat{l}_{t-1} + B_1 \eta_{t-1}. \tag{2.21}$$

Substitute, like before:

$$\alpha_0 \left[\Lambda_l \hat{l}_t + \gamma_0 \eta_{t+1} + B_0 \eta_t \right] + \alpha_1 \hat{l}_t = \tilde{\xi}_0 \eta_{t+1} + \xi_1 \eta_t$$

so that

$$\Lambda_l = -\frac{\alpha_1}{\alpha_0},\ \gamma_0 = \tilde{\xi}_0/\alpha_0,\ B_0 = \xi_1/\alpha_0. \tag{2.22}$$

Also,

$$\Lambda_d \hat{l}_{t-1} + B_1 \eta_{t-1} = a_0 \left[\Lambda_l \hat{l}_{t-1} + \gamma_0 \eta_t + B_0 \eta_{t-1} \right] + a_1 \hat{l}_{t-1} + \tilde{a}_2 \eta_t$$

so that

$$\Lambda_d = a_0 \Lambda_l + a_1,\ B_1 = a_0 B_0 = a_0 \xi_1/\alpha_0,\ a_0 \gamma_0 + \tilde{a}_2 = 0. \tag{2.23}$$

The only thing we need to think about is whether the two solutions for γ_0 just given are consistent. We require

$$\frac{\tilde{\xi}_0}{\alpha_0} = -\frac{\tilde{a}_2}{a_0},$$

or,

$$\frac{\xi_0 + \varphi\beta_0}{\alpha_0} = -\frac{a_2\varphi}{a_0}.$$

In this expression, there are two free parameters, ξ_0 and φ. Solve for the second in terms of the first:

$$\varphi = -\frac{\xi_0}{\frac{\alpha_0 a_2}{a_0} + \beta_0}. \tag{2.24}$$

So, the solution is (2.20)-(2.21) with parameter values given by (2.22)-(2.23). This is a three-dimensional continuum of solutions, indexed by the choice of stochastic process for η_t and by choice of the values for ξ_0, ξ_1. In addition, the initial values of employment and η, l_{-1} and η_{-1}, also matter. However, their influence dissipates over time if $|\Lambda_l| < 1$. We may be confident that this is an equilibrium if $|\Lambda_l| < 1$ (i.e., the nonstochastic steady state is indeterminate), because the degree of variation is limited.

C. Produce a set of parameter values for the model which imply that the inflation target equilibrium is indeterminate (always set $\alpha = 1.5$). Construct values for the sunspot variables so that $R_t > 0$ is guaranteed, for all possible realizations of the sunspot variables (see the Christiano-Rostagno reference mentioned above for guidance here). Simulate the model in response to a long (i.e., 1000 observations) realization of the sunspot variables. What is the variance of output, employment? What is the correlation between output and the interest rate, between output and money growth? Do the simulated data resemble the business cycle?

D. To construct a sunspot equilibrium around the target inflation equilibrium, is it necessary that that equilibrium be indeterminate? (The analysis in C above should persuade you of sufficiency.)