1. Construct a non-explosive solution for the model of question 1 of the previous homework, in which a sunspot is present. Simulate 100 observations from the resulting system.

2. Consider the economy of question 2 of the previous homework.

Solve the model again, but this time do so without substituting out for labor in the dynamic Euler equation. In this question, set the externality parameter, $\gamma$ to zero. Set up the state space system. Note that the exact procedure applied before cannot be applied now because $a$ is not invertible. There are two ways to proceed. The brute force way is to do the substitutions you did in the previous homework. That has the effect of eliminating the troublesome singularity in the state-space form and converting the system into one in which $a$ is invertible after all.

But, this is an awkward approach in more complicated environments. An alternative strategy that works quite generally is based on the QZ decomposition. It also removes the singularity in a system, reduces its size, and produces a new system where the $a$ matrix is invertible. This approach is described in detail in the section, ‘the non-invertible $a$ case’ in Christiano, Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients.

The idea is to find the QZ decomposition of $a$ and $b$, i.e., orthonormal matrices $Q$ and $Z$ with the properties

$$QaZ = H_0, \quad QbZ = H_1,$$

where $H_0$ and $H_1$ are upper triangular matrices. All these matrices are $3 \times 3$. The matrix $H_0$ is structured so that the $l$ zeros on its diagonal are located in the lower right part of $H_0$ (you should determine what the value of $l$ is in the model of the question).\footnote{The QZ decomposition can be implemented on $a$ and $b$ with MATLAB routine QZ. A problem with MATLAB’s routine is that it does not order $H_0$ so that the zeros on} Denote the upper
\((3 - l) \times (3 - l)\) block of \(H_0\) by \(G_0\). This matrix must be non-singular. Let the corresponding upper left \((3 - l) \times (3 - l)\) block in \(H_1\) be denoted \(G_1\). I assume that the diagonal terms in the lower right \(l \times l\) block of \(H_1\) are non-zero. Also, it is useful to partition \(Z'\) as follows:

\[
Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},
\]

where \(L_1\) is \((3 - l) \times 3\) and \(L_2\) is \(l \times 3\).

Inserting \(ZZ'\) (= \(I\)) before \(Y_{t+1}\) and \(Y_t\) in (1), defining \(\gamma_t \equiv Z'Y_t\), and pre-multiplying (1) by \(Q\), (1) becomes:

\[
H_0\gamma_{t+1} + H_1\gamma_t = 0, \ t = 0, 1, \ldots.
\] (1)

Partition \(\gamma_t\) as follows:

\[
\gamma_t = \begin{pmatrix} \gamma_1^t \\ \gamma_2^t \end{pmatrix},
\] (2)

where \(\gamma_1^t\) is \((3 - l) \times 1\) and \(\gamma_2^t\) is \(l \times 1\). It is easy to verify that (1) implies \(\gamma_2^t = 0, \ t \geq 0\), i.e.,

\[
L_2Y_t = 0, \ t = 0, 1, \ldots.
\] (3)

With (3) imposed, the last \(l\) equations in (1) are redundant, so (1) can be written

\[
G_0\gamma_{t+1}^1 + G_1\gamma_1^t = 0, \ t = 0, 1, \ldots.
\] (4)

This system looks just like (1), except that its dimension is less than 6. Also, the analog of the \(a\) matrix, \(G_0\), is now nonsingular.

The set of solutions to the new system can be expressed as \(\gamma_1^t = (-G_0^{-1}G_1)^t\gamma_0^1, \ t \geq 0, \) or,

\[
P^{-1}\gamma_1^t = \Lambda P^{-1}\gamma_0^1,
\] (5)

the diagonal are located along the bottom right part of the diagonal. This can be done by following MATLAB’s \(QZ\) program with programs written by Chris Sims to do the reordering. The programs are \(QZDIV.M\) and \(QZSWITCH.M\). The latter is called by \(QZDIV.M\). Here is the way I implement the decomposition:

```java
[H0,H1,q,z,v]=qz(a,b);
stake1 = 1e+08;
stake2 = 1e-05;
[H0,H1,q,z] = qzdiv(stake1,H0,H1,q,z);
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where \( P \Lambda P^{-1} = -G_0^{-1}G_1 \) is the eigenvector, eigenvalue decomposition of \(-G_0^{-1}G_1\). The \( \gamma_1^t \) that solve (5) converge to zero asymptotically if, and only if, \( \tilde{p}\gamma_0^1 = 0 \), where \( \tilde{p} \) is composed of the rows of \( P^{-1} \) corresponding to diagonal terms in \( \Lambda \) that exceed 1 in absolute value. Taking into account the definition of \( \gamma_1^t \), this condition is:

\[
\tilde{p}L_1Y_0 = 0. \tag{6}
\]

The number of free elements in \( Y_0 \) is 2 (i.e., \( n_0 \) and \( k_1 \)). Equation (3) for \( t = 0 \) represents \( l \) (actually, \( l \) had better be unity in this example) restrictions on \( Y_0 \), so that to pin \( Y_0 \) down uniquely, \( 2-l \) more restrictions are needed. Thus, uniqueness of convergent paths requires that there be \( 2-l \) explosive eigenvalues in \( \Lambda \), i.e., that \( \tilde{p}L_1 \) contain \( 2-l \) rows. Then, define

\[
D = \begin{bmatrix} \tilde{p}L_1 \\ L_2 \end{bmatrix}. \tag{7}
\]

Use the condition, \( DY_t = 0 \), to express the minimal state solutions in the form \( \tilde{k}_{t+1} = d\tilde{k}_t, \tilde{n}_t = q\tilde{k}_t \). How many are there? How do they compare with what you got before, when you substituted out for hours worked?

3. Redo the analysis with a \( \gamma \) that produces multiplicity. Note how that manifests itself in the present system, using the QZ decomposition.