Banking and Financial Frictions in a Dynamic, General Equilibrium Model

Lawrence Christiano, Roberto Motto, and Massimo Rostagno

October 15, 2002

Contents

1. Introduction

We describe a dynamic, general equlibrium model with banking and financial frictions. We combine features from three papers. First, we incorporate the features of the real economy analyzed in Christiano, Eichenbaum and Evans (2001). We do this because of their finding that the features that they emphasize are helpful for matching empirical evidence on the monetary transmission mechanism. Second, we incorporate the neoclassical model of banking studied in Chari, Christiano and Eichenbaum (1995). We do this because we are interested in a model which allows us to study the demand for different monetary aggregates, such as demand deposits, time deposits, reserves and excess reserves. Third, we incorporate a version of the costly state verification (CSV) setup described in Bernanke, Gertler and Gilchrist (1999).[1] We do this because we wish to explore the role of asset prices in the propagation of shocks. Asset prices play a role in the propagation of shocks through their impact on net worth. Net worth matters in the determination of aggregate economic activity because it determines the amount of lending that banks do.

The model has a range of shocks. There are disturbances to the banking services and production technologies. There are also various types of velocity shocks to the transactions technology. These include shocks to household preferences for holding demand deposits versus currency. In addition, we have various shocks to money demand by firms. The presence of these shocks and the various banking frictions make the model suitable for evaluating the operating characteristics of alternative monetary policies presently under discussion. We think the analysis will be especially interesting because, although the various model features have been discussed before, now is the first time that they are being incorporated into a single framework. The analysis will be made particularly relevant for the Euro area because we plan to estimate model parameters using Euro-wide data.

Our model is sufficiently developed that it will allow us to go beyond the usual analysis of monetary policy rules. It provides a framework for studying major problems in the control of money that are reputed to have occurred in the past. A particularly famous example is the Friedman and Schwartz hypothesis about the severity of the US Great Depression in the 1930s. They argued much of the responsibility for this lays with the US Federal Reserve, which made a policy mistake when it targeted the monetary base rather than $M1$ in the face of a shift away from demand deposits and towards currency. Our model is sufficiently rich that it can be used to credibly address this hypothesis. We think that this episode, though it occurred long ago, continues to hold important lessons for monetary policy makers today.

On the empirical dimension, this project will confront some unique challenges and op-

[1] This work builds on Townsend (1979), Gale and Hellwig (1985), Williamson (1987). Other recent contributions to this literature include Fisher (1996) and Carlstrom and Fuerst (1997, 2000).

portunities. First, we do this project in the middle of a historic change in monetary policy. We cannot treat the past 25 years' of data as being drawn from a single regime. This fact must be accommodated in the econometric strategy that we use to estimate and test the model. One challenge is that in specifying the model we must take a stand on the nature of the sources of long-term growth. For example, we must decide whether the exogenous shocks are trend stationary or have a unit root. The literature reports that it is difficult to distinguish between these two specifications, even in data sets with a span as long at 50 years. Second, the model we plan to work on incorporates a working capital channel into the monetary transmission mechanism. That is, in addition to the usual demand channels, the model specifies that interest rate changes operate on the economy via a supply-side mechanism arising from firms' needs for working capital. Evidence from US data suggests that the amount of borrowing to finance short-term variable costs is quite high. To parameterize our model, we will have to gather the same evidence for the Euro area. This type of evidence and this channel for monetary policy has, up until recently, received relatively little attention.

The following section describes a benchmark model economy, which we will use at the start of our analysis. After that there is a very brief indication of the econometric analysis we plan to do. Once we have an empirically defensible model in hand, we plan to apply it to analyze monetary policy questions.

2. The Model Economy

In this section we describe our model economy and display the problems solved by intermediate and final good firms, entrepreneurs, producers of physical capital, banks and households. Final output is produced using the usual Dixit-Stiglitz aggregator of intermediate inputs. Intermediate inputs are produced by monopolists who set prices using a variant of the approach described in Calvo (1983). These firms use the services of capital and labor. We assume that a fraction of these variable costs ('working capital') must be financed in advance by banks. Capital services are supplied by entrepreneurs who own the physical capital and determine its rate of utilization. They finance their acquisition of physical capital partially using their own net worth and partially using the variant on the costly state verification (CSV) technology described in Bernanke, Gertler and Gilchrist (1999) (BGG). As is standard in the CSV literature with net worth, we need to make assumptions to guarantee that entrepreneurs do not accumulate enough net worth to make the CSV technology unnecessary. We accomplish this by assuming that a part of net worth is exogenously destroyed in each period. Physical capital is produced by firms which combine old capital and investment goods to produce new, installed, capital.

The model has banks which are the entities that make working capital loans to intermediate good firms and which provide the standard CSV debt contracts to entrepeneurs to help them finance their acquisition of new capital.

The timing of decisions during a period is important in the model. At the beginning of the period, the shocks to the various technologies in the model are realized. Then, wage, price, consumption, investment and capital utilization decisions are made. After this various financial market shocks are realized and the monetary action occurs. Finally, goods and asset markets meet and clear. See Figure 1 for reference.

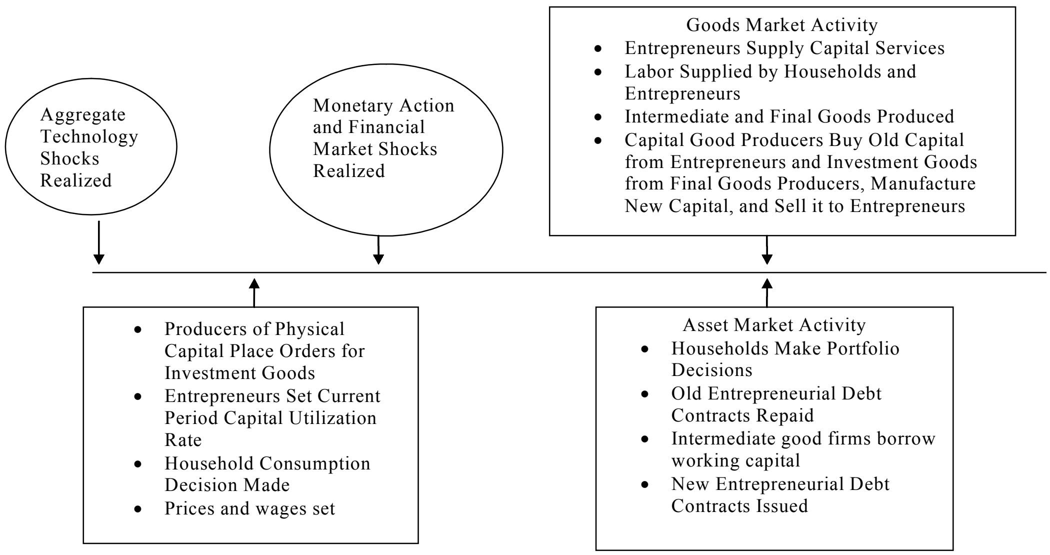


Figure 1: Timing in Model

2.1. Information

We divide up the shocks in the model into financial market shocks - money demand (by banks, households and firms) and monetary policy shocks - and non-financial market shocks (technology, government spending, preference for leisure, elasticities of demand for differentiated products and labor, etc.). The time t information set which includes period $t-s$, $s>0$, and period t observations on the non-financial shocks is denoted Ω_t. The information set which includes Ω_t plus the current period financial market shocks is denoted Ω_t^μ. Also,

$$\begin{aligned} E\left[X_t|\Omega_t\right] &= E_t X_t \\ E\left[X_t|\Omega_t^\mu\right] &= E_t^\mu X_t. \end{aligned}$$

2.2. Firm Sector

We describe three approaches to Calvo pricing here. They are differentiated according to how people set their prices when they don't have an opportunity to reoptimize. In the first case, we suppose they can't change their price at all. When steady state inflation is different from zero, this leads to a very different system of equations than the norm. After this, we review the standard Calvo equations.

We may in the future also want to consider the possibility that there is a costly state-verification setup for financing firm inputs. This has been explored in Carlstrom and Fuerst

(2000, Federal Reserve Bank of Cleveland Working Paper 0011). A potential advantage of this approach is that it may rationalize an 'efficiency wedge' in the resource constraint. This is suggested by the Carlstrom-Fuerst result in equation (13) of page 13. There are a couple of drawbacks to this approach, however. First, their impulse response functions suggest that the CSV setup may not have a big quantitative effect. For example, the efficiency wedge effects suggested by the results in figure 1 seem small. At the same time, it may be that for the size shocks we are interested in considering, perhaps the effects are quantitatively large. Second, by adopting a CSV approach we suspect we have to abandon Calvo pricing. Under the CSV approach, firms are told how much they can borrow and so there is no more discretion on how much to produce. But then, if firms set prices they will generically be off their demand curve. This may introduce complicated non-linearities, depending on how we handled disequilibrium phenomena. Third, by abandoning the Calvo model we lose a potential efficiency wedge. This wedge does not show up with first order approximations (although, for an exception, see the first subsection below), but it may be quantitatively large if we consider second order approximations and shocks that are big enough. Fourth, by abandoning the Calvo setup, we also lose heterogeneity among firms, and then we lose the demand elasticity parameter, λ_f. Shocks in this parameter represent direct shocks to the price level (see below), and shocks like this will be useful for our analysis.

We adopt a standard Dixit-Stiglitz formulation. A homogeneous final good is produced by a representative, competitive firm using a linear homogeneous production technology that uses a continuum of differentiated intermediate goods as inputs. Each intermediate good is produced by a monopolist who sets prices using two variants of the approach described in Calvo (1983). The two variants are distinguished according to the stand they take how firms set prices when they cannot reoptimize. In the version where they simply follow the previous period's aggregate inflation rate, the reduced form for inflation is:

$$(**)\ \hat{\pi}_t = \frac{1}{1+\beta}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta}E_t\hat{\pi}_{t+1} + \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta)\,\xi_p}\left[E_t\left(\hat{s}_t\right) + \hat{\lambda}_{f,t}\right]. \tag{2.1}$$

Here, a hat over a variable indicates percent deviation from steady state. Also, s_t denotes real marginal cost of production for intermediate good firms and $\pi_t = P_t/P_{t-1}$ is the inflation rate in the price, P_t, of the final good. In addition, ξ_p is the fraction of firms that cannot reoptimize their price in a given period. Also, $\beta \in (0,1)$ is the household's discount rate. Finally, E_t indicates expectation, conditional on the date t nonfinancial market shocks only (not on the financial market shocks). This timing reflects our assumption that period t intermediate good prices are set at after the realization of the financial market shocks, but before the realization of the financial market shocks (i.e., the money demand and money supply shocks).

Note that a shock, $\hat{\lambda}_{f,t}$, has been added to λ_f. That it ends up this way, symmetric with

$\hat{s}_t$, is simple to explain. There are three ways for $\hat{\lambda}_{f,t}$ to show up in (2.1). To see this, consider first the version of (C.1) that applies to the present setup. There, λ_f shows up in two ways. One is symmetric with s_t. The other is as a power. The power term can in fact be ignored because, in the version of (C.1) that is of interest here, the object in square brackets is zero in steady state. So, of the two channels for $\hat{\lambda}_{f,t}$ to show up in (2.1), only the one where it is symmetric with $\hat{s}_t$ is operative. The third channel for $\hat{\lambda}_{f,t}$ to show up in (2.1) operates in principle via (C.2), which provides a mapping from inflation to $\tilde{p}_t$. Note that λ_f shows up as a power there. Perturbations in λ_f have no impact there, because, in the version of (C.2) relevant to the current situation, the derivative of (C.2) with respect to λ_f involves a log, and that log is evaluated at unity in steady state. That is, it is evaluated at the steady state value of $\tilde{p}_t$ and π_{t-1}/π_t, both of which are unity in steady state.

The alternative formulation of Calvo pricing assumes that firms who cannot reoptimize their price follow the steady state inflation rate. This leads to the following reduced form:

$$(**)\ \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}\left[E_t\left(\hat{s}_t\right) + \hat{\lambda}_{f,t}\right]. \tag{2.2}$$

The argument for why it is that shocks to λ_f show up in the way indicated in (2.2) is the same as the argument justifying (2.1).

We assume that intermediate good firms are commited to supply whatever demand occurs at the price that they set. Once prices have been set, and after the realization of current period uncertainty, they decide how to produce the required level of output at minimum cost. The production function of the j^{th} intermediate good firm is:

$$Y_{jt} = \begin{cases} \epsilon_t K_{jt}^{\alpha}\left(z_t l_{jt}\right)^{1-\alpha} - \Phi z_t & \text{if } \epsilon_t K_{jt}^{\alpha}\left(z_t l_{jt}\right)^{1-\alpha} > \Phi z_t \\ 0, & \text{otherwise} \end{cases},\ 0 < \alpha < 1,$$

where Φ is a fixed cost and K_{jt} and l_{jt} denote the services of capital and labor. The variable, z_t, is a shock to technology, which has a covariance stationary growth rate. The variable, ϵ_t, is a stationary shock to technology. The time series representations for z_t and ϵ_t are discussed below. Firms are competive in factor markets, where they confront a rental rate, Pr_t^k, on capital services and a wage rate, W_t, on labor services. Each of these is expressed in units of money. Also, each firm must finance a fraction, $\psi_{k,t}$, of its capital services expenses in advance. Similarly, it must finance a fraction, $\psi_{l,t}$, of its labor services in advance. The interest rate it faces is R_t. Working capital includes the wage bill, $W_t l_{jt}$, and the rent on capital services, $P_t r_t^k K_t$. As a result, the marginal cost - after dividing by P_t - of producing one unit of Y_{jt} is:

$$s_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{\left(r_t^k\left[1+\psi_{k,t}R_t\right]\right)^{\alpha}\left(w_t\left[1+\psi_{l,t}R_t\right]\right)^{1-\alpha}}{\epsilon_t}, \tag{2.3}$$

where

$$w_t = \frac{W_t}{z_t P_t}.$$

Linearizing this about steady state:

$$\begin{aligned}
(**)\ \hat{s}_t &= \alpha \left[\hat{r}_t^k + (1 + \widehat{\psi_{k,t} R_t}) \right] + (1-\alpha) \left[\hat{w}_t + (1 + \widehat{\psi_{l,t} R_t}) \right] - \hat{\epsilon}_t \\
&= \alpha \left[\hat{r}_t^k + \frac{\psi_k R \left(\hat{\psi}_{k,t} + \hat{R}_t \right)}{1 + \psi_k R} \right] + (1-\alpha) \left[\hat{w}_t + \frac{\psi_l R \left(\hat{\psi}_{l,t} + \hat{R}_t \right)}{1 + \psi_l R} \right] - \hat{\epsilon}_t \\
&= \alpha \hat{r}_t^k + \alpha \frac{\psi_k R}{1 + \psi_k R} \left(\hat{\psi}_{k,t} + \hat{R}_t \right) + (1-\alpha) \hat{w}_t + (1-\alpha) \frac{\psi_l R}{1 + \psi_l R} \left(\hat{\psi}_{l,t} + \hat{R}_t \right) - \hat{\epsilon}_t \\
&= \alpha \hat{r}_t^k + \alpha \frac{\psi_k R}{1 + \psi_k R} \hat{\psi}_{k,t} + (1-\alpha) \hat{w}_t + (1-\alpha) \frac{\psi_l R}{1 + \psi_l R} \hat{\psi}_{l,t} \\
&\quad + \left[\frac{\alpha \psi_k R}{1 + \psi_k R} + \frac{(1-\alpha) \psi_l R}{1 + \psi_l R} \right] \hat{R}_t - \hat{\epsilon}_t
\end{aligned}$$

Marginal cost must also satisfy another condition:

$$s_t = \frac{r_t^k \left[1 + \psi_{k,t} R_t \right]}{\alpha \epsilon_t \left(\frac{z_t l_{jt}}{K_{jt}} \right)^{1-\alpha}} = \frac{r_t^k \left[1 + \psi_{k,t} R_t \right]}{\alpha \epsilon_t \left(\frac{z_t \nu^l l_t}{\nu^k K_t} \right)^{1-\alpha}} = \frac{r_t^k \left[1 + \psi_{k,t} R_t \right]}{\alpha \epsilon_t \left(\frac{\mu_{z,t} l_t}{k_t} \right)^{1-\alpha}}, \tag{2.4}$$

where ν^l and ν^k are, respectively, the share of aggregate labor in the intermediate good sector and the share of aggregate capital in the intermediate good sector. We have imposed here that $\nu^l = \nu^k$ since the production function in the firm sector is the same as the (value-added) production function in the banking sector. Also, l_t and K_t are the unweighted integrals of employment and capital services hired by individual intermediate good producers. Then,

$$\begin{aligned}
(**)\ \hat{s}_t &= \hat{r}_t^k + [1 + \widehat{\psi_{k,t} R_t}] - \hat{\epsilon}_t - (1-\alpha) \left(\hat{\mu}_{z,t} + \hat{l}_t - \hat{k}_t \right) \\
&= \hat{r}_t^k + \frac{\psi_k R \left(\hat{\psi}_{k,t} + \hat{R}_t \right)}{1 + \psi_k R} - \hat{\epsilon}_t - (1-\alpha) \left(\hat{\mu}_{z,t} + \hat{l}_t - \hat{k}_t \right)
\end{aligned}$$

Final output is produced according to the following production function, by the representative final good firm:

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{1}{\lambda_f}} dj \right]^{\lambda_f}.$$

Total labor and capital services used by the intermediate good firms is:

$$K_t^f = \int_0^1 K_{jt} dj, \; l_t = \int_0^1 l_{jt} dj.$$

2.3. Capital Producers

There is a large, fixed, number of identical capital producers, who take prices as given. They are owned by households and any profits or losses are transmitted in a lump-sum fashion to households. The capital producer must commit to a level of investment, I_t, before the period t realization of the monetary policy shock and after the period t realization of the other shocks. Investment goods are actually purchased in the goods market which meets after the monetary policy shock. The price of investment goods in that market is P_t, and this is a function of the realization of the monetary policy shock. The capital producer also purchases old capital in the amount, x, at the time the goods market meets. Old capital and investment goods are combined to produce new capital, x', using the following technology:

$$x' = x + F(I_t, I_{t-1}),$$

where the presence of lagged investment reflects that there are costs to changing the flow of investment. We denote the price of new capital by $Q_{\bar{K}',t}$, and this is a function of the realized value of the monetary policy shock. Since the marginal rate of transformation from old capital into new capital is unity, the price of old capital is also $Q_{\bar{K}',t}$. The firm's time t profits, after the realization of the monetary policy shock are:

$$\Pi_t^k = Q_{\bar{K}',t} \left[x + F(I_t, I_{t-1}) \right] - Q_{\bar{K}',t} x - P_t I_t.$$

This expression for profits is a function of the realization of the period t monetary policy shock, because $Q_{\bar{K}',t}$, x, and P_t are. Since the choice of I_t influences profits in period $t+1$, the firm must incorporate that into the objective as well. But, that term involves I_{t+1} and x_{t+1}. So, state contingent choices for those variables must be made for the firm to be able to select I_t and x_t. Evidently, the problem choosing x_t and I_t expands into the problem of solving an infinite horizon optimization problem:

$$\max_{\{I_{t+j}, x_{t+j}\}} E \left\{ \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \left(Q_{\bar{K}',t+j} \left[x_{t+j} + F(I_{t+j}, I_{t+j-1}) \right] - Q_{\bar{K}',t+j} x_{t+j} - P_{t+j} I_{t+j} \right) | \Omega_t \right\},$$

where it is understood that I_{t+j} is a function of all shocks up to period $t+j$ except the $t+j$ financial market shocks and x_{t+j} is a function of all the shocks up to period $t+j$. Also, Ω_t

includes all shocks up to period t, except the period t financial market shocks. These are composed of shocks to monetary policy and to money demand.

From this problem it is evident that any value of x_{t+j} whatsoever is profit maximizing. Thus, setting $x_{t+j} = (1-\delta)\bar{K}_{t+j}$ is consistent with both profit maximization by firms and with market clearing.

The first order necessary condition for maximization of I_t is:

$$E\left[\lambda_t P_t q_t F_{1,t} - \lambda_t P_t + \beta \lambda_{t+1} P_{t+1} q_{t+1} F_{2,t+1} | \Omega_t\right] = 0,$$

where q_t is Tobin's q :

$$q_t = \frac{Q_{\bar{K},t}}{P_t}.$$

Multiply by z_t :

$$E\left[\lambda_{zt} q_t F_{1,t} - \lambda_{z,t} + \frac{\beta}{\mu_{z,t+1}} \lambda_{z,t+1} q_{t+1} F_{2,t+1} | \Omega_t\right] = 0. \tag{2.5}$$

We have that:[2]

$$F(I_t, I_{t-1}) = \left[1 - S(I_t/I_{t-1})\right] I_t$$

As a result:

$$F_1(I_t, I_{t-1}) = -S'(I_t/I_{t-1}) I_t/I_{t-1} + 1 - S(I_t/I_{t-1}),$$

or, after scaling variables[3],

$$F_1(I_t, I_{t-1}) = -S'(\frac{i_t \mu_{z,t}}{i_{t-1}}) \frac{i_t \mu_{z,t}}{i_{t-1}} + 1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}}) = f_t^1,$$

say. Totally differentiating:

$$f^1 \hat{f}_t^1 = -S''(\frac{i_t \mu_{z,t}}{i_{t-1}}) \frac{i_t \mu_{z,t}}{i_{t-1}} \left[\frac{i \mu_{z,t}}{i_{t-1}} \hat{\imath}_t + \frac{i_t \mu_z}{i_{t-1}} \hat{\mu}_{z,t} - \frac{i_t \mu_{z,t}}{i_{t-1}^2} i \hat{\imath}_{t-1}\right].$$

---

[2] The following function, S, satisfies $S = S' = 0$ in steady state, and S'' in steady state is a given parameter:

$$S(x) = \exp\left[a(x - x^*)\right] + \exp\left[-a(x - x^*)\right] - 2,$$

with $a = \sqrt{S''/2}$, and where x^* is the steady state value of x.

[3] Using the adjustment cost function of the previous footnote,

$$S(\frac{I}{I_{-1}}) = \exp\left[a(\frac{I}{I_{-1}} - \mu_z)\right] + \exp\left[-a(\frac{I}{I_{-1}} - \mu_z)\right] - 2,$$

we have, using $F(I, I_{-1}) = [1 - S(I/I_{-1})]I$, Here, we have anticipated that when the derivatives are evaluated in steady state, the other terms in the total derivative disappear because $S = S' = 0$. Evaluating in steady state:

$$f^1 \hat{f}_t^1 = -S'' \mu_z^2 \left[\hat{\imath}_t + \hat{\mu}_{z,t} - \hat{\imath}_{t-1} \right], \ f^1 = 1.$$

Now consider the other derivative:

$$F_2(I_{t+1}, I_t) = S'(\frac{i_{t+1}\mu_{z,t+1}}{i_t}) \left(\frac{i_{t+1}\mu_{z,t+1}}{i_t} \right)^2 = f_{t+1}^2,$$

say. Totally differentiating:

$$f^2 \hat{f}_{t+1}^2 = S''(\frac{i_{t+1}\mu_{z,t+1}}{i_t}) \left(\frac{i_{t+1}\mu_{z,t+1}}{i_t} \right)^2 \left[\frac{i\mu_{z,t+1}}{i_t} \hat{\imath}_{t+1} + \frac{i_{t+1}\mu_z}{i_t} \hat{\mu}_{z,t+1} - \frac{i_{t+1}\mu_{z,t+1}}{i_t^2} i \hat{\imath}_t \right].$$

As before, there is no need to include the rest in this derivative, because it disappears when we evaluate it in steady state due to our specification, $S' = 0$. Evaluating in steady state: (corrected)

$$f^2 \hat{f}_{t+1}^2 = S'' \mu_z^3 \left[\hat{\imath}_{t+1} + \hat{\mu}_{z,t+1} - \hat{\imath}_t \right].$$

With these results in hand, we proceed now to totally differentiate the object in braces in (2.5). Rewriting it first:

$$\lambda_{zt} q_t f_t^1 - \lambda_{zt} + \frac{\beta}{\mu_{z,t+1}} \lambda_{zt+1} q_{t+1} f_{t+1}^2$$

Totally differentiating:

$$\lambda_z q f^1 \left[\hat{\lambda}_{zt} + \hat{q}_t + \hat{f}_t^1 \right] - \lambda_z \hat{\lambda}_{zt} + \frac{\beta}{\mu_z} \lambda_z q \left[f^2 \hat{\lambda}_{zt+1} + f^2 \hat{q}_{t+1} + f^2 \hat{f}_{t+1}^2 - f^2 \hat{\mu}_{z,t+1} \right]$$

or, taking into account $\hat{f}^2 = 0$ and the results derived for $\hat{f}_t^1$, $f^2 \hat{f}_{t+1}^2$:

$$\lambda_z \left[\hat{\lambda}_{zt} + \hat{q}_t - S'' \mu_z^2 \left(\hat{\imath}_t + \hat{\mu}_{z,t} - \hat{\imath}_{t-1} \right) \right] - \lambda_z \hat{\lambda}_{zt} + \frac{\beta}{\mu_z} \lambda_z q S'' \mu_z^3 \left[\hat{\imath}_{t+1} + \hat{\mu}_{z,t+1} - \hat{\imath}_t \right]$$

Now, divide by λ_z

$$\hat{\lambda}_{zt} + \hat{q}_t - S'' \mu_z^2 \left(\hat{\imath}_t + \hat{\mu}_{z,t} - \hat{\imath}_{t-1} \right) - \hat{\lambda}_{zt} + \frac{\beta}{\mu_z} q S'' \mu_z^3 \left[\hat{\imath}_{t+1} + \hat{\mu}_{z,t+1} - \hat{\imath}_t \right],$$

or,

$$(**) \ E \left\{ \hat{q}_t - S'' \mu_z^2 (1+\beta) \hat{\imath}_t - S'' \mu_z^2 \hat{\mu}_{z,t} + S'' \mu_z^2 \hat{\imath}_{t-1} + \beta S'' \mu_z^2 \hat{\imath}_{t+1} + \beta S'' \mu_z^2 \hat{\mu}_{z,t+1} | \Omega_t \right\} = 0$$

The coefficients in the canonical form are:

$$\begin{aligned}
\alpha_1(4,9) &= 1 \\
\alpha_1(4,4) &= -S''\mu_z^2(1+\beta) \\
\alpha_2(4,4) &= S''\mu_z^2 \\
\alpha_0(4,4) &= \beta S''\mu_z^2 \\
\beta_0(4,46) &= \beta S''\mu_z^2 \\
\beta_1(4,46) &= -S''\mu_z^2
\end{aligned}$$

We need an equation linking investment and the capital stock:

$$\bar{K}_{t+1} = (1-\delta)\bar{K}_t + \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t,$$

or, after taking into account $\bar{K}_{t+1} = z_t \bar{k}_{t+1}$ and the scaling of I_t :

$$z_t\bar{k}_{t+1} = (1-\delta)z_{t-1}\bar{k}_t + \left[1 - S\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)\right] i_t z_t.$$

Divide both sides by z_t :

$$\bar{k}_{t+1} = (1-\delta)\frac{1}{\mu_{z,t}}\bar{k}_t + \left[1 - S\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)\right] i_t.$$

Now, expand this:

$$\begin{aligned}
\bar{k}\widehat{\bar{k}}_{t+1} &= (1-\delta)\frac{1}{\mu_z}\bar{k}\left(\widehat{\bar{k}}_t - \hat{\mu}_{z,t}\right) - S'\mu_z\left[\hat{\imath}_t + \hat{\mu}_{z,t} - \hat{\imath}_{t-1}\right] + [1-S]\, i\hat{\imath}_t \\
&= (1-\delta)\frac{1}{\mu_z}\bar{k}\left(\widehat{\bar{k}}_t - \hat{\mu}_{z,t}\right) + i\hat{\imath}_t,
\end{aligned}$$

since $S' = S = 0$. Dividing by $\bar{k}$:

$$\widehat{\bar{k}}_{t+1} = \frac{1-\delta}{\mu_z}\left(\widehat{\bar{k}}_t - \hat{\mu}_{z,t}\right) + \frac{i}{\bar{k}}\hat{\imath}_t.$$

2.4. Entrepreneurs

There is a large population of entrepreneurs. Consider the j^{th} entrepreneur (see Figure 2). During the period t goods market, the j^{th} entrepreneur accumulates net worth, N_{t+1}^j. This abstract purchasing power, which is denominated in units of money, is determined as follows. The sources of funds are the rent earned as a consequence of supplying capital services to the period t capital rental market, the sales proceeds from selling the undepreciated component of the physical stock of capital to capital goods producers. The uses of funds include repayment on debt incurred on loans in period $t-1$ and expenses for capital utilization. Net worth is composed of these sources minus these uses of funds.

At this point, $1-\gamma$ entrepreneurs die and γ survive to live another day. The newly produced stock of physical capital is purchased by the γ entrepreneurs who survive and $1-\gamma$ newly-born entrepreneurs. The surviving entrepreneurs finance their purchases with their net worth and loans from the bank. The newly-born entrepreneurs finance their purchases with a transfer payment received from the government and a loan from the bank. We actually allow γ to be a random variable, but we delete the time subscript here to keep from cluttering the notation too much.

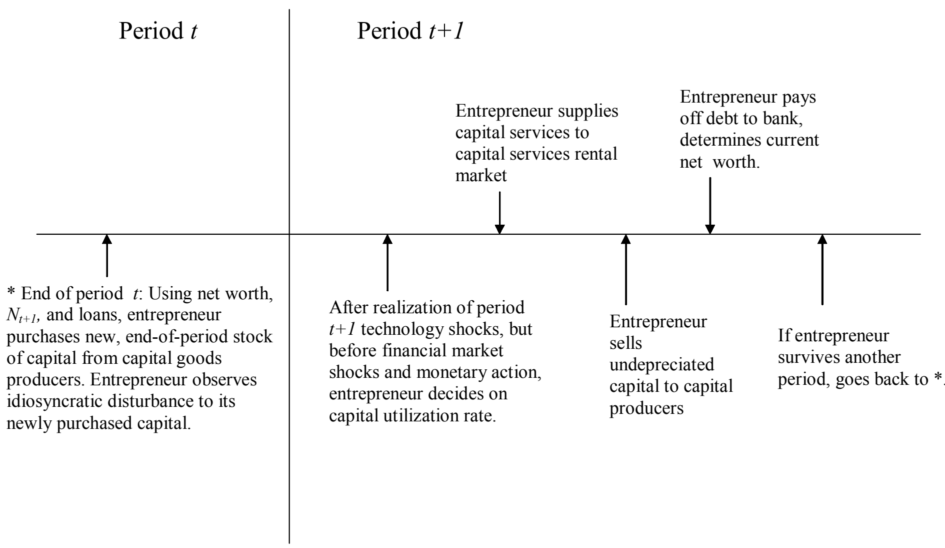
The j^{th} entrepreneur who purchases capital, $\bar{K}_{t+1}^j$, from the capital goods producers at the price, $Q_{\bar{K}',t}$ in period t experiences an idiosyncratic shock to the size of his purchase. Just after the purchase, the size of capital changes from $\bar{K}_{t+1}^j$ to $\omega\bar{K}_{t+1}^j$. Here, ω is a unit mean, non-negative random variable distributed independently across entrepreneurs. After observing the realization of the non-financial market shocks, but before observing the financial market shock, the j^{th} entrepreneur decides on the level of capital utilitzation in period $t+1$, and then rents capital services. At the end of the period $t+1$ goods market, the entrepreneur sells its undepreciated capital. At this point, the entrepeneur's net worth, N_{t+2}^j, is the rent earned in period $t+1$, minus the utilization costs on capital, minus debt repayment, plus the proceeds of the sale of the undepreciated capital, $(1-\delta)\omega\bar{K}_{t+1}^j$. As indicated above, the entrepreneur then proceeds to die with probability $1-\gamma$, and to survive to live another day with the complementary probability, γ.

The $1-\gamma$ entrepreneurs who are born and the γ who survive receive a subsidy, W_t^e. There is a technical reason for this. The standard debt contract in the entrepreneurial loan market has the property that entrepreneurs with no net worth receive no loans. If new-born entrepreneurs received no transfers, they would have no net worth and would therefore not be able to purchase any capital. In effect, without the transfer they could not enter the population of entrepeneurs. Regarding the surviving entrepreneurs, in each period a fraction loses everything, and they would have no net worth in the absence of a transfer. Absent a transfer, these entrepreneurs would in effect leave the population of entrepreneurs. Absent transfers, the population of entrepreneurs would be empty. The transfers are designed to avoid this. They are financed by a lump sum tax on households.

Entrepeneurial death in the model is a device to ensure that net worth does not grow to the point where the CSV setup becomes redundant. Presumably, this corresponds to the real-world observation that enormous concentrations of wealth, for various reasons, do not survive for long.

We need to allocate the net worth of the entrepreneurs who die. We assume that a fraction, Θ, of a dead entrepreneur's net worth is used to finance the purchase of C_t^e of final output. The complementary fraction is redistributed as a lump-sum transfer to the household. In practice, Θ will be small or zero.

FIGURE 2: A Day in the Life of an Entrepreneur



2.4.1. The Production Technology of the Entrepreneur

We now go into the details of the entrepreneur's situation. The j^{th} entrepreneur produces capital services, K_{t+1}^j, from physical capital using using the following technology:

$$K_{t+1}^j = u_{t+1}^j \omega \bar{K}_{t+1}^j,$$

where u_{t+1}^j denotes the capital utilization rate chosen by the j^{th} entrepreneur. Here, ω is drawn from a distribution with mean unity and distribution function, F :

$$\Pr[\omega \leq x] = F(x).$$

Each entrepreneur draws independently from this distribution immediately after $\bar{K}_{t+1}^j$ has been purchased. Capital services are supplied to the capital services market in period $t+1$, where they earn the rental rate, r_{t+1}^k.

The capital utilization rate chosen by the j^{th} entrepreneur, u_{t+1}^{j}, must be chosen before period $t+1$ financial market shocks, and after the other period $t+1$ shocks. Higher rates of utilization are associated with higher costs as follows:

$$P_{t+1}a(u_{t+1}^{j})\omega\bar{K}_{t+1}^{j},\ a',a''>0.$$

As in BGG, we suppose that the entrepreneur is risk neutral. As a result, the j^{th} entrepreneur chooses u_{t+1}^{j} to solve:

$$\max_{u_{t+1}^{j}} E\left\{\left[u_{t+1}^{j}r_{t+1}^{k}-a(u_{t+1}^{j})\right]\omega\bar{K}_{t+1}^{j}P_{t+1}|\Omega_{t+1}\right\}.$$

The first order necessary condition for optimization is:

$$E_t\left[r_t^k-a'(u_t)\right]=0.$$

This reflects that $\bar{K}_{t+1}^{j}$ P_{t+1} are contained in Ω_{t+1}. That P_{t+1} is in Ω_{t+1} is due to our assumption that prices are set before the realization of the financial market shocks. Totally differentiating the expression inside the conditional expectation:

$$r^k\hat{r}_t^k-a''u\hat{u}_t,$$

and evaluating this in steady state when $r^k=a'$

$$\hat{r}_t^k-\frac{a''}{a'}\hat{u}_t.$$

Putting this back into the expectation operator, and letting $\sigma_a=a''/a'$:[4]

$$(**)\ E_t\left[\hat{r}_t^k-\sigma_a\hat{u}_t\right]=0.$$

After the capital has been rented in period $t+1$, the j^{th} entrepreneur sells the undepreciated part, $(1-\delta)\omega\bar{K}_{t+1}^{j}$, to the capital goods producer.

Below we introduce taxation on capital income. This does not enter into the above first order condition because capital income taxation affects rental income and the cost of utilization symmetrically. In addition, the capital income tax rate that applies to the utilization rate at time $t+1$ is contained in the information set, Ω_{t+1}.

[4]An a function that has the properties that we use is:

$$a(u)=\frac{r^k}{\sigma_a}\left[\exp(\sigma_a\left[u-1\right])-1\right].$$

Note, $a(1)=0$, $a'(u)=r^k\exp(\sigma_a\left[u-1\right])=r^k$ when $u=1$. Also, $a''(u)=\sigma_a r^k\exp(\sigma_a\left[u-1\right])=\sigma_a r^k$, when $u=1$. Then, $a''/a'=\sigma_a$. Here, r^k is the steady state value of the rental rate of capital.

2.4.2. Taxation of Capital Income

We now discuss the taxation of capital income. It is convenient to begin the discussion by considering, as a benchmark, an approach to the taxation of capital which parallels the treatment of taxes on interest-bearing securities. This is convenient because it allows us to compare the differential impact on entrepreneurial capital accumulation and on household asset accumulation of inflation. To simplify, suppose for the moment that the return on capital is simply r^k, the real price of capital in period t is q_t and the price of a consumption good in period t is P_t. We temporarily abstract from variable capital utilization and idiosyncratic variation across entrepreneurs. The pretax nominal rate of return on capital from t to $t+1$ is

$$1 + R^k_{t+1} = \frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t}\frac{P_{t+1}}{P_t}.$$

Suppose that capital income is taxed in such a way that the after tax return is $1+(1-\tau)R^k_{t+1}$:

$$1 + (1-\tau)R^k_{t+1} = 1 + \left[\frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t}\frac{P_{t+1}}{P_t} - 1\right](1-\tau).$$

The real after tax return, then, is:

$$\left[1 + (1-\tau)R^k_{t+1}\right]\frac{P_t}{P_{t+1}} = \frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t}(1-\tau) + \frac{P_t}{P_{t+1}}\tau.$$

So, if the pre-tax real rate of return on capital, $\left[r^k_{t+1} + (1-\delta)q_{t+1}\right]/q_t$, were invariant to inflation, then the after tax rate of return obviously would not be invariant.

The 'normal' way of treating capital income taxes is the following:

$$1 + R^k_{\tau,t+1} = \frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t}\frac{P_{t+1}}{P_t} - \frac{\tau_{t+1}(r^k_{t+1}P_{t+1} - \delta q_t P_t)}{q_t P_t}.$$

Note that we value depreciated capital at its historic cost. Then, the after tax gross real return is:

$$\begin{aligned}\left(1 + R^k_{\tau,t+1}\right)\frac{P_t}{P_{t+1}} &= \frac{r^k_{t+1} + (1-\delta)q_{t+1}}{q_t} - \frac{\tau_{t+1}(r^k_{t+1}P_{t+1} - \delta q_t P_t)}{q_t P_t}\frac{P_t}{P_{t+1}} \\ &= \frac{(1-\tau^k_{t+1})r^k_{t+1} + (1-\delta)q_{t+1}}{q_t} + \tau_{t+1}\delta\frac{P_t}{P_{t+1}}\end{aligned}$$

Here too, the after tax rate of return is decreasing in inflation. This is because of the way depreciation is treated. If instead capital could be depreciated at current market cost, then, we would have

$$\left(1+R^k_{\tau,t+1}\right)\frac{P_t}{P_{t+1}} = \frac{(1-\tau^k_{t+1})r^k_{t+1}+(1-\delta)q_{t+1}}{q_t}+\tau_{t+1}\delta\frac{q_{t+1}}{q_t},$$

and the after tax real return would be invariant to the rate of inflation (as long as the pre-trax return were invariant).

We now turn back to our model, with variable capital utilization and idiosyncratically different returns for different entrepreneurs. Following is the after tax rate of return on capital, when the capital tax rate is τ^k_t and all the details of our model are included, for the entrepreneur with productivity, ω. The expression assumes that depreciation occurs at historic cost:

$$\begin{aligned}
1+R^{k,\omega}_{t+1} &= \frac{\left[u_{t+1}\omega r^k_{t+1}-a(u_{t+1})\omega\right]P_{t+1}+(1-\delta)\omega Q_{\bar{K}',t+1}}{Q_{\bar{K}',t}} \\
&\quad -\frac{\tau^k_t\left[u_{t+1}\omega r^k_{t+1}-a(u_{t+1})\omega\right]P_{t+1}-\tau^k_t\delta Q_{\bar{K}',t}}{Q_{\bar{K}',t}} \\
&= \frac{u_{t+1}\omega r^k_{t+1}-a(u_{t+1})\omega+(1-\delta)\omega q_{t+1}-\tau^k_t\left[u_{t+1}\omega r^k_{t+1}-a(u_{t+1})\omega\right]}{q_t}\frac{P_{t+1}}{P_t}+\tau^k_t\delta \\
&= \frac{(1-\tau^k_t)\left[u_{t+1}r^k_{t+1}-a(u_{t+1})\right]+(1-\delta)q_{t+1}}{q_t}\frac{P_{t+1}}{P_t}\omega+\tau^k_t\delta \\
&= (1+\tilde{R}^k_{t+1})\omega+\tau^k_t\delta.
\end{aligned}$$

The latter expression allows the possibility that total tax payments are negative. This would occur when

$$\frac{\omega\left[u_{t+1}r^k_{t+1}-a(u_{t+1})\right]P_{t+1}-\delta Q_{\bar{K}',t}}{Q_{\bar{K}',t}}<0.$$

We could restrict tax payments to be non-negative, simply by setting those payments to zero whenever the above condition holds. This implies a critical value of ω, the one that sets the above to zero:

$$\omega^*_{t+1}=\frac{\delta Q_{\bar{K}',t}}{\left[u_{t+1}r^k_{t+1}-a(u_{t+1})\right]P_{t+1}}.$$

The appropriate formula for the rate of return now is, for the entrepreneur who receives productivity ω :

$$1+R^{k,\omega}_{t+1}=(1+\tilde{R}^k_{t+1})\omega+\tau^k_t\delta\times 1_{[\omega\geq\omega^*_{t+1}]},$$

where

$$1_{[\omega\geq\omega^*_{t+1}]}=\begin{cases}1 \text{ if } \omega\geq\omega^*_{t+1}\\ 0 \text{ if } \omega<\omega^*_{t+1}\end{cases}$$

A shortcoming of this specification is that when ω is low, the current owner loses part of the depreciation allowances. Capital changes hands every period, and so throughout its life it will periodically end up in the hands of someone with low income. As a result, capital is never fully depreciated in this way of setting things up. In principle, 'lost depreciation allowances' could be carried forward, but this would be very awkward in our environment, with capital changing hands. The model would have to include a way of carrying forward unclaimed depreciation allowances. There is another problem with this specification. It appears that this indicator function causes the cutoff productivity level in the CSV contract (discussed below) to become a discontinuous function of the state. This is completely inconsistent with the basic solution strategy we adopt. This makes this option essentially infeasible.

An alternative possibility is to work with the case where it's the capital that's available after ω is realized which can be depreciated, then the gross rate of return on capital is proportional to ω. In this case, the rate of return for an $\omega-$type entrepreneur in this case is:

$$\begin{aligned} 1+R_{t+1}^{k,\omega} &= \left\{\frac{(1-\tau_t^k)\left[u_{t+1}r_{t+1}^k - a(u_{t+1})\right] + (1-\delta)q_{t+1}}{q_t}\frac{P_{t+1}}{P_t} + \tau_t^k\delta\right\}\omega \\ &= (1+R_{t+1}^k)\omega. \end{aligned}$$

Note how the rate of return on $\bar{K}_{t+1}^{\omega}$ is a product of a rate of return, R_{t+1}^k, which is the same across all entrepreneurs and ω. The shortcoming of this specification is that you can't depreciate the full amount of the initial capital purchase, when ω is low. An interpretation of this is that it captures the notion that you lose depreciation allowances when your income is too low to deduct the full amount. It's an awkward way to capture this, but it has the advantage of being tractable.

Linearizing the previous measure of the rate of return on capital, which we rewrite here:

$$R_{t+1}^k = \frac{(1-\tau_t^k)\left[u_{t+1}r_{t+1}^k - a(u_{t+1})\right] + (1-\delta)q_{t+1}}{q_t}\pi_{t+1} + \tau_t^k\delta - 1$$

Linearizing:

$$\begin{aligned} (**)\ \hat{R}_{t+1}^k &= \frac{(1-\tau^k)r^k + (1-\delta)q}{R^k q}\pi\left[\widehat{(1-\tau_t^k)\left[u_{t+1}r_{t+1}^k - a(u_{t+1})\right] + (1-\delta)q_{t+1}} + \hat{\pi}_{t+1} - \hat{q}_t\right] + \frac{\delta\tau^k\hat{\tau}_t^k}{R^k} \\ &= \frac{(1-\tau^k)r^k + (1-\delta)q}{R^k q}\pi\left[\frac{\left(1-\tau^k\right)r^k\hat{r}_{t+1}^k - \tau^k r^k\hat{\tau}_t^k + (1-\delta)q\hat{q}_{t+1}}{(1-\tau^k)r^k + (1-\delta)q} + \hat{\pi}_{t+1} - \hat{q}_t\right] + \frac{\delta\tau^k\hat{\tau}_t^k}{R^k} \end{aligned}$$

2.4.3. The Financing Arrangement for the Entrepreneur

How is the j^{th} entrepreneur's level of capital, $\bar{K}_{t+1}^j$, determined? At the moment the entrepreneur enters the loan market, it's state variable is its net worth. It is has nothing else. It owns no capital, for example. Apart from net worth, no other aspect of the entrepreneur's history is relevant at this point.

There are many entrepreneurs, all with different amounts of net worth. We imagine that corresponding to each possible value of net worth, there are many entrepreneurs. They participate in a competitive loan market with banks. That is, there is a competitive loan market corresponding to each different level of net worth, N_{t+1}. In the usual CSV way, the contracts traded in the loan market specify an interest rate and a loan amount. The contracts are competitively determined. This means that they must satisfy a zero profit condition on banks and they must be utility maximizing for entrepreneurs. Equilibrium is incompatible with positive profits because of free entry and incompatible with negative profits because of free exit. In addition, contracts must be utility maximizing (subject to zero profits) for entrepreneurs because of competition. Equilibrium is incompatible with contracts that fail to do so, because in any candidate equilibrium like this, an individual bank could offer a better contract, one that makes positive profits, and take over the market.

The CSV contracts that we study are known to be optimal when there is no aggregate uncertainty. However, the way we have set up our environment, there is such uncertainty. We do this in part because we are interested exploring phenomena like the 'debt deflation hypothesis' discussed by Irving Fisher. We interpret this hypothesis as corresponding to a situation in which a shock (in this case, to the price level) occurs after entrepreneurs have borrowed from banks, but before they have paid back what they owe. A problem with what we do is that the contract we study is not known to be the optimal one. However, we share BGG's conjecture that in fact the contract is optimal, at least for sufficiently risk averse households. This is because the contract has the property that uncertainty associated with an aggregate shock is absorbed by entrepreneurs, while households receive a state-noncontingent rate of return on their loans to entrepeneurs (these loans actually are intermediated by banks). The reason this arrangement may not be optimal is as follows. We have not ruled out the possibility that there could be a return for households which is state contingent but compensates them for this, and which permits a CSV loan contract to entrepreneurs that increases their welfare.

We now discuss the contracts offered in equilibrium to entrepreneurs with level of net worth, N_{t+1}. Denote the level of capital purchases by such an entrepreneur by $\bar{K}_{t+1}^{N}$. To finance such a purchase an $N_{t+1}-$type entrepreneur must borrow

$$B_{t+1}^{N} = Q_{\bar{K}',t}\bar{K}_{t+1}^{N} - N_{t+1}. \tag{2.6}$$

The standard debt contract specifies a loan amount, B_{t+1}^{N}, and a gross rate of interest, Z_{t+1}^{N}, to be paid if ω is high enough that the entreprenuer can do so. Entrepreneurs who cannot pay this interest rate, because they have a low value of ω must give everything they have to the bank. The parameters of the $N_{t+1}-$type standard debt contract, B_{t+1}^{N} Z_{t+1}^{N}, imply a cutoff value of ω, $\bar{\omega}_{t+1}^{N}$, as follows:[5]

$$\bar{\omega}_{t+1}^{N}\left(1+R_{t+1}^{k}\right)Q_{\bar{K}',t}\bar{K}_{t+1}^{N}=Z_{t+1}^{N}B_{t+1}^{N}. \tag{2.7}$$

The amount of the loan, B_{t+1}^{N}, extended to an $N_{t+1}-$type entrepreneur is obviously not dependent on the realization of the period $t+1$ shocks. For reasons explained below, the interest rate on the loan, Z_{t+1}^{N}, is dependent on those shocks. Since R_{t+1}^{k} and Z_{t+1}^{N} are dependent on the period $t+1$ shocks, it follows from the previous expression that $\bar{\omega}_{t+1}^{N}$ is in principle also dependent upon those shocks.

For $\omega<\bar{\omega}_{t+1}^{N}$, the entrepreneur pays all its revenues to the bank:

$$\left(1+R_{t+1}^{k}\right)\omega Q_{\bar{K}',t}\bar{K}_{t+1}^{N},$$

which is less than $Z_{t+1}^{N}B_{t+1}^{N}$. In this case, the bank must monitor the enterpreneur, at cost

$$\mu\left(1+R_{t+1}^{k}\right)\omega Q_{\bar{K}',t}\bar{K}_{t+1}^{N}.$$

We now describe how the parameters, B_{t+1}^{N} and Z_{t+1}^{N}, of the standard debt contract that is offered in equilibrium to entrepreneurs with net worth N_{t+1} are chosen.

We suppose that banks have access to funds at the end of the period t goods market at a nominal rate of interest, R_{t+1}^{e}. This interest rate is contingent on all shocks realized in period t, and is not contingent on the realization of the idiosyncratic shocks to individual $N_{t+1}-$type entrepreneurs, and is also not contingent on the $t+1$ aggregate shocks. Banks obtain these funds for lending to entrepreneurs by issuing time deposits at the end of the goods market in period t, which is when the entrepreneurs need funds for the purchase of $\bar{K}_{t+1}^{N}$. Zero profits for banks implies:

$$\left[1-F\left(\bar{\omega}_{t+1}^{N}\right)\right]Z_{t+1}^{N}B_{t+1}^{N}+(1-\mu)\int_{0}^{\bar{\omega}_{t+1}^{N}}\omega dF(\omega)\left(1+R_{t+1}^{k}\right)Q_{\bar{K}',t}\bar{K}_{t+1}^{N}=\left(1+R_{t+1}^{e}\right)B_{t+1}^{N}, \tag{2.8}$$

or,

$$\left[1-F\left(\bar{\omega}_{t+1}^{N}\right)\right]\bar{\omega}_{t+1}^{N}+(1-\mu)\int_{0}^{\bar{\omega}_{t+1}^{N}}\omega dF(\omega)=\frac{1+R_{t+1}^{e}}{1+R_{t+1}^{k}}\frac{B_{t+1}^{N}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}}. \tag{2.9}$$

[5] With the alternative treatment of depreciation, this expression becomes:

$$\left(\left[1+\tilde{R}_{t+1}^{k}\right]\bar{\omega}_{t+1}^{N}+\tau_{t}^{k}\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^{N}=Z_{t+1}^{N}B_{t+1}^{N}.$$

BGG argue that, given a mild regularity condition on F, the expression on the left of the equality has an inverted U shape. There is some unique interior maximum, $\bar{\omega}^*$. It is increasing for $\bar{\omega}_{t+1}^N < \bar{\omega}^*$ and decreasing for $\bar{\omega}_{t+1}^N > \bar{\omega}^*$. Conditional on a given ratio, $B_{t+1}^N / \left(Q_{\bar{K}',t}\bar{K}_{t+1}^N\right)$, the right side fluctuates with R_{t+1}^k. The setup resembles the usual Laffer-curve setup, with the right side playing the role of the financing requirement and the left the role of tax revenues as a function of function of the 'tax rate', $\bar{\omega}_{t+1}^N$. So, we see that, generically, there are two $\bar{\omega}_{t+1}^N$'s that solve the above equation for given $B_{t+1}^N / \left(Q_{\bar{K}',t}\bar{K}_{t+1}^N\right)$. Between these two, the smaller one is preferred to entrepreneurs, so this is a candidate CSV. The implication is that in a CSV, $\bar{\omega}_{t+1}^N \leq \bar{\omega}^*$. Since, for $\bar{\omega}_{t+1}^N < \bar{\omega}^*$ the left side is increasing in a CSV, we conclude that any shock that drives up R_{t+1}^k will simultaneously drive down $\bar{\omega}_{t+1}^N$.

From (2.8), it is possible to see why Z_{t+1}^N must be dependent upon the realization of the period $t+1$ shocks. Substitute out for $\left(1+R_{t+1}^k\right) Q_{\bar{K}',t}\bar{K}_{t+1}^N$ using (2.7), to obtain:

$$\left[1 - F(\bar{\omega}_{t+1}^N) + \frac{1-\mu}{\bar{\omega}_{t+1}^N}\int_0^{\bar{\omega}_{t+1}^N} \omega dF(\omega)\right] Z_{t+1}^N = \left(1+R_{t+1}^e\right),$$

after dividing both sides by B_{t+1}^N. Recall our specification that R_{t+1}^e is not dependent on the period $t+1$ realization of shocks. The last expression then implies that if Z_{t+1}^N is not dependent on the period $t+1$ shocks, then $\bar{\omega}_{t+1}^N$ must not be either. In this case, it is impossible for (2.7) to hold for all date $t+1$ states of nature. So, Z_{t+1}^N must be dependent on the period $t+1$ shocks.[6] Of course, if R_{t+1}^e were state dependent, then perhaps we could specify Z_{t+1}^N to be period $t+1$ state independent.

[6] This may appear implausible, at first glance. In practice, when banks extend loans the rate of interest that is to be paid is specified in advance. One interpretation of the fact that Z_t^N is contingent on the realization of the aggregate shock is that banks are unwilling to extend loans whose duration spans the whole period of the entrepreneur's project. Instead, they extend the loan for a part of the period, and that allows them to back out before too many funds are commited, in case it looks like the project is going bad. This is closely related to the interpretation offered in Bernanke, Gertler and Gilchrist (1999, footnote 10).

Substituting out for $Z_{t+1}^{N}B_{t+1}^{N}$ from (2.7) in the bank's zero profit condition, we obtain:[7]

$$\begin{aligned}\left(1+R_{t+1}^{e}\right)B_{t+1}^{N} &= \left[1-F(\bar{\omega}_{t+1}^{N})\right]\bar{\omega}_{t+1}^{N}\left(1+R_{t+1}^{k}\right)Q_{\bar{K}',t}\bar{K}_{t+1}^{N} \\ &\quad+\int_{0}^{\bar{\omega}_{t+1}^{N}}(1-\mu)\left(1+R_{t+1}^{k}\right)\omega Q_{\bar{K}',t}\bar{K}_{t+1}^{j}dF(\omega) \\ &= \left[\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})\right]\left(1+R_{t+1}^{k}\right)Q_{\bar{K}',t}\bar{K}_{t+1}^{N},\end{aligned} \tag{2.10}$$

where $\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})$ is the expected share of profits, net of monitoring costs, accruing to the bank and

$$\begin{aligned}G(\bar{\omega}_{t+1}^{N}) &= \int_{0}^{\bar{\omega}_{t+1}^{N}}\omega dF(\omega). \\ \Gamma(\bar{\omega}_{t+1}^{N}) &= \bar{\omega}_{t+1}^{N}\left[1-F(\bar{\omega}_{t+1}^{N})\right]+G(\bar{\omega}_{t+1}^{N})\end{aligned}$$

It is useful to work out the derivative of Γ :

$$\begin{aligned}\Gamma'(\bar{\omega}_{t+1}^{N}) &= 1-F(\bar{\omega}_{t+1}^{N})-\bar{\omega}_{t+1}^{N}F'(\bar{\omega}_{t+1}^{N})+G'(\bar{\omega}_{t+1}^{N}) \\ &= 1-F(\bar{\omega}_{t+1}^{N})>0.\end{aligned} \tag{2.11}$$

Dividing both sides of (2.10) by $Q_{\bar{K}',t}\bar{K}_{t+1}^{N}\left(1+R_{t+1}^{k}\right)$:

$$\frac{1+R_{t+1}^{e}}{1+R_{t+1}^{k}}\left(1-\frac{N_{t+1}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}}\right)=\left[\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})\right]$$

[7] Under the alternative treatment of depreciation,

$$\begin{aligned}\left(1+R_{t+1}^{e}\right)B_{t+1}^{N} &= \left[1-F(\bar{\omega}_{t+1}^{N})\right]\left[(1+\tilde{R}_{t+1}^{k})\bar{\omega}_{t+1}+\tau_{t}^{k}\delta\right]Q_{\bar{K}',t}\bar{K}_{t+1}^{N} \\ &\quad+\int_{0}^{\bar{\omega}_{t+1}^{N}}(1-\mu)\left[(1+\tilde{R}_{t+1}^{k})\omega+\tau_{t}^{k}\delta\right]Q_{\bar{K}',t}\bar{K}_{t+1}^{j}dF(\omega) \\ &= \left[1-F(\bar{\omega}_{t+1}^{N})\right]\left[(1+\tilde{R}_{t+1}^{k})\bar{\omega}_{t+1}+\tau_{t}^{k}\delta\right]Q_{\bar{K}',t}\bar{K}_{t+1}^{N} \\ &\quad+G(\bar{\omega}_{t+1}^{N})(1-\mu)(1+\tilde{R}_{t+1}^{k})Q_{\bar{K}',t}\bar{K}_{t+1}^{j}+F(\bar{\omega}_{t+1}^{N})(1-\mu)\tau_{t}^{k}\delta Q_{\bar{K}',t}\bar{K}_{t+1}^{j} \\ &= \left[\left(1-F(\bar{\omega}_{t+1}^{N})\right)\bar{\omega}_{t+1}+G(\bar{\omega}_{t+1}^{N})(1-\mu)\right](1+\tilde{R}_{t+1}^{k})Q_{\bar{K}',t}\bar{K}_{t+1}^{N}+\tau_{t}^{k}\delta Q_{\bar{K}',t}\bar{K}_{t+1}^{N}\left[1-F(\bar{\omega}_{t+1}^{N})\mu\right] \\ &= \left[\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})\right](1+\tilde{R}_{t+1}^{k})Q_{\bar{K}',t}\bar{K}_{t+1}^{N}+\tau_{t}^{k}\delta Q_{\bar{K}',t}\bar{K}_{t+1}^{N}\left[1-F(\bar{\omega}_{t+1}^{N})\mu\right]\end{aligned}$$

or, after dividing:

$$\frac{\left(1+R_{t+1}^{e}\right)B_{t+1}^{N}}{(1+\tilde{R}_{t+1}^{k})Q_{\bar{K}',t}\bar{K}_{t+1}^{N}}=\left[\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})\right]+\frac{\tau_{t}^{k}\delta\left[1-F(\bar{\omega}_{t+1}^{N})\mu\right]}{(1+\tilde{R}_{t+1}^{k})}$$

Multiply this expression by $\left(Q_{\bar{K}',t}\bar{K}_{t+1}^N/N_{t+1}\right)(1+R_{t+1}^k)/(1+R_{t+1}^e)$, to obtain:

$$\frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}} - 1 = \frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}}\frac{1+R_{t+1}^k}{1+R_{t+1}^e}\left[\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N)\right].$$

Let

$$\tilde{u}_{t+1} \equiv \frac{1+R_{t+1}^k}{E\left(1+R_{t+1}^k|\Omega_t^\mu\right)},\ s_{t+1} \equiv \frac{E\left(1+R_{t+1}^k|\Omega_t^\mu\right)}{1+R_{t+1}^e}.$$

Then, the non-negativity constraint on bank profits is:

$$\frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}} - 1 \leq \frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}}\tilde{u}_{t+1}s_{t+1}\left[\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N)\right], \tag{2.12}$$

From this we can see that $\bar{\omega}_{t+1}^N$ is a function of the capital to net worth ratio and $\left(1+R_{t+1}^e\right)/\left(1+R_{t+1}^k\right)$ only:

$$\bar{\omega}_{t+1}^N = g\left(\frac{1+R_{t+1}^e}{1+R_{t+1}^k}\left(1-\frac{N_{t+1}}{Q_{\bar{K}',t}\bar{K}_{t+1}^N}\right)\right). \tag{2.13}$$

As noted above, competition implies that the loan contract is the best possible one, from the point of view of the entrepreneur. That is, it maximizes the entrepreneur's 'utility' subject to the zero profit constraint just stated. The entrepreneur's expected revenues over the period in which the standard debt contract applies are:[8]

$$\begin{aligned} & E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\left(1+R_{t+1}^k\right)\omega Q_{\bar{K}',t}\bar{K}_{t+1}^N - Z_{t+1}^N B_{t+1}^N\right]dF(\omega)|\Omega_t, X_t\right\} \\ = \ & E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\omega - \bar{\omega}_{t+1}^N\right]dF(\omega)\left(1+R_{t+1}^k\right)|\Omega_t, X_t\right\}Q_{\bar{K}',t}\bar{K}_{t+1}^N. \end{aligned}$$

[8]We treat this as the entrepreneur's utility function, even though the entrepreneur will be around in the future (either he will be around as a condemmed person eating his last meal in the next period, or he will be around with at least one more period after that). Still, we drop all reference to the future in our expression of his utility function. A possible rationale for this is that future utility is a linear function of future net worth. We hope to show this in a future draft.

Note that[9]

$$1 = \int_0^\infty \omega dF(\omega) = \int_{\bar{\omega}_{t+1}^N}^\infty \omega dF(\omega) + G(\bar{\omega}_{t+1}^N),$$

so that the objective can be written:

$$E\left\{\left[1 - \Gamma(\bar{\omega}_{t+1}^N)\right]\left(1 + R_{t+1}^k\right)|\Omega_t^\mu\right\} Q_{\bar{K}',t}\bar{K}_{t+1}^N,$$

or, after dividing by $(1 + R_{t+1}^e)N_{t+1}$ (which is a constant with respect to date $t+1$ aggregate uncertainty), and rewriting:

$$E\left\{\left[1 - \Gamma(\bar{\omega}_{t+1}^N)\right]\tilde{u}_{t+1}|\Omega_t^\mu\right\} s_{t+1}\frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}},\ \tilde{u}_{t+1} = \frac{1 + R_{t+1}^k}{E\left(1 + R_{t+1}^k|\Omega_t^\mu\right)},\ s_{t+1} = \frac{E\left(1 + R_{t+1}^k|\Omega_t^\mu\right)}{1 + R_{t+1}^e}, \quad (2.14)$$

where Ω_t^μ denotes all period t shocks. From this expression and the fact, $\Gamma' > 0$, it is evident that the objective is decreasing in $\bar{\omega}_{t+1}^N$ for given $Q_{\bar{K}',t}\bar{K}_{t+1}^N/N_{t+1}$. This property of the objective was alluded to above.

The debt contract selects $Q_{\bar{K}',t}\bar{K}_{t+1}^N/N_{t+1}$ and $\bar{\omega}_{t+1}^N$ to optimize (2.14) subject to (2.12). It is convenient to denote:

$$k_{t+1}^N = \frac{Q_{\bar{K}',t}\bar{K}_{t+1}^N}{N_{t+1}}.$$

Writing the CSV problem in Lagrangian form,

$$\max_{\bar{\omega}^N, k^N} E\left\{\left[1 - \Gamma(\bar{\omega}^N)\right]\tilde{u}_{t+1}s_{t+1}k^N + \lambda^N\left[k^N\tilde{u}_{t+1}s_{t+1}\left(\Gamma(\bar{\omega}^N) - \mu G(\bar{\omega}^N)\right) - k^N + 1\right]|\Omega_t^\mu\right\}.$$

The single first order condition for k^N is:

$$E\left\{\left[1 - \Gamma(\bar{\omega}_{t+1}^N)\right]\tilde{u}_{t+1}s_{t+1} + \lambda_{t+1}^N\left[\tilde{u}_{t+1}s_{t+1}\left(\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N)\right) - 1\right]|\Omega_t^\mu\right\} = 0. \quad (2.15)$$

[9] Under the alternative treatment of depreciation,

$$\begin{aligned} & E\left\{\int_{\bar{\omega}_{t+1}^N}^\infty \left[\left((1 + \tilde{R}_{t+1}^k)\omega + \tau_t^k\delta\right) Q_{\bar{K}',t}\bar{K}_{t+1}^N - \left(\left[1 + \tilde{R}_{t+1}^k\right]\bar{\omega}_{t+1}^N + \tau_t^k\delta\right) Q_{\bar{K}',t}\bar{K}_{t+1}^N\right] dF(\omega)|\Omega_t, X_t\right\} \\ = \ & E\left\{\int_{\bar{\omega}_{t+1}^N}^\infty \left[\omega - \bar{\omega}_{t+1}^N\right] dF(\omega)\left(1 + \tilde{R}_{t+1}^k\right)|\Omega_t, X_t\right\} Q_{\bar{K}',t}\bar{K}_{t+1}^N. \end{aligned}$$

The first order conditions for $\bar{\omega}^N$ are, after dividing by $\tilde{u}_{t+1} s_{t+1} k_{t+1}^N$:

$$\Gamma'(\bar{\omega}_{t+1}^N) = \lambda_{t+1}^N \left[\Gamma'(\bar{\omega}_{t+1}^N) - \mu G'(\bar{\omega}_{t+1}^N) \right]. \tag{2.16}$$

Finally, there is the complementary slackness condition, $\lambda^N \left[k^N \tilde{u}_{t+1} s_{t+1} \left(\Gamma(\bar{\omega}^N) - \mu G(\bar{\omega}^N) \right) - k^N + 1 \right] = 0$. Assuming the constraint is binding, so that $\lambda^N > 0$, this reduces to:

$$k_{t+1}^N \tilde{u}_{t+1} s_{t+1} \left(\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N) \right) - k_{t+1}^N + 1 = 0. \tag{2.17}$$

It should be understood that λ_{t+1}^N in (2.15) is defined by (2.16). We can think of (2.15)-(2.17) as defining functions relating k_{t+1}^N and $\bar{\omega}_{t+1}^N$to s_{t+1}. Remember, k_{t+1}^N is not indexed by $\tilde{u}_{t+1}$, while $\bar{\omega}_{t+1}^N$ is. So, we think of $\bar{\omega}_{t+1}^N$ as a family of functions of s_{t+1}, each function being indexed by a different realization of $\tilde{u}_{t+1}$. Note that N_{t+1} does not appear in the equations that define k_{t+1}^N and $\bar{\omega}_{t+1}^N$. This establishes that the values of these variables in the CSV contract is the same for each value of N_{t+1}. For this reason, we can drop the superscript notation, N. That is, the functions we are concerned with are k_{t+1} and $\bar{\omega}_{t+1}$.

We find it convenient to denote the function relating k_{t+1} to s_{t+1} by:

$$k_{t+1} = \psi(s_{t+1}). \tag{2.18}$$

This function (at least, its Taylor expansion around the steady state value of s_{t+1}) could be used to play an important role in the computations. In general, it is difficult to characterize ψ analytically, from (2.15)-(2.17). One has to solve for this function by jointly solving for it and the functions relating λ_{t+1} and $\bar{\omega}_{t+1}$ to s_{t+1}. For purposes of computation it is not necessary to characterize ψ analytically. We only need its value and derivative in steady state, which we denote by ψ and ψ'. We obtain its value by solving (2.15)-(2.17) in steady state, and we obtain its derivative by differentiation.

We find it convenient to drop time subscripts to keep the notation simple, and because it should entail no confusion. The equations that concern us are:

$$E\left\{ \left[1 - \Gamma(\bar{\omega})\right] \tilde{u} s + \lambda \left[\tilde{u} s \left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) \right) - 1 \right] \right\} = 0, \tag{2.19}$$

$$\Gamma'(\bar{\omega}) = \lambda \left[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) \right], \tag{2.20}$$

$$k \tilde{u} s \left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) \right) - k + 1 = 0. \tag{2.21}$$

It is understood that the expectation operator is over different values of $\tilde{u}$, and k is constant across $\tilde{u}$ while λ and $\bar{\omega}$ vary with $\tilde{u}$. Let $\bar{\omega}_s$ denote the derivative of $\bar{\omega}$ with respect to s and define k_s similarly. So, the values of k and k_s are the ψ and ψ' that we seek.

To obtain the steady state value of $\bar{\omega}$ substitute out for λ from (2.20) into (2.19) and evaluate in steady state, with $\tilde{u} = 1$:

$$[1-\Gamma(\bar{\omega})]\,s+\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\left[s\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)-1\right]=0, \tag{2.22}$$

$$[1-\Gamma(\bar{\omega})]\,s+\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\left[s\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)-s+s-1\right]=0,$$

$$[1-\Gamma(\bar{\omega})]\,s-\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}[1-\Gamma(\bar{\omega})]\,s+\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\left[-s\mu G(\bar{\omega})+s-1\right]=0,$$

$$[1-\Gamma(\bar{\omega})]\,s\left\{1-\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\right\}+\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\left[-s\mu G(\bar{\omega})+s-1\right]=0,$$

$$[1-\Gamma(\bar{\omega})]\,s\frac{-\mu G'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}=\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})}\left[s\mu G(\bar{\omega})-s+1\right],$$

$$-\mu G'(\bar{\omega})\,[1-\Gamma(\bar{\omega})]\,s=\Gamma'(\bar{\omega})\left[s\mu G(\bar{\omega})+1-s\right],$$

$$\mu\bar{\omega}F'(\bar{\omega})\,[1-\Gamma(\bar{\omega})]\,s+[1-F(\bar{\omega})]\left[s\left(\mu G(\bar{\omega})-1\right)+1\right]=0.$$

Dividing by $1-F(\bar{\omega})$:

$$\mu\frac{\bar{\omega}F'(\bar{\omega})}{1-F(\bar{\omega})}[1-\Gamma(\bar{\omega})]\,s+s\left(\mu G(\bar{\omega})-1\right)+1=0.$$

$$\left\{\mu\frac{\bar{\omega}F'(\bar{\omega})}{1-F(\bar{\omega})}[1-\Gamma(\bar{\omega})]+\mu G(\bar{\omega})-1\right\}s+1=0.$$

Note that if $s = 1$, then the object on the left is the sum of two positive numbers. This cannot be zero. So, $s = 1$ cannot be a steady state equilibrium. BGG argue that a steady state equilibrium requires $s > 1$. Presumably, this means that the object in braces in the preceding expression is negative. That remains to be shown.

Taking into account, $\Gamma' = 1 - F$ and $G' = \bar{\omega}F'$, we obtain from (2.22):

$$[1-\Gamma(\bar{\omega})]\,s+\frac{1-F(\bar{\omega})}{1-F(\bar{\omega})-\mu\bar{\omega}F'(\bar{\omega})}\left[s\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)-1\right]=0.$$

We follow BGG in specifying that F corresponds to a log-normal distribution. This has two parameters. However, the requirement, $E\omega = 1$, pins down one of them. In addition, $F(\bar{\omega})$ is treated as an observable variable. So, for purposes of computing the steady state, we think of there being two unknowns: $\bar{\omega}$ and the free parameter of F. These are solved for by enforcing the previous equation and the desired value of $F(\bar{\omega})$.[10] After this, ψ may be computed by solving (2.21) setting $\tilde{u} = 1$.

Next, we consider ψ'. Differentiating (2.19)-(2.21), we obtain:

$$E\{[-\Gamma'(\bar{\omega})\tilde{u}s + \lambda\tilde{u}s\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)]\bar{\omega}_s + [1 - \Gamma(\bar{\omega})]\tilde{u} + \lambda\left[\tilde{u}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)\right] \\ +\lambda_s\left[\tilde{u}s\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right]\} = 0,$$

$$\{\Gamma''(\bar{\omega}) - \lambda\left[\Gamma''(\bar{\omega}) - \mu G''(\bar{\omega})\right]\}\bar{\omega}_s - \lambda_s\left[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right] = 0$$
$$\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)k_s + k\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)\bar{\omega}_s = 0$$

Evaluating these equations in steady state, and making use of (2.16):

$$[1 - \Gamma(\bar{\omega})] + \lambda\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) \\ +\lambda_s\left[s\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right] = 0,$$

$$\{\Gamma''(\bar{\omega}) - \lambda\left[\Gamma''(\bar{\omega}) - \mu G''(\bar{\omega})\right]\}\bar{\omega}_s - \lambda_s\left[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right] = 0$$
$$\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)k_s + k\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)\bar{\omega}_s = 0$$

To find the object of interest, k_s, we need to solve these three equations for k_s, $\bar{\omega}_s$ and λ_s. The first of these three equations can be solved for λ_s, the second $\bar{\omega}_s$. Then, the last equation can be solved for k_s. This requires knowing $\Gamma''(\bar{\omega})$ and $G''(\bar{\omega})$. From (2.11) we obtain that $\Gamma''(\bar{\omega}) = -F'(\bar{\omega})$. Also, since $G'(\bar{\omega}) = \bar{\omega}F'(\bar{\omega})$, so that

$$G''(\bar{\omega}) = F'(\bar{\omega}) + \bar{\omega}F''(\bar{\omega}).$$

This requires evaluating $F''(\bar{\omega})$.[11]

2.4.4. Aggregating Across Entrepreneurs

We now discuss the evolution of the aggregate net worth of all entrepreneurs. In terms of the previous notation, if $f_{t+1}(N)$ is the density of entrepreneurs having net worth N_{t+1}, then aggregate net worth, $\bar{N}_{t+1}$, is:

$$\bar{N}_{t+1} = \int_0^\infty N f_{t+1}(N)dN.$$

[10] The MATLAB function, logncdf.m, can be used to compute F. The function, lognpdf.m, can be used to compute F'. Computing G will require cooking a quadrature integration routine.

[11] To obtain $F''(\bar{\omega})$, differentiate the log-normal density function,

$$F'(x) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left[\frac{-(\log x - \mu)^2}{2\sigma^2}\right],$$

with respect to x, and evaluate the result at $x = \bar{\omega}$.

Applying this integral to (2.18), we obtain

$$Q_{\bar{K}',t}\bar{K}_{t+1} = \psi(\frac{E\left[\left(1+R_{t+1}^{k}\right)|\Omega_{t}^{\mu}\right]}{\left(1+R_{t+1}^{e}\right)})\bar{N}_{t+1},$$

where BGG argue $\psi(1) = 1$, $\psi'(.) > 0$, and where $\bar{K}_{t+1}$ is the aggregate, end of period t stock of physical capital. We now linearize this, following the argument in BGG (page 1361). First, however, we need to scale the variables:

$$q_t\bar{k}_{t+1} = \psi(\frac{E\left[\left(1+R_{t+1}^{k}\right)|\Omega_{t}^{\mu}\right]}{\left(1+R_{t+1}^{e}\right)})n_{t+1},$$

where

$$\bar{k}_{t+1} = \frac{\bar{K}_{t+1}}{z_t},\ n_{t+1} = \frac{\bar{N}_{t+1}}{z_t P_t}. \tag{2.23}$$

Linearizing about steady state:

$$\bar{k}q\hat{q}_t + q\bar{k}\widehat{\bar{k}}_{t+1} = \psi n\hat{n}_{t+1} + n\psi'\left(\frac{1+R^k}{1+R^e}\right)\frac{1+R^k}{1+R^e}\left[\frac{R^k}{1+R^k}\hat{R}_{t+1}^{k} - \frac{R^e}{1+R^e}\hat{R}_{t+1}^{e}\right]$$

$$\bar{k}q\hat{q}_t + q\bar{k}\widehat{\bar{k}}_{t+1} = \frac{\bar{k}q}{n}n\hat{n}_{t+1} + n\psi'\left(\frac{1+R^k}{1+R^e}\right)\frac{1+R^k}{1+R^e}\left[\frac{R^k}{1+R^k}\hat{R}_{t+1}^{k} - \frac{R^e}{1+R^e}\hat{R}_{t+1}^{e}\right]$$

So, the linearized solution to the contracting problem is:

$$CSV1\ (**)\ \frac{R^k}{1+R^k}E\left[\hat{R}_{t+1}^{k}|\Omega_t^{\mu}\right] - \frac{R^e}{1+R^e}\hat{R}_{t+1}^{e} = -v_{bgg}\left[\hat{n}_{t+1} - \left(\hat{q}_t + \widehat{\bar{k}}_{t+1}\right)\right]$$

where

$$v_{bgg} = \frac{\psi}{\psi'}\frac{1+R^e}{1+R^k}.$$

This is an equation emphasized in *BGG*. However, we don't use it in our solution procedure.

We now discuss the law of motion of aggregate net worth, $\hat{n}_{t+1}$. Suppose $\bar{N}_t$ is given. Let V_t^N denote the average of profits of N_t-type entrepreneurs, net of repayments to banks:

$$V_t^N = \left(1+R_t^k\right)Q_{\bar{K}',t-1}\bar{K}_t^N - \Gamma(\bar{\omega}_t)\left(1+R_t^k\right)Q_{\bar{K}',t-1}\bar{K}_t^N.$$

The aggregate capital stock is:

$$\bar{K}_t = \int_0^{\infty} f_t(N)\bar{K}_t^N dN$$

Given that R_t^k and $\bar{\omega}_t$ are independent of N_t, we have:

$$V_t \equiv \int_0^\infty f_t(N) V_t^N dN = \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t - \Gamma(\bar{\omega}_t) \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t$$

Writing this out more fully:

$$\begin{aligned} V_t &= \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t - \left\{ [1 - F(\bar{\omega}_t)] \bar{\omega}_t + \int_0^{\bar{\omega}_t} \omega dF(\omega) \right\} \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t \\ &= \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t \\ &\quad - \left\{ [1 - F(\bar{\omega}_t)] \bar{\omega}_t + (1-\mu) \int_0^{\bar{\omega}_t} \omega dF(\omega) + \mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \right\} \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t. \end{aligned}$$

Notice that the first two terms in braces correspond to the net revenues of the bank, which must equal $(1 + R_t^e)(Q_{\bar{K}',t-1}\bar{K}_t - \bar{N}_t)$. Substituting:

$$V_t = \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t - \left\{ 1 + R_t^e + \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t}{Q_{\bar{K}',t-1} \bar{K}_t - \bar{N}_t} \right\} (Q_{\bar{K}',t-1} \bar{K}_t - \bar{N}_t). \tag{2.24}$$

Since entrepreneurs are selected randomly for death, the integral over entrepreneurs' net profits is just γV_t. So, the law of motion for $\bar{N}_t$ is:

$$\begin{aligned} \bar{N}_{t+1} &= \gamma \left\{ \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t - \left[1 + R_t^e + \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t}{Q_{\bar{K}',t-1} \bar{K}_t - \bar{N}_t} \right] (Q_{\bar{K}',t-1} \bar{K}_t - \bar{N}_t) \right\} \\ &\quad + W_t^e, \end{aligned}$$

where W_t^e is the transfer payment to entrepreneurs. The $(1-\gamma)$ entrepreneurs who are selected for death, consume:

$$P_t C_t^e = \Theta(1-\gamma) V_t.$$

Finally, there seem to be at least two objects that could be called the 'external finance premium'. One is the ratio involving μ in square brackets above. The other is $Z_t - (1 + R_t^e)$. Either one is straightforward to compute. The former appears to correspond to the 'average' external finance premium, while the latter is only the external finance premium for entrepreneurs who are able to repay Z_t. In the case of the former, the 'premium' paid by some is actually negative. For those entrepreneurs with ω sufficiently small, they are paying essentially nothing, and so in particular they pay less than R_t^e and so they have an ex post negative premium. BGG refer to s as the external finance premium.

We move now to the linear representation of (2.25). Simplifying that expression:

$$\bar{N}_{t+1} = \gamma_t Q_{\bar{K}',t-1}\bar{K}_t\left\{R_t^k - R_t^e - \mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R_t^k\right)\right\} + W_t^e + \gamma_t\left(1+R_t^e\right)\bar{N}_t,$$

where a time subscript has been added to γ to capture the possibility that there are random disturbances to the death rate of entrepreneurs. By putting a coefficient of unity in front of W_t^e, we are implicitly making the assumption that when there is a shock to the death rate of entrepreneurs, say γ_t falls, then there is an equal shock in the other direction in the rate of arrival of new entrepreneurs.

Dividing by $z_t P_t$ and taking into account (2.23):

$$n_{t+1} = \frac{\gamma_t}{\pi_t\mu_{z,t}}\left\{R_t^k - R_t^e - \mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R_t^k\right)\right\}\bar{k}_t q_{t-1} + w_t^e + \gamma_t\left(\frac{1+R_t^e}{\pi_t}\right)\frac{1}{\mu_{z,t}}n_t, \quad (2.25)$$

where

$$w_t^e = \frac{W_t^e}{z_t P_t},\ \pi_t = \frac{P_t}{P_{t-1}},\ q_t = \frac{Q_{\bar{K}',t}}{P_t}.$$

In this last expression, we see the fundamental reason for setting $\gamma < 1$. The real interest rate divided by the growth rate is $1/\beta$ in steady state, which would imply that n_t explodes when $\gamma = 1$. This in turn implies that real net worth grows faster than the economy and, hence, the capital stock. That means that eventually, net worth exceeds the capital stock and the CSV arrangement becomes irrelevant. It is to avoid this outcome that γ is assumed to be small.

We proceed now to linearize the equations. We do not want $\hat{\lambda}_t$ to be among the variables to be solved for. So, we linearize (2.15) and (2.17), and use (2.16) to substitute out for the multiplier. Writing out the object in braces in (2.15), and replacing $\tilde{u}_{t+1}s_{t+1}$:

$$[1-\Gamma(\bar{\omega}_{t+1})]\frac{1+R_{t+1}^k}{1+R_{t+1}^e} + \lambda_{t+1}\left[\frac{1+R_{t+1}^k}{1+R_{t+1}^e}\left(\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})\right) - 1\right]$$

Log-linearly expanding this about steady state:

$$\begin{aligned}
&-\Gamma'(\bar{\omega})\frac{1+R^k}{1+R^e}\bar{\omega}\hat{\bar{\omega}}_{t+1} \\
&+[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}\left(\widehat{\left[1+R_{t+1}^k\right]} - \widehat{\left[1+R_{t+1}^e\right]}\right) \\
&+\lambda\hat{\lambda}_{t+1}\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right] \\
&+\lambda\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right]\widehat{\left[\frac{1+R_{t+1}^k}{1+R_{t+1}^e}\left(\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})\right) - 1\right]}
\end{aligned}$$

We have:

$$\widehat{1+R^k_{t+1}} = \frac{R^k \hat{R}^k_{t+1}}{1+R^k},\ \widehat{1+R^e_{t+1}} = \frac{R^e \hat{R}^e_{t+1}}{1+R^e}.$$

Also,

$$\begin{aligned}
&\widehat{\left[\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})\right) - 1\right]} \\
= \quad & \frac{d\left[\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})\right) - 1\right]}{\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1} \\
= \quad & \frac{\frac{1+R^k}{1+R^e}\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)\bar{\omega}\widehat{\bar{\omega}}_{t+1} + \frac{R^k \hat{R}^k_{t+1}}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - \frac{1+R^k}{(1+R^e)^2}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) R^e \hat{R}^e_{t+1}}{\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1}
\end{aligned}$$

Substituting

$$\begin{aligned}
&-\Gamma'(\bar{\omega})\frac{1+R^k}{1+R^e}\bar{\omega}\widehat{\bar{\omega}}_{t+1} \\
&+\left[1-\Gamma(\bar{\omega})\right]\frac{1+R^k}{1+R^e}\left(\frac{R^k \hat{R}^k_{t+1}}{1+R^k} - \frac{R^e \hat{R}^e_{t+1}}{1+R^e}\right) \\
&+\lambda\hat{\lambda}_{t+1}\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right] \\
&+\lambda[\frac{1+R^k}{1+R^e}\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)\bar{\omega}\widehat{\bar{\omega}}_{t+1} \\
&+\frac{R^k \hat{R}^k_{t+1}}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - \frac{1+R^k}{(1+R^e)^2}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) R^e \hat{R}^e_{t+1}]
\end{aligned}$$

Collecting terms:

$$\begin{aligned}
&\left[-\Gamma'(\bar{\omega})\frac{1+R^k}{1+R^e}\bar{\omega} + \lambda\frac{1+R^k}{1+R^e}\left(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right)\bar{\omega}\right]\widehat{\bar{\omega}}_{t+1} \\
&+\left[1-\Gamma(\bar{\omega}) + \lambda\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)\right]\frac{1+R^k}{1+R^e}\left(\frac{R^k \hat{R}^k_{t+1}}{1+R^k} - \frac{R^e \hat{R}^e_{t+1}}{1+R^e}\right) \\
&+\lambda\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right]\hat{\lambda}_{t+1}
\end{aligned}$$

Simplify this, using the steady state relation (see (2.15)):

$$[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}+\lambda\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)-1\right]=0,$$

we obtain

$$\begin{aligned}&\left[-\Gamma'(\bar{\omega})\frac{1+R^k}{1+R^e}\bar{\omega}+\lambda\frac{1+R^k}{1+R^e}\left(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})\right)\bar{\omega}\right]\widehat{\bar{\omega}}_{t+1}\\&+\lambda\left(\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}\right)\\&-[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}\hat{\lambda}_{t+1}.\end{aligned}$$

From (2.16):

$$\Gamma'(\bar{\omega})=\lambda\left[\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})\right].$$

Substituting this into the previous expression, the coefficient on $\widehat{\bar{\omega}}_{t+1}$ turns out to be zero. So, we are left with:

$$E\{\lambda\left(\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}\right)+[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}\hat{\lambda}_{t+1}|\Omega^\mu_t\}=0.$$

The correct terms is actually:

$$E\{\lambda\left(\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}\right)-[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}\hat{\lambda}_{t+1}|\Omega^\mu_t\}=0.$$

$$\Gamma'(\bar{\omega}^N_{t+1})=\lambda^N_{t+1}\left[\Gamma'(\bar{\omega}^N_{t+1})-\mu G'(\bar{\omega}^N_{t+1})\right]$$

Expanding (2.16) and making use of (2.20):

$$\Gamma''(\bar{\omega})\bar{\omega}\widehat{\bar{\omega}}_{t+1}=\hat{\lambda}_{t+1}\Gamma'(\bar{\omega})+\lambda\left[\Gamma''(\bar{\omega})-\mu G''(\bar{\omega})\right]\bar{\omega}\widehat{\bar{\omega}}_{t+1},$$

or,

$$\hat{\lambda}_{t+1}=\left[\frac{\Gamma''(\bar{\omega})\bar{\omega}}{\Gamma'(\bar{\omega})}-\frac{\lambda\left[\Gamma''(\bar{\omega})-\mu G''(\bar{\omega})\right]\bar{\omega}}{\Gamma'(\bar{\omega})}\right]\widehat{\bar{\omega}}_{t+1}.$$

Substituting, we obtain the log-linearized version of (2.15):

$$(**)\ E\{\lambda\left(\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}\right)$$

$$-\left[1-\Gamma(\bar{\omega})\right]\frac{1+R^k}{1+R^e}\left[\frac{\Gamma''(\bar{\omega})\bar{\omega}}{\Gamma'(\bar{\omega})}-\frac{\lambda\left[\Gamma''(\bar{\omega})-\mu G''(\bar{\omega})\right]\bar{\omega}}{\Gamma'(\bar{\omega})}\right]\widehat{\bar{\omega}}_{t+1}|\Omega^\mu_t\} \quad = \quad 0.$$

Now we log-linearize (2.17). Writing (2.17) out:

$$\frac{Q_{\bar{K}',t}\bar{K}^N_{t+1}}{N_{t+1}}\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}^N_{t+1})-\mu G(\bar{\omega}^N_{t+1})\right)-\frac{Q_{\bar{K}',t}\bar{K}^N_{t+1}}{N_{t+1}}+1=0.$$

Take into account

$$\bar{K}_{t+1}=z_t\bar{k}_{t+1},\ \bar{N}_{t+1}=z_tP_tn_{t+1},\ Q_{\bar{K}',t}=q_tP_t,$$

so that

$$\frac{Q_{\bar{K}',t}\bar{K}^N_{t+1}}{N_{t+1}}=\frac{q_tP_tz_t\bar{k}_{t+1}}{z_tP_tn_{t+1}}=\frac{q_t\bar{k}_{t+1}}{n_{t+1}},$$

and

$$\frac{q_t\bar{k}_{t+1}}{n_{t+1}}\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}^N_{t+1})-\mu G(\bar{\omega}^N_{t+1})\right)-\frac{q_t\bar{k}_{t+1}}{n_{t+1}}+1=0. \tag{2.26}$$

Then,

$$\left[\frac{q\bar{k}}{n}-1\right]\widehat{\frac{q_t\bar{k}_{t+1}}{n_{t+1}}\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}^N_{t+1})-\mu G(\bar{\omega}^N_{t+1})\right)}=\left[\frac{q\bar{k}}{n}-1\right]\left[\widehat{\frac{q_t\bar{k}_{t+1}}{n_{t+1}}}-1\right].$$

Now,

$$\begin{aligned}
&\widehat{\frac{q_t\bar{k}_{t+1}}{n_{t+1}}\frac{1+R^k_{t+1}}{1+R^e_{t+1}}\left(\Gamma(\bar{\omega}^N_{t+1})-\mu G(\bar{\omega}^N_{t+1})\right)}\\
=\ &\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}+\widehat{\frac{1+R^k_{t+1}}{1+R^e_{t+1}}}+\widehat{\left(\Gamma(\bar{\omega}^N_{t+1})-\mu G(\bar{\omega}^N_{t+1})\right)}\\
=\ &\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}+\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}\\
&+\frac{\left(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega})\right)}{\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)}\bar{\omega}\widehat{\bar{\omega}}^N_{t+1}
\end{aligned}$$

and

$$\left[\frac{\widehat{q_t\bar{k}_{t+1}}}{n_{t+1}}-1\right]=\frac{\frac{q\bar{k}}{n}\left[\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}\right]}{\left[\frac{q\bar{k}}{n}-1\right]}$$

Putting all this together, the log-linearized expression is:

$$\begin{aligned}&\left[\frac{q\bar{k}}{n}-1\right]\left[\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}+\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}+\frac{(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega}))}{(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))}\bar{\omega}\widehat{\bar{\omega}}^N_{t+1}\right]\\ =\ &\frac{q\bar{k}}{n}\left[\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}\right],\end{aligned}$$

or,

$$\begin{aligned}\left[\frac{q\bar{k}}{n}-1\right]\left[\frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}+\frac{(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega}))}{(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))}\bar{\omega}\widehat{\bar{\omega}}^N_{t+1}\right]&\\ -\left(\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}\right)=&\qquad 0.\end{aligned}$$

Note that this must hold in every realized state of nature. Perhaps the best way to implement this equation is to require that it hold in the first period and in later periods, rather than just starting in the second period. It is different from the following expression, which is what showed up in the initial version of the linearized writeup of the (2.26):

$$\begin{aligned}(**)\ \frac{R^k\hat{R}^k_{t+1}}{1+R^k}-\frac{R^e\hat{R}^e_{t+1}}{1+R^e}+\frac{(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega}))}{(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))}\bar{\omega}\widehat{\bar{\omega}}^N_{t+1}&\\ -\left(\hat{q}_t+\widehat{\bar{k}}_{t+1}-\hat{n}_{t+1}\right)&=0.\end{aligned}$$

Finally, we log-linearize the law of motion of aggregate net worth, (2.25), which we repeat here for convenience:

$$n_{t+1}=\frac{\gamma_t}{\pi_t\mu_{z,t}}\left\{R^k_t-R^e_t-\mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R^k_t\right)\right\}\bar{k}_tq_{t-1}+w^e_t+\gamma_t\left(\frac{1+R^e_t}{\pi_t}\right)\frac{1}{\mu_{z,t}}n_t,$$

so that:

$$
\begin{aligned}
\hat{n}_{t+1} \quad = \quad & \left\{ \frac{\gamma}{\pi\mu_z} \left[R^k - R^e - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \left(1 + R^k\right) \right] \frac{\bar{k}q}{n} + \gamma \left(\frac{1 + R^e}{\pi} \right) \frac{1}{\mu_z} \right\} \hat{\gamma}_t \\
& - \left\{ \frac{\gamma}{\pi\mu_z} \left[R^k - R^e - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \left(1 + R^k\right) \right] \frac{\bar{k}q}{n} + \gamma \left(\frac{1 + R^e}{\pi} \right) \frac{1}{\mu_z} \right\} \hat{\mu}_{z,t} \\
& - \left\{ \frac{\gamma}{\pi\mu_z} \left[R^k - R^e - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \left(1 + R^k\right) \right] \frac{\bar{k}q}{n} + \gamma \frac{1 + R^e}{\pi} \frac{1}{\mu_z} \right\} \hat{\pi}_t \\
& + \frac{\gamma}{\pi\mu_z} \left(1 - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \right) \frac{\bar{k}q}{n} R^k \hat{R}_t^k + \left[\gamma \frac{1}{\pi} \frac{1}{\mu_z} R^e - \frac{\gamma}{\pi\mu_z} \frac{\bar{k}q}{n} R^e \right] \hat{R}_t^e \\
& - \frac{\gamma}{\pi\mu_z} \mu \bar{\omega}^2 F'(\bar{\omega}) \left(1 + R^k\right) \frac{\bar{k}q}{n} \widehat{\bar{\omega}}_t \\
& + \frac{\gamma}{\pi\mu_z} \left\{ R^k - R^e - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \left(1 + R^k\right) \right\} \frac{\bar{k}q}{n} \left[\widehat{\bar{k}}_t + \hat{q}_{t-1} \right] + \frac{w^e}{n} \hat{w}_t^e \\
& + \gamma \left(\frac{1 + R^e}{\pi} \right) \frac{1}{\mu_z} \hat{n}_t,
\end{aligned}
$$

or:

$$
(**) \ \hat{n}_{t+1} = a_0 \hat{R}_t^k + a_1 \hat{R}_t^e + a_2 \widehat{\bar{k}}_t + a_3 \hat{w}_t^e + a_4 \hat{\gamma}_t + a_5 \hat{\pi}_t + a_6 \hat{\mu}_{z,t} + a_7 \hat{q}_{t-1} + a_8 \widehat{\bar{\omega}}_t + a_9 \hat{n}_t,
$$

where

$$
\begin{aligned}
a_0 &= \frac{\gamma}{\pi\mu_z}\left(1-\mu\int_0^{\bar{\omega}}\omega dF(\omega)\right)\frac{\bar{k}q}{n}R^k \\
a_1 &= \left(1-\frac{\bar{k}q}{n}\right)\frac{\gamma R^e}{\pi\mu_z} \\
a_2 &= \frac{\gamma}{\pi\mu_z}\left\{R^k-R^e-\mu\int_0^{\bar{\omega}}\omega dF(\omega)\left(1+R^k\right)\right\}\frac{\bar{k}q}{n} \\
a_3 &= \frac{w^e}{n} \\
a_4 &= a_2+\frac{\gamma\left(1+R^e\right)}{\mu_z\pi} \\
a_5 &= -a_4 \\
a_6 &= -a_4 \\
a_7 &= a_2 \\
a_8 &= -\frac{\gamma}{\pi\mu_z}\mu\bar{\omega}^2F'(\bar{\omega})\left(1+R^k\right)\frac{\bar{k}q}{n} \\
a_9 &= \gamma\left(\frac{1+R^e}{\pi}\right)\frac{1}{\mu_z}
\end{aligned}
$$

2.5. Banks

We assume that there is a continuum of identical, competitive banks. All bank decisions are taken after the realization of the current period shocks. Each bank operates a technology to convert capital, K_t^b, labor, l_t^b, and excess reserves into real deposit services, D_t/P_t. The production function is:

$$
\frac{D_t}{P_t}=a^b x_t^b\left(\left(K_t^b\right)^{\alpha}\left(z_t l_t^b\right)^{1-\alpha}\right)^{\xi_t}\left(\frac{E_t^r}{P_t}\right)^{1-\xi_t} \tag{2.27}
$$

Here a^b is a positive scalar, and $0<\alpha<1$. Also, x_t^b is a unit-mean technology shock that is specific to the banking sector. In addition, $\xi_t\in(0,1)$ is a shock to the relative value of excess reserves, E_t^r. The stochastic process governing these shocks will be discussed later. We include excess reserves as an input to the production of demand deposit services as a reduced form way to capture the precautionary motive of a bank concerned about the possibility of unexpected withdrawals.

We now discuss a typical bank's balance sheet. The bank's assets consist of cash reserves and loans. It obtains cash reserves from two sources. Households deposit A_t dollars and the monetary authority credits households' checking accounts with X_t dollars. Consequently, total time t cash reserves of the banking system equal $A_t + X_t$. Bank loans are extended to firms and other banks to cover their working capital needs, and to entrepreneurs to finance purchases of capital.

The bank has two types of liabilities: demand deposits, D_t, and time deposits, T_t. Demand deposits, which pay interest, R_{at}, are created for two reasons. First, there are the household deposits, $A_t + X_t$ mentioned above. We denote this by D_t^h. Second, working capital loans made by banks to firms and other banks are granted in the form of demand deposits. We denote firm and bank demand deposits by D_t^f. Total deposits, then, are:

$$D_t = D_t^h + D_t^f.$$

Time deposit liabilities are issued by the bank to finance the standard debt contracts that they offer to entrepreneurs. Time and demand deposits differ in three respects. First, demand deposits yield transactions services, while time deposits do not. Second, time deposits have a longer maturity structure. Third, demand deposits are backed by working capital loans and reserves, while time deposits are backed by standard debt contracts to entrepreneurs.

We now discuss the demand deposit liabilities. We suppose that the interest on demand deposits that are created when firms and banks receive working capital loans, are paid to the recipient of the loans. Firms and banks just sit on these demand deposits. The wage bill isn't actually paid until a settlement period that occurs after the goods market.

We denote the interest payment on working capital loans, net of interest on the associated demand deposits, by R_t. Since each borrower receives interest on the deposit associated with their loan, the gross interest payment on loans is $R_t + R_{at}$. Put differently, the spread between the interest on working capital loans and the interest on demand deposits is R_t.

The maturity of period t working capital loans and the associated demand deposit liabilities coincide. A period t working capital loan is extended just prior to production in period t, and then paid off after production. The household deposits funds into the bank just prior to production in period t and then liquidates the deposit after production.

We now discuss the time deposit liabilities. Unlike in the case of demand deposits, we assume that the cost of maintaining time deposit liabilities is zero. Competition among banks in the provision of time deposits and entrepreneurial loans drives the interest rate on time deposits to the return the bank earns (net of expenses, including monitoring costs) on the loans, R_t^e. The maturity structure of time deposits coincides with that of the standard debt contract, and differs from that of demand deposits and working capital loans. The maturity structure of the two types of assets can be seen in Figure 3. Time deposits and entrepeneurial loans are created at the end of a given period's goods market. This is the time when newly constructed capital is sold by capital producers to entrepreneurs. Time deposits and entrepreneurial loans pay off at the end of next period's goods market, when the entrepreneurs sell their undepreciated capital to capital producers (who use it as a raw material in the production of next period's capital). The payoff on the entrepreneurial loan coincides with the payoff on time deposits. Competition in the provision of time deposits guarantees that these payoffs coincide.

The maturity difference between demand and time deposits implies that the return on the latter carries risks not present in the former. In the case of demand deposits, no shocks are realized between the creation of a deposit and its payoff. In the case of time deposits, there are shocks whose value is realized between creation and payoff (see Figure 3). So, the household that acquires a time deposit bears the uncertainty in the payoff of the underlying physical capital whose accumulation is being financed with the time deposit. The entrepreneur also bears risk, for the same reason.

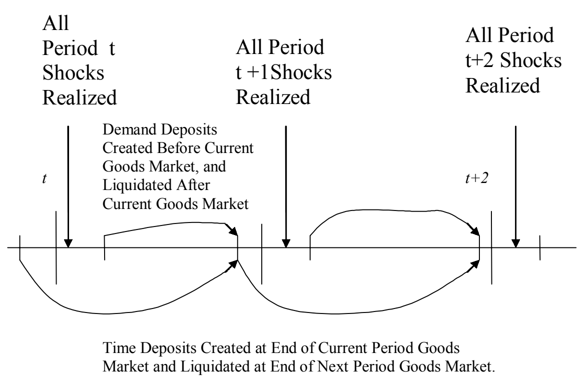


Figure 3: Maturity Structure of Time and Demand Deposits

We now discuss the assets and liabilities of the bank in greater detail. We describe the banks' books at two points in time within the period: just before the goods market, when the market for working capital loans and demand deposits is open, and just after the goods market. At the latter point in time, the market for time deposits and entrepreneurial loans is open. Liabilities and assets just before the goods market are:

$$D_t + T_{t-1} = A_t + X_t + S_t^w + B_t, \tag{2.28}$$

where S_t^w denotes working capital loans. The monetary authority imposes a reserve requirement that banks must hold at least a fraction τ of their demand deposits in the form of currency. Consequently, nominal excess reserves, E_t^r, are given by

$$E_t^r = A_t + X_t - \tau_t D_t. \tag{2.29}$$

The bank's 'T' accounts are as follows:

Assets	Liabilities
Reserves	
A_t	D_t
X_t	
Short-term Working Capital Loans	
S_t^w	
Long-term, Entrepreneural Loans	
B_t	T_{t-1}

After the goods market, demand deposits are liquidated, so that $D_t = 0$ and $A_t + X_t$ is returned to the households, so this no longer appears on the bank's balance sheet. Similarly, working capital loans, S_t^w, and 'old' entrepreneurial loans, B_t, are liquidated at the end of the goods market and also do not appear on the bank's balance sheet. At this point, the assets on the bank's balance sheet are the new entrepreneurial loans issued at the end of the goods market, B_{t+1}, and the bank liabilities are the new time deposits, T_t.

At the end of the goods market, the bank settles claims for transactions that occured in the goods market and that arose from it's activities in the previous period's entrepreneurial loan and time deposit market. The bank's sources of funds at this time are: net interest from borrowers and $A_t + X_t$ of high-powered money (i.e., a mix of vault cash and claims on the central bank).[12] Working capital loans coming due at the end of the period pay R_t in interest and so the associated principal and interest is

$$(1 + R_t)S_t^w = (1 + R_t)\left(\psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t\right).$$

Loans to entrepreneurs coming due at the end of the period are the ones that were extended in the previous period, $Q_{\bar{K}',t-1}\bar{K}_t - N_t$, and they pay the interest rate from the previous period, after monitoring costs:

$$(1 + R_t^e)\left(Q_{\bar{K}',t-1}\bar{K}_t - N_t\right)$$

Its uses of funds are (i) interest and principle obligations on demand deposits and time deposits, $(1 + R_{at})D_t$ and $(1 + R_t^e)T_{t-1}$, respectively, and (ii) interest and principal expenses

[12]For now, we suppose that interest is not paid by the central bank on high-powered money.

on working capital, i.e., capital and labor services. Interest and principal expenses on factor payments in the banking sector are handled in the same way as in the goods sector. In particular, banks must finance a fraction, $\psi_{k,t}$, of capital services and a fraction, $\psi_{l,t}$, of labor services, in advance, so that total factor costs as of the end of the period, are $(1+\psi_{k,t}R_t)\,P_t r_t^k K_t^b$. The bank's net source of funds, Π_t^b, is:

$$\begin{aligned}\Pi_t^b \;=\;& (A_t+X_t)+(1+R_t+R_{at})S_t^w-(1+R_{at})D_t \\ & -\left[(1+\psi_{k,t}R_t)\,P_t r_t^k K_t^b\right]-\left[(1+\psi_{l,t}R_t)\,W_t l_t^b\right] \\ & +\left[1+R_t^e+\frac{\mu\int_0^{\bar\omega_t}\omega dF(\omega)\left(1+R_t^k\right)Q_{\bar K',t-1}\bar K_t}{Q_{\bar K',t-1}\bar K_t-N_t}\right]B_t \\ & -\mu\int_0^{\bar\omega_t}\omega dF(\omega)\left(1+R_t^k\right)Q_{\bar K',t-1}\bar K_t-\left(1+R_t^e\right)T_{t-1} \\ & +T_t-B_{t+1}\end{aligned}$$

Because of competition, the bank takes all wages and prices and interest rates as given and beyond its control.

We now describe the bank's optimization problem. The bank pays Π_t^b to households in the form of dividends. It's objective is to maximize the present discounted value of these dividends. In period 0, it's objective is:

$$E_0\sum_{t=0}^{\infty}\beta^t\lambda_t\Pi_t^b,$$

where λ_t is the multiplier on Π_t^b in the Lagrangian representation of the household's optimization problem. It takes as given its time deposit liabilities from the previous period, T_{-1}, and its entrepreneurial loans issued in the previous period, B_0. In addition, the bank takes all rates of return and λ_t as given. The bank optimizes its objective by choice of $\left\{S_t^w,\ B_{t+1},\ D_t,\ T_t,\ K_t^b,\ E_t^r; t\geq 0\right\}$, subject to (2.27)-(2.29).

In the previous section, we discussed the determination of the variables relating to entrepreneurial loans. There is no further need to discuss them here, and so we take those as given. To discuss the variables of concern here, we adopt a Lagrangian representation of the bank problem which uses a version of (2.30) that ignores variables pertaining to the entrepeneur. The Lagrangian representation of the problem that we work with is:

$$\begin{aligned}&\max_{A_t,S_t^w,K_t^b,l_t^b}\left\{R_tS_t^w-R_{at}\left(A_t+X_t\right)-\left[(1+\psi_{k,t}R_t)\,P_t r_t^k K_t^b\right]-\left[(1+\psi_{l,t}R_t)\,W_t l_t^b\right]\right\} \\ &+\lambda_t^b\left[h(x_t^b,K_t^b,l_t^b,\frac{A_t+X_t-\tau_t\left(A_t+X_t+S_t^w\right)}{P_t},\xi_t,x_t^b,z_t)-\frac{A_t+X_t+S_t^w}{P_t}\right]\end{aligned}$$

$$\max_{A_t,S_t^w,K_t^b,l_t^b}\{R_tS_t^w - R_{at}\left(A_t+X_t\right) - R_t^bF_t - \left[\left(1+\psi_{k,t}R_t\right)P_tr_t^kK_t^b\right] - \left[\left(1+\psi_{l,t}R_t\right)W_tl_t^b\right]\}$$
$$+\lambda_t^b\left[h(x_t^b,K_t^b,l_t^b,\frac{A_t+X_t+F_t-\tau_t\left(A_t+X_t+S_t^w\right)}{P_t},\xi_t,x_t^b,z_t) - \frac{\left(A_t+X_t+S_t^w\right)}{P_t}\right]$$

fonc for F_t :

$$R_t^b = \lambda_t^b h_{e,t}\frac{1}{P_t} = \frac{R_t h_{e^r,t}}{\tau_t h_{e^r,t}+1}$$

where

$$\begin{aligned} h(x_t^b,K_t^b,l_t^b,e_t^r,\xi_t,x_t^b,z_t) &= a^b x_t^b\left(\left(K_t^b\right)^{\alpha}\left(z_tl_t^b\right)^{1-\alpha}\right)^{\xi_t}\left(e_t^r\right)^{1-\xi_t} \\ e_t^r &= \frac{E_t^r}{P_t} = \frac{A_t+X_t-\tau_t\left(A_t+X_t+S_t^w\right)}{P_t} \end{aligned}$$

The first order conditions are, for A_t, S_t^w, K_t^b, l_t^b, respectively:

$$-R_{at}+\lambda_t^b\frac{1}{P_t}\left[\left(1-\tau_t\right)h_{e^r,t}-1\right] = 0 \quad (2.30)$$
$$R_t-\lambda_t^b\frac{1}{P_t}\left[\tau_t h_{e^r,t}+1\right] = 0 \quad (2.31)$$
$$-\left(1+\psi_{k,t}R_t\right)P_tr_t^k+\lambda_t^b h_{K^b,t} = 0 \quad (2.32)$$
$$-\left(1+\psi_{l,t}R_t\right)W_t+\lambda_t^b h_{l^b,t} = 0 \quad (2.33)$$

Substituting for λ_t^b in (2.32) and (2.33) from (2.31), we obtain:

$$\left(1+\psi_{k,t}R_t\right)r_t^k = \frac{R_t h_{K^b,t}}{1+\tau_t h_{e^r,t}},$$

and

$$\left(1+\psi_{l,t}R_t\right)\frac{W_t}{P_t} = \frac{R_t h_{l^b,t}}{1+\tau_t h_{e^r,t}}.$$

These are the first order conditions associated with the bank's choice of capital and labor. Each says that the bank attempts to equate the marginal product - in terms of extra loans - of an additional factor of production, with the associated marginal cost. The marginal product in producing loans must take into account two things: an increase in S^w requires an equal increase in deposits and an increase in deposits raises required reserves. The first raises loans by the marginal product of the factor in h, while the reserve implication works in the other direction. To see that the capital-labor ratio in the banking and intermediate good sectors coincide, take the ratio of the above two equations:

$$\frac{(1+\psi_{k,t}R_t)\,r_t^k}{(1+\psi_{l,t}R_t)\,\frac{W_t}{P_t}} = \frac{h_{K^b,t}}{h_{l^b,t}} = \frac{\alpha\xi_t a^b x_t^b\,(e_{v,t})^{1-\xi_t}\left(\frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t}\right)^{1-\alpha}}{(1-\alpha)\,\xi_t a^b x_t^b\,(e_{v,t})^{1-\xi_t}\left(\frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t}\right)^{-\alpha}z_t}$$

$$= \frac{\alpha}{(1-\alpha)\,z_t}\left(\frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t}\right).$$

From (2.4),

$$\frac{r_t^k\,[1+\psi_{k,t}R_t]}{(1+\psi_{l,t}R_t)\,\frac{W_t}{P_t}} = \frac{\alpha\epsilon_t\left(\frac{z_t\nu^l l_t}{\nu^k K_t}\right)^{1-\alpha}}{(1-\alpha)\epsilon_t\left(\frac{z_t\nu^l l_t}{\nu^k K_t}\right)^{-\alpha}z_t} = \frac{\alpha}{(1-\alpha)z_t}\frac{z_t\nu^l l_t}{\nu^k K_t}$$

$$= \frac{\alpha}{(1-\alpha)z_t}\frac{z_t\nu^l l_t}{\nu^k z_{t-1}k_t}$$

$$= \frac{\alpha}{(1-\alpha)z_t}\frac{\mu_{z,t}\nu^l l_t}{\nu^k k_t}.$$

Equating the previous two expressions, and cancelling:

$$\frac{\nu_t^k}{(1-\nu_t^k)} = \frac{\nu_t^l}{(1-\nu_t^l)}.$$

Note that the object on the left and right are each monotone increasing functions of ν_t^l and ν_t^k, respectively. As a result, they can only be equal for $\nu_t^l = \nu_t^k$.

Taking the ratio of (2.31) to (2.30), we obtain:

$$R_{at} = \frac{(1-\tau_t)\,h_{e^r,t}-1}{\tau_t h_{e^r,t}+1}R_t. \tag{2.34}$$

This can be thought of as the first order condition associated with the bank's choice of A_t. The object multiplying R_t is the increase in S^w the bank can offer for one unit increase in A. The term on the right indicates the net interest earnings from those loans. The term on the left indicates the cost. Recall that R_t represents *net* interest on loans, because the actual interest is $R_t + R_{at}$, so that R_t represents the spread between the interest rate charged by banks on their loans and the cost to them of the underlying funds. Since loans are made in the form of deposits, and deposits earn R_{at} in interest, the net cost of a loan to a borrower is R_t.

We now proceed to linearize. For this, we need expressions for the derivatives of h with respect to capital, labor and excess reserves. One expression that appears in all of these is the ratio of real excess reserves to value-added, which we denote by $e_{v,t}$:

$$
\begin{aligned}
e_{v,t} &= \frac{\frac{A_t+X_t-\tau_t(A_t+X_t+S_t^w)}{P_t}}{\left(z_t\frac{z_{t-1}}{z_t}k_t^b\right)^{\alpha}\left(z_tl_t^b\right)^{1-\alpha}} \\
&= \frac{\frac{A_t+X_t-\tau_t(A_t+X_t+S_t^w)}{z_tP_t}}{\left(\frac{1}{\mu_{z,t}}k_t^b\right)^{\alpha}\left(l_t^b\right)^{1-\alpha}} \\
&= \frac{\frac{M_t^b-M_t+X_t-\tau_t\left(M_t^b-M_t+X_t+S_t^w\right)}{z_tP_t}}{\left(\frac{1}{\mu_{z,t}}k_t^b\right)^{\alpha}\left(l_t^b\right)^{1-\alpha}} \\
&= \frac{M_t^b}{P_tz_t}\frac{1-m_t+x_t-\tau_t\left(1-m_t+x_t+\psi_{l,t}w_t\frac{z_tP_t}{M_t^b}l_t+\psi_{k,t}\frac{z_tP_t}{M_t^b}r_t^k\frac{1}{\mu_{z,t}}k_t\right)}{\left(\frac{1}{\mu_{z,t}}k_t^b\right)^{\alpha}\left(l_t^b\right)^{1-\alpha}} \\
&= m_t^b\frac{1-m_t+x_t-\tau_t\left(1-m_t+x_t+\frac{\psi_{l,t}w_t}{m_t^b}l_t+\frac{\psi_{k,t}r_t^k}{m_t^b}\frac{1}{\mu_{z,t}}k_t\right)}{\left(\frac{1}{\mu_{z,t}}k_t^b\right)^{\alpha}\left(l_t^b\right)^{1-\alpha}} \\
&= \frac{\left(1-\tau_t\right)m_t^b\left(1-m_t+x_t\right)-\tau_t\left(\psi_{l,t}w_tl_t+\frac{1}{\mu_{z,t}}\psi_{k,t}r_t^kk_t\right)}{\left(\frac{1}{\mu_{z,t}}(1-\nu_t^k)k_t\right)^{\alpha}\left((1-\nu_t^l)l_t\right)^{1-\alpha}}
\end{aligned}
$$

To linearize this, it is useful to first linearize the numerator, n_t, and denominator, d_t, separately. Thus,

$$
\begin{aligned}
n_t &= \left(1-\tau_t\right)m_t^b\left(1-m_t+x_t\right)-\tau_t\left(\psi_{l,t}w_tl_t+\frac{1}{\mu_{z,t}}\psi_{k,t}r_t^kk_t\right) \\
d_t &= \left(\frac{1}{\mu_{z,t}}(1-\nu_t^k)k_t\right)^{\alpha}\left((1-\nu_t^l)l_t\right)^{1-\alpha}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
n\hat{n}_t &= -\tau m^b\left(1-m+x\right)\hat{\tau}_t+\left(1-\tau\right)m^b\left(1-m+x\right)\hat{m}_t^b \\
&\quad -\left(1-\tau\right)m^bm\hat{m}_t+\left(1-\tau\right)m^bx\hat{x}_t-\tau\left(\psi_lwl+\frac{1}{\mu_z}\psi_kr^kk\right)\hat{\tau}_t \\
&\quad -\tau\psi_lwl\left[\hat{\psi}_{l,t}+\hat{w}_t+\hat{l}_t\right]-\tau\frac{1}{\mu_z}\psi_kr^kk\left[-\hat{\mu}_{z,t}+\hat{\psi}_{k,t}+\hat{r}_t^k+\hat{k}_t\right],
\end{aligned}
$$

or,

$$\begin{aligned}\hat{n}_t &= n_\tau \hat{\tau}_t + n_{m^b}\hat{m}_t^b + n_m \hat{m}_t + n_x \hat{x}_t \\ &\quad + n_{\psi_l}\hat{\psi}_{l,t} + n_{\psi k}\hat{\psi}_{k,t} + n_k \hat{k}_t \\ &\quad + n_{r^k}\hat{r}_t^k + n_w \hat{w}_t + n_l \hat{l}_t + n_{\mu_z}\hat{\mu}_{z,t}\end{aligned}$$

where

$$\begin{aligned}n_\tau &= \frac{-\tau m^b\left(1-m+x\right)-\tau\left(\psi_l wl+\frac{1}{\mu_z}\psi_k r^k k\right)}{n}, \\ n &= \left(1-\tau\right)m^b\left(1-m+x\right)-\tau\left(\psi_l wl+\frac{1}{\mu_z}\psi_k r^k k\right), \\ n_{m^b} &= \left(1-\tau\right)m^b\left(1-m+x\right)/n \\ n_m &= -\left(1-\tau\right)m^b m/n \\ n_x &= \left(1-\tau\right)m^b x/n \\ n_{\psi_l} &= n_w = n_l = -\tau\psi_l wl/n \\ n_{\psi_k} &= n_{r^k} = n_k = -\tau\frac{1}{\mu_z}\psi_k r^k k/n \\ n_{\mu_z} &= \tau\frac{1}{\mu_z}\psi_k r^k k/n\end{aligned}$$

Turning to the denominator of $e_{v,t}$,

$$\begin{aligned}d\hat{d}_t &= \alpha\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^\alpha\left((1-\nu^l)l\right)^{1-\alpha}\left[-\hat{\mu}_{z,t}+\frac{-\nu^k\hat{\nu}_t^k}{1-\nu^k}+\hat{k}_t\right] \\ &\quad +(1-\alpha)\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^\alpha\left((1-\nu^l)l\right)^{1-\alpha}\left[\frac{-\nu^l\hat{\nu}_t^l}{1-\nu^l}+\hat{l}_t\right],\end{aligned}$$

or,

$$\hat{d}_t = d_{\mu_z}\hat{\mu}_{z,t} + d_k\hat{k}_t + d_{\nu^k}\hat{\nu}_t^k + d_{\nu l}\hat{\nu}_t^l + d_l\hat{l}_t,$$

where

$$
\begin{aligned}
d &= \left(\frac{1}{\mu_z}(1-\nu^k)k\right)^{\alpha}\left((1-\nu^l)l\right)^{1-\alpha} \\
d_{\mu_z} &= \frac{-\alpha\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^{\alpha}\left((1-\nu^l)l\right)^{1-\alpha}}{\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^{\alpha}\left((1-\nu^l)l\right)^{1-\alpha}} = -\alpha \\
d_k &= \alpha \\
d_{\nu^k} &= -\alpha\frac{\nu^k}{1-\nu^k} \\
d_l &= 1-\alpha \\
d_{\nu^l} &= -(1-\alpha)\frac{\nu^l}{1-\nu^l}
\end{aligned}
$$

Also, since the capital labor ratios in the banking sector and the rest of the economy are the same, $(\nu^k = \nu^l)$ we know that

$$
\hat{\nu}_t^k = \hat{\nu}_t^l. \tag{2.35}
$$

Then, with $e_{v,t} = n_t/d_t$,

$$
\hat{e}_{v,t} = \hat{n}_t - \hat{d}_t,
$$

or,

$$
\begin{aligned}
(**)\ \hat{e}_{v,t} &= n_\tau\hat{\tau}_t + n_{m^b}\hat{m}_t^b + n_m\hat{m}_t + n_x\hat{x}_t + n_{\psi_l}\hat{\psi}_{l,t} \\
&\quad + n_{\psi k}\hat{\psi}_{k,t} + (n_k - d_k)\,\hat{k}_t + n_{r^k}\hat{r}_t^k + n_w\hat{w}_t \\
&\quad + (n_l - d_l)\,\hat{l}_t + (n_{\mu_z} - d_{\mu_z})\,\hat{\mu}_{z,t} - d_{\nu^k}\hat{\nu}_t^k - d_{\nu^l}\hat{\nu}_t^l,
\end{aligned}
$$

where (2.35) is to be recalled. It is also useful to have an expression for $z_t l_t^b / K_t^b$:

$$
l_t^k = \frac{z_t l_t^b}{K_t^b} = \frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t},
$$

linearizing this:

$$
\hat{l}_t^k = \hat{\mu}_{z,t} - \frac{\nu^l\hat{\nu}_t^l}{1-\nu^l} + \hat{l}_t + \frac{\nu^k\hat{\nu}_t^k}{1-\nu^k} - \hat{k}_t.
$$

The partial derivative of h with respect to capital is:

$$
h_{K^b,t} = \alpha\xi_t a^b x_t^b \left(e_{v,t}\right)^{1-\xi_t}\left(\frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t}\right)^{1-\alpha}.
$$

Linearizing this:

$$\hat{h}_{k^b,t} = [1 - \log(e_v)\,\xi]\,\hat{\xi}_t + \hat{x}_t^b + (1-\xi)\,\hat{e}_{v,t} + (1-\alpha)\left[\hat{\mu}_{z,t} - \frac{\nu^l \hat{\nu}_t^l}{1-\nu^l} + \hat{l}_t + \frac{\nu^k \hat{\nu}_t^k}{1-\nu^k} - \hat{k}_t\right]$$

The derivative of h with respect to excess reserves is:

$$h_{e^r,t} = (1-\xi_t)\,a^b x_t^b\,(e_{v,t})^{-\xi_t}.$$

We now linearize this. First, note:

$$\begin{aligned}
f(\xi_t) &= (1-\xi_t)\,a^b x_t^b\,(e_{v,t})^{-\xi_t} \\
df(\xi_t) &= -a^b x^b\,(e_{v,t})^{-\xi} + (1-\xi)\,a^b x^b d\exp[-\xi_t \log(e_{v,t})]d\xi_t \\
&= \left[-a^b x^b\,(e_v)^{-\xi} - (1-\xi)\,a^b x^b \log(e_v)\,(e_v)^{-\xi}\right] d\xi_t \\
&= -f\left[\frac{1}{1-\xi} + \log(e_v)\right] d\xi_t \\
\hat{f}(\xi_t) &= \frac{df(\xi_t)}{f} = -\left[\frac{1}{1-\xi} + \log(e_v)\right]\xi\hat{\xi}_t.
\end{aligned}$$

Using this, we obtain:

$$\hat{h}_{e^r,t} = -\left[\frac{1}{1-\xi} + \log(e_v)\right]\xi\hat{\xi}_t + \hat{x}_t^b - \xi\hat{e}_{v,t}$$

The derivative of h with respect to labor is:

$$h_{l^b,t} = (1-\alpha)\,\xi_t a^b x_t^b\,(e_{v,t})^{1-\xi_t}\left(\frac{\mu_{z,t}(1-\nu_t^l)l_t}{(1-\nu_t^k)k_t}\right)^{-\alpha} z_t.$$

Linearizing $h_{z,l^b,t} = h_{l^b,t}/z_t$:

$$\hat{h}_{z,l^b,t} = [1 - \log(e_v)\,\xi]\,\hat{\xi}_t + \hat{x}_t^b + (1-\xi)\,\hat{e}_{v,t} - \alpha\left[\hat{\mu}_{z,t} - \frac{\nu^l \hat{\nu}_t^l}{1-\nu^l} + \hat{l}_t + \frac{\nu^k \hat{\nu}_t^k}{1-\nu^k} - \hat{k}_t\right],$$

or,

$$\hat{h}_{k^b,t} = \hat{h}_{z,l^b,t} + \hat{\mu}_{z,t} - \frac{\nu^l \hat{\nu}_t^l}{1-\nu^l} + \hat{l}_t + \frac{\nu^k \hat{\nu}_t^k}{1-\nu^k} - \hat{k}_t \tag{2.36}$$

Always, recall (2.35).

We can use the previous intermediate results to obtain linearizations of the first order conditions for capital and labor, which we repeat here for convenience. The first order condition for capital is:

$$\left(1+\psi_{k,t}R_t\right)r_t^k = \frac{R_t h_{k^b,t}}{1+\tau_t h_{e^r,t}},$$

which in linearized form is:

$$\widehat{(1+\psi_{k,t}R_t)} + \hat{r}_t^k = \hat{R}_t + \hat{h}_{k^b,t} - \widehat{(1+\tau_t h_{e^r,t})}$$

or,

$$\frac{\psi_k R\left[\hat{\psi}_{k,t}+\hat{R}_t\right]}{1+\psi_k R} + \hat{r}_t^k = \hat{R}_t + \hat{h}_{k^b,t} - \frac{\tau h_{e^r}\left[\hat{\tau}_t + \hat{h}_{e^r,t}\right]}{1+\tau h_{e^r}}. \tag{2.37}$$

Substituting,

$$\begin{aligned}\frac{\psi_k R\left[\hat{\psi}_{k,t}+\hat{R}_t\right]}{1+\psi_k R} + \hat{r}_t^k &= \hat{R}_t + \left[1-\log\left(e_v\right)\xi\right]\hat{\xi}_t + \hat{x}_t^b + (1-\xi)\,\hat{e}_{v,t} \\ &\quad +(1-\alpha)\left[\hat{\mu}_{z,t} - \frac{\nu^l\hat{\nu}_t^l}{1-\nu^l} + \hat{l}_t + \frac{\nu^k\hat{\nu}_t^k}{1-\nu^k} - \hat{k}_t\right] \\ &\quad -\frac{\tau h_{e^r}\left(\hat{\tau}_t - \left[\frac{1}{1-\xi}+\log\left(e_v\right)\right]\xi\hat{\xi}_t + \hat{x}_t^b - \xi\hat{e}_{v,t}\right)}{1+\tau h_{e^r}}.\end{aligned}$$

Collecting terms:

$$\begin{aligned}(**)\ 0 &= k_R\hat{R}_t + k_\xi\hat{\xi}_t - \hat{r}_t^k + k_x\hat{x}_t^b + k_e\hat{e}_{v,t} + k_\mu\hat{\mu}_{z,t} \\ &\quad + k_{\nu^l}\hat{\nu}_t^l + k_{\nu^k}\hat{\nu}_t^k + k_l\hat{l}_t + k_k\hat{k}_t + k_\tau\hat{\tau}_t + k_{\psi_k}\hat{\psi}_{k,t}\end{aligned} \tag{2.38}$$

where

$$\begin{aligned}k_R &= \left[1-\frac{\psi_k R}{1+\psi_k R}\right],\ k_\xi = 1-\log\left(e_v\right)\xi + \frac{\tau h_{e^r}\left[\frac{1}{1-\xi}+\log\left(e_v\right)\right]\xi}{1+\tau h_{e^r}} \\ k_x &= \frac{1}{1+\tau h_{e^r}},\ k_e = 1-\xi+\frac{\tau h_{e^r}\xi}{1+\tau h_{e^r}},\ k_\mu = (1-\alpha) \\ k_{\nu^l} &= -(1-\alpha)\frac{\nu^l}{1-\nu^l},\ k_{\nu^k} = (1-\alpha)\frac{\nu^k}{1-\nu^k},\ k_l = (1-\alpha),\ k_k = -(1-\alpha) \\ k_\tau &= -\frac{\tau h_{e^r}}{1+\tau h_{e^r}},\ k_{\psi_k} = -\frac{\psi_k R}{1+\psi_k R}.\end{aligned}$$

The first order condition for labor is:

$$0 = \frac{R_t h_{z,l^b,t}}{1 + \tau_t h_{e^r,t}} - (1 + \psi_{l,t} R_t)\, w_t.$$

Linearizing this:

$$(1 \widehat{+ \psi_{l,t}} R_t) + \hat{w}_t = \hat{R}_t + \hat{h}_{z,l^b,t} - (1 \widehat{+ \tau_t h_{e^r}},_t)$$

or,

$$\frac{\psi_l R \left(\hat{\psi}_{l,t} + \hat{R}_t\right)}{1 + \psi_l R} + \hat{w}_t = \hat{R}_t + \hat{h}_{z,l^b,t} - \frac{\tau h_{e^r} \left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{1 + \tau h_{e^r}}$$

Substituting from (2.36),

$$\frac{\psi_k R \left[\hat{\psi}_{k,t} + \hat{R}_t\right]}{1 + \psi_k R} + \hat{r}_t^k = \hat{R}_t + \hat{h}_{k^b,t} - \frac{\tau h_{e^r} \left[\hat{\tau}_t + \hat{h}_{e^r,t}\right]}{1 + \tau h_{e^r}}$$

$$\begin{aligned} \frac{\psi_l R \left(\hat{\psi}_{l,t} + \hat{R}_t\right)}{1 + \psi_l R} + \hat{w}_t &= \hat{R}_t + \hat{h}_{k^b,t} - \frac{\tau h_{e^r} \left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{1 + \tau h_{e^r}} \\ &\quad - \left[\hat{\mu}_{z,t} - \frac{\nu^l \hat{\nu}_t^l}{1 - \nu^l} + \hat{l}_t + \frac{\nu^k \hat{\nu}_t^k}{1 - \nu^k} - \hat{k}_t\right] \end{aligned}$$

Note that the first line of this looks like (2.37) with ψ_l replacing ψ_k and $\hat{w}_t$ replacing $\hat{r}_t^k$. We can exploit this fact when collecting terms in the previous expression. In particular,

$$\begin{aligned} (**)\ 0 &= l_R \hat{R}_t + l_\xi \hat{\xi}_t - \hat{w}_t + l_x \hat{x}_t^b + l_e \hat{e}_{v,t} + l_\mu \hat{\mu}_{z,t} \\ &\quad + l_{\nu^l} \hat{\nu}_t^l + l_{\nu^k} \hat{\nu}_t^k + l_l \hat{l}_t + l_k \hat{k}_t + l_\tau \hat{\tau}_t + l_{\psi_l} \hat{\psi}_{l,t}, \end{aligned}$$

where

$$\begin{aligned} l_i &= k_i \text{ for all } i, \text{ except} \\ l_R &= \left[1 - \frac{\psi_l R}{1 + \psi_l R}\right],\ l_{\psi l} = -\frac{\psi_l R}{1 + \psi_l R} \\ l_\mu &= k_\mu - 1,\ l_{\nu^l} = k_{\nu^l} + \frac{\nu^l}{1 - \nu^l},\ l_l = k_l - 1, \\ l_{\nu^k} &= k_{\nu^k} - \frac{\nu^k}{1 - \nu^k},\ l_k = k_k + 1. \end{aligned}$$

The production function for deposits is:

$$a^b x_t^b \left(e_{v,t}\right)^{-\xi_t} e_t^r = \frac{M_t^b - M_t + X_t + S_t^w}{P_t}$$

Scaling this:

$$\begin{aligned}
a^b x_t^b \left(e_{v,t}\right)^{-\xi_t} \frac{e_t^r}{z_t} &= \frac{M_t^b - M_t + X_t + \left(\psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t\right)}{z_t P_t} \\
&= \frac{m_t^b z_t P_t - \left(M_t/M_t^b\right) M_t^b + \left(X_t/M_t^b\right) M_t^b + \left(\psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k z_t \left(z_{t-1}/z_t\right) k_t\right)}{z_t P_t} \\
&= m_t^b \left(1 - m_t + x_t\right) + \psi_{l,t} w_t l_t + \psi_{k,t} r_t^k k_t / \mu_{zt} \\
&= m_{1t} + m_{2t},
\end{aligned}$$

where

$$\begin{aligned}
m_{1t} &= m_t^b \left(1 - m_t + x_t\right) \\
m_{2t} &= \psi_{l,t} w_t l_t + \psi_{k,t} r_t^k k_t / \mu_{zt}.
\end{aligned}$$

Linearing these pieces:

$$\begin{aligned}
\hat{m}_{1t} &= \hat{m}_t^b + (1 - \widehat{m_t + x_t}) \\
&= \hat{m}_t^b + \frac{-m\hat{m}_t + x\hat{x}_t}{1 - m + x} \\
\hat{m}_{2t} &= \frac{\psi_l w l}{\psi_l w l + \psi_k r^k k/\mu_z} \left(\hat{\psi}_{l,t} + \hat{w}_t + \hat{l}_t\right) \\
&\quad + \frac{\psi_k r^k k/\mu_z}{\psi_l w l + \psi_k r^k k/\mu_z} \left(\hat{\psi}_{k,t} + \hat{r}_t^k + \hat{k}_t - \hat{\mu}_{zt}\right)
\end{aligned}$$

It is convenient to derive an expression for the linearization of e_t^r / z_t.

$$\begin{aligned}
e_{z,t}^r &= \frac{E_t^r}{P_t z_t} = \frac{e_t^r}{z_t} = \frac{M_t^b - M_t + X_t - \tau_t \left(M_t^b - M_t + X_t + S_t^w\right)}{z_t P_t} \\
&= (1 - \tau_t) m_t^b \left(1 - m_t + x_t\right) - \tau_t \left(\psi_{l,t} w_t l_t + \psi_{k,t} r_t^k k_t / \mu_{zt}\right) \\
&= (1 - \tau_t) m_{1t} - \tau_t m_{2t}
\end{aligned}$$

Linearizing this:

$$\hat{e}_{z,t}^r = \frac{(1-\tau) m_1}{(1-\tau) m_1 - \tau m_2} \left[\frac{-\tau \hat{\tau}_t}{1 - \tau} + \hat{m}_{1t}\right] - \frac{\tau m_2}{(1-\tau) m_1 - \tau m_2} \left[\hat{\tau}_t + \hat{m}_{2t}\right].$$

We can use these results in linearly expanding the scaled production function, which we repeat here for convenience:

$$a^b x_t^b \left(e_{v,t}\right)^{-\xi_t} e_{z,t}^r = m_{1t} + m_{2t}.$$

Expanding this:

$$\hat{x}_t^b - \xi \hat{e}_{v,t} - \log\left(e_{v,t}\right) \xi \hat{\xi}_t + \hat{e}_{z,t}^r = \frac{m_1}{m_1 + m_2} \hat{m}_{1t} + \frac{m_2}{m_1 + m_2} \hat{m}_{2t}.$$

Substituting out:

$$\begin{aligned} & \hat{x}_t^b - \xi \hat{e}_{v,t} - \log\left(e_{v,t}\right) \xi \hat{\xi}_t + \frac{(1-\tau) m_1}{(1-\tau) m_1 - \tau m_2} \left[\frac{-\tau \hat{\tau}_t}{1-\tau} + \hat{m}_{1t} \right] - \frac{\tau m_2}{(1-\tau) m_1 - \tau m_2} \left[\hat{\tau}_t + \hat{m}_{2t} \right] \\ = \quad & \frac{m_1}{m_1 + m_2} \hat{m}_{1t} + \frac{m_2}{m_1 + m_2} \hat{m}_{2t}. \end{aligned}$$

Collecting terms:

$$\begin{aligned} & \hat{x}_t^b - \xi \hat{e}_{v,t} - \log\left(e_{v,t}\right) \xi \hat{\xi}_t - \frac{\tau\left(m_1 + m_2\right)}{(1-\tau) m_1 - \tau m_2} \hat{\tau}_t \\ = \quad & \left[\frac{m_1}{m_1 + m_2} - \frac{(1-\tau) m_1}{(1-\tau) m_1 - \tau m_2} \right] \hat{m}_{1t} + \left[\frac{m_2}{m_1 + m_2} + \frac{\tau m_2}{(1-\tau) m_1 - \tau m_2} \right] \hat{m}_{2t}. \end{aligned}$$

Substituting:

$$\begin{aligned} (**) \qquad & \hat{x}_t^b - \xi \hat{e}_{v,t} - \log\left(e_{v,t}\right) \xi \hat{\xi}_t - \frac{\tau\left(m_1 + m_2\right)}{(1-\tau) m_1 - \tau m_2} \hat{\tau}_t \\ = \quad & \left[\frac{m_1}{m_1 + m_2} - \frac{(1-\tau) m_1}{(1-\tau) m_1 - \tau m_2} \right] \left[\hat{m}_t^b + \frac{-m \hat{m}_t + x \hat{x}_t}{1 - m + x} \right] \\ & + \left[\frac{m_2}{m_1 + m_2} + \frac{\tau m_2}{(1-\tau) m_1 - \tau m_2} \right] \\ & \times [\frac{\psi_l w l}{\psi_l w l + \psi_k r^k k / \mu_z} \left(\hat{\psi}_{l,t} + \hat{w}_t + \hat{l}_t \right) + \frac{\psi_k r^k k / \mu_z}{\psi_l w l + \psi_k r^k k / \mu_z} \left(\hat{\psi}_{k,t} + \hat{r}_t^k + \hat{k}_t - \hat{\mu}_{zt} \right)]. \end{aligned}$$

We now linearize the equation linking R_{at} and R_t

$$R_{at} = \frac{\left(1 - \tau_t\right) h_{e^r,t} - 1}{\tau_t h_{e^r,t} + 1} R_t.$$

$$\hat{R}_{at} = \left(\left(1 - \widehat{\tau_t) h_{e^r,t}} - 1 \right) + \hat{R}_t - \left(\tau_t \widehat{h_{e^r,t} + 1} \right)$$

But,

$$
\begin{aligned}
\left(\left(1-\widehat{\tau_t\right) h_{e^r,t}} - 1 \right) &= \frac{h_{e^r}\hat{h}_{e^r,t} - \tau h_{e^r}\left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{\left(1-\tau\right) h_{e^r} - 1} \\
\left(\widehat{\tau_t h_{e^r,t} + 1}\right) &= \frac{\tau h_{e^r}\left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{\tau h_{e^r} + 1},
\end{aligned}
$$

then, using the previous expressions, as well as the expression for $\hat{h}_{e^r,t}$ obtained above,

$$
\begin{aligned}
(**)\ \hat{R}_{at} &= \frac{h_{e^r}\hat{h}_{e^r,t} - \tau h_{e^r}\left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{\left(1-\tau\right) h_{e^r} - 1} + \hat{R}_t - \frac{\tau h_{e^r}\left(\hat{\tau}_t + \hat{h}_{e^r,t}\right)}{\tau h_{e^r} + 1} \\
&= \left[\frac{h_{e^r} - \tau h_{e^r}}{\left(1-\tau\right) h_{e^r} - 1} - \frac{\tau h_{e^r}}{\tau h_{e^r} + 1}\right]\hat{h}_{e^r,t} - \left[\frac{\tau h_{e^r}}{\left(1-\tau\right) h_{e^r} - 1} + \frac{\tau h_{e^r}}{\tau h_{e^r} + 1}\right]\hat{\tau}_t + \hat{R}_t \\
&= \left[\frac{h_{e^r} - \tau h_{e^r}}{\left(1-\tau\right) h_{e^r} - 1} - \frac{\tau h_{e^r}}{\tau h_{e^r} + 1}\right]\left[-\left(\frac{1}{1-\xi} + \log\left(e_v\right)\right)\xi\hat{\xi}_t + \hat{x}_t^b - \xi\hat{e}_{v,t}\right] \\
&\quad - \left[\frac{\tau h_{e^r}}{\left(1-\tau\right) h_{e^r} - 1} + \frac{\tau h_{e^r}}{\tau h_{e^r} + 1}\right]\hat{\tau}_t + \hat{R}_t
\end{aligned}
$$

This completes the linearization of the four banking equations. The variables associated with these are $\hat{e}_{v,t}$, $\hat{\nu}_t^k$, $\hat{\nu}_t^l$, $\hat{R}_t$.

The clearing condition in the market for working capital loans is:

$$
S_t^w = \psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t \tag{2.39}
$$

Here, S_t^w represents the supply of loans, and the terms on the right of the equality in (2.39) represent total demand.

We close our discussion of the banking sector with an illustration to clarify the nature of the money multiplier in the model. Consider the following table with gross assets and liabilities for the banking sector as a whole:

Assets	Liabilities
X=50	D=1000
A=200	T=100
S^w=750	
B=100	

The assets and liabilities in this balance sheet are 'gross' in that they do not net out interbank claims. Thus, D includes demand deposits liabilities by some banks to others, corresponding to bank working capital loans. The corresponding assets are included on the asset side of the balance sheet in S. Suppose the reserve requirement, τ, is 0.20. The balance sheet shows that total reserves are 250, of which 200 are required and 50 are excess. So, the excess reserve ratio in this example, i.e., the ratio of excess reserves to deposits, is 5 percent. To see how the money multiplier works, suppose X were 60 instead of 50,while A and the excess reserve ratio do not change. Then the balance sheet would look like this:

Assets	Liabilities
X=60	D=1040
A=200	T=100
S^w=780	
B=100	

Now, total reserves are 260, of which $\tau D = 208$ are required and the rest, 52, are excess. So, in this example, the money multiplier is $1/(.2 + .05) = 4$. In our model, D, A, T, S and the excess reserve ratio are all endogenous variables.

We adjust the bank's production function so that it is expressed in terms of people, rather than the homogeneous labor produced by the contractor. First, write this in terms of the aggregate factor inputs and shares used in the banking sector:

$$\begin{aligned} \frac{D_t}{P_t} &= a^b x_t^b \left(\left(K_t^b\right)^\alpha \left(z_t l_t^b\right)^{1-\alpha} \right)^{\xi_t} \left(\frac{E_t^r}{P_t}\right)^{1-\xi_t} \\ &= a^b x_t^b \left(\left(\left(1-\nu_t^k\right) K_t\right)^\alpha \left(z_t \left(1-\nu_t^l\right) l_t\right)^{1-\alpha} \right)^{\xi_t} \left(\frac{E_t^r}{P_t}\right)^{1-\xi_t} \end{aligned}$$

We express this in terms of unweighted hours of households using a result from section 2.9

$$\frac{D_t}{P_t} = a^b x_t^b \left(\left(\left(1-\nu_t^k\right) K_t\right)^\alpha \left(z_t \left(1-\nu_t^l\right) \left(\frac{W_t^*}{W_t}\right)^{\frac{\lambda_w}{\lambda_w - 1}} l_t^* \right)^{1-\alpha} \right)^{\xi_t} \left(\frac{E_t^r}{P_t}\right)^{1-\xi_t}$$

In section 2.9, we show that W_t^*/W_t can - to a first approximation - be treated as a constant, equal to unity.

2.6. Households

There is a continuum of households, indexed by $j \in (0, 1)$. Households consume, save and supply a differentiated labor input. They set their wages using the variant on the Calvo (1983) technology described by Erceg, Henderson and Levin (2000).

The sequence of decisions by the household during a period is as follows. First, it makes its consumption decision after the non-monetary shocks are realized. Second, it purchases securities whose payoffs are contingent upon whether it can reoptimize its wage decision. Third, it sets its wage rate after finding out whether it can reoptimize or not. Fourth, the current period monetary action is realized. Fifth, after the monetary action, and before the goods market, the household decides how much of its financial assets to hold in the form of currency and demand deposits. At this point, the time deposits purchased by the household in the previous period are fixed and beyond its control. Sixth, the household goes to the goods market, where labor services are supplied and goods are purchased. Seventh, after the goods market, the household settles claims arising from its goods market experience and makes its current period time deposit decision.

Since the uncertainty faced by the household over whether it can reoptimize its wage is idiosyncratic in nature, households work different amounts and earn different wage rates. So, in principle they are also heterogeneous with respect to consumption and asset holdings. A straightforward extension of arguments in Erceg, Henderson and Levin (2000) and Woodford (1996), establish that the existence of state contingent securities ensures that in equilibrium households are homogeneous with respect to consumption and asset holdings. Reflecting this result, our notation assumes that households are homogeneous with respect to consumption and asset holdings, and heterogeneous with respect to the wage rate that they earn and hours worked. The preferences of the j^{th} household are given by:

$$E_t^j \sum_{l=0}^{\infty} \beta^{l-t} \left\{ u(C_{t+l} - bC_{t+l-1}) - \zeta_{t+l} z(h_{j,t+l}) - \upsilon_{t+l} \frac{\left[\left(\frac{P_{t+l}C_{t+l}}{M_{t+l}} \right)^{\theta_{t+l}} \left(\frac{P_{t+l}C_{t+l}}{D^h_{t+l}} \right)^{1-\theta_{t+l}} \right]^{1-\sigma_q}}{1-\sigma_q} \right\}, \tag{2.40}$$

where E_t^j is the expectation operator, conditional on aggregate and household j idiosyncratic information up to, and including, time $t-1$; C_t denotes time t consumption; and h_{jt} denotes time t hours worked. In order to help assure that our model has a balanced growth path, we specify that u is the natural logarithm. When $b > 0$, (2.40) allows for habit formation in consumption preferences. Various authors, such as Fuhrer (2000), and McCallum and Nelson (1998), have argued that this is important for understanding the monetary transmission mechanism. In addition, habit formation is useful for understanding other aspects of the economy, including the size of the premium on equity. Finally, the term in square brackets captures the notion that currency and demand deposits contribute to utility by providing transactions services. Those services are an increasing function of the level of consumption.

The specification in (2.40) is our 'benchmark' specification of preferences. We also consider a second specification. In one, the ACEL specification, we make the marginal utility of money independent of consumption. To do this, and preserve balanced growth, we replace C_{t+l} in (2.40) by z_{t+l} :

$$E_t^j \sum_{l=0}^{\infty} \beta^{l-t} \left\{ u(C_{t+l} - bC_{t+l-1}) - \zeta_{t+l} z(h_{j,t+l}) - \upsilon_{t+l} \frac{\left[\left(\frac{P_{t+l} z_{t+l}}{M_{t+l}} \right)^{\theta_{t+l}} \left(\frac{P_{t+l} z_{t+l}}{D_{t+l}^h} \right)^{1-\theta_{t+l}} \right]^{1-\sigma_q}}{1-\sigma_q} \right\} \tag{2.41}$$

The basic text below describes analysis of the benchmark specification. As appropriate, we indicate how things change for the ACEL specification.

We now discuss the household's period t uses and sources of funds. Just before the goods market in period t, after the realization of all shocks, the household has M_t^b units of high powered money which it splits into currency, M_t, and deposits with the bank:

$$M_t^b - (M_t + A_t) \geq 0. \tag{2.42}$$

The household deposits A_t with the bank, in exchange for a demand deposit. Demand deposits pay the relatively low interest rate, R_{at}, but offer transactions services.

The central bank credits the household's bank deposit with X_t units of high powered money, which automatically augments the household's demand deposits. So, household demand deposits are D_t^h :

$$D_t^h = A_t + X_t.$$

As noted in the previous section, the household only receives interest on the non-wage component of its demand deposits, since the interest on the wage component is earned by intermediate good firms.

The household also can acquire a time deposit. This can be acquired at the end of the period t goods market and pays a rate of return, $1+R_{t+1}^e$, at the end of the period $t+1$ goods market. The rate of return, R_{t+1}^e, is known at the time that the time deposit is purchased. It is not contingent on the realization of any of the period $t+1$ shocks.

The household also uses its funds to pay for consumption goods, P_tC_t and to acquire high powered money, Q_{t+1}, for use in the following period. Additional sources of funds include profits from producers of capital, Π_t^k, from banks, Π_t^b, from intermediate good firms, $\int \Pi_t^j dj$, and $A_{j,t}$, the net payoff on the state contingent securities that the household purchases to insulate itself from uncertainty associated with being able to reoptimize its wage rate. Households also receive lump-sum transfers, $1-\Theta$, corresponding to the net worth of the $1-\gamma$ entrepreneurs which die in the current period. Finally, the households pay a lump-sum tax to finance the transfer payments made to the γ entrepreneurs that survive and to the $1-\gamma$ newly born entrepeneurs. These observations are summarized in the following asset accumulation equation:

$$\begin{aligned}&\left[1+\left(1-\tau_t^D\right)R_{at}\right]\left(M_t^b-M_t+X_t\right)-T_t\\&-\left(1+\tau_t^c\right)P_tC_t+(1-\Theta)(1-\gamma)V_t-W_t^e+Lump_t\\&+\left[1+\left(1-\tau_t^T\right)R_t^e\right]T_{t-1}+\left(1-\tau_t^l\right)W_{j,t}h_{j,t}+M_t+\Pi_t^b+\Pi_t^k+\int\Pi_t^f df+A_{j,t}-M_{t+1}^b\geq 0.\end{aligned}$$

The household's problem is to maximize (2.40) subject to the timing constraints mentioned above, the various non-negativity constraints, and (2.43).

We consider the Lagrangian representation of the household problem, in which $\lambda_t \geq 0$ is the multiplier on (2.43). The consumption and the wage decisions are taken before the realization of the financial market shocks. The other decisions, M_{t+1}^b, M_t and T_t are taken after the realization of all shocks during the period. The period t multipliers are functions of all the date t shocks. We now consider the first order conditions associated with C_t, M_{t+1}^b, M_t and T_t. The Lagrangian representation of the problem, ignoring constant terms in the asset evolution equation, is:

$$\begin{aligned}&E_0^j\sum_{t=0}^{\infty}\beta^t\{u(C_t-bC_{t-1})-\zeta_t z(h_{j,t})-\upsilon_t\frac{\left[P_tC_t\left(\frac{1}{M_t}\right)^{\theta_t}\left(\frac{1}{M_t^b-M_t+X_t}\right)^{1-\theta_t}\right]^{1-\sigma_q}}{1-\sigma_q}\\&+\lambda_t[\left[1+\left(1-\tau_t^D\right)R_{at}\right]\left(M_t^b-M_t\right)-T_t-\left(1+\tau_t^c\right)P_tC_t\\&+\left[1+\left(1-\tau_t^T\right)R_t^e\right]T_{t-1}+\left(1-\tau_t^l\right)W_{j,t}h_{j,t}+M_t-M_{t+1}^b]\}\end{aligned}$$

We now consider the various first order conditions associated with this maximization problem.

2.6.1. The T_t First Order Condition

The first order condition with respect to T_t is:

$$E\left\{-\lambda_t+\beta\lambda_{t+1}\left[1+\left(1-\tau_{t+1}^T\right)R_{t+1}^e\right]|\Omega_t^\mu\right\}=0$$

To scale this, multiply by z_tP_t :

$$E\left\{-\lambda_{z,t}+\frac{\beta}{\mu_{z,t+1}\pi_{t+1}}\lambda_{z,t+1}\left[1+\left(1-\tau_{t+1}^T\right)R_{t+1}^e\right]|\Omega_t^\mu\right\}=0.$$

Linearize the expression in braces:

$$-\lambda_z\hat{\lambda}_{z,t}+\frac{\beta}{\mu_z\pi}\lambda_z\left[1+\left(1-\tau^T\right)R^e\right]\left[\hat{\lambda}_{z,t+1}-\hat{\mu}_{z,t+1}-\hat{\pi}_{t+1}+\left[1+\widehat{\left(1-\tau_{t+1}^T\right)R_{t+1}^e}\right]\right]$$

Note,

$$\left[1+\left(1-\widehat{\tau_{t+1}^{T}}\right)R_{t+1}^{e}\right]=\frac{-R^{e}\tau^{T}\hat{\tau}_{t+1}^{T}+R^{e}\left(1-\tau^{T}\right)\hat{R}_{t+1}^{e}}{1+\left(1-\tau^{T}\right)R^{e}},$$

so that

$$-\lambda_{z}\hat{\lambda}_{z,t}+\frac{\beta}{\mu_{z}\pi}\lambda_{z}\left[1+\left(1-\tau^{T}\right)R^{e}\right]\left[\hat{\lambda}_{z,t+1}-\hat{\mu}_{z,t+1}-\hat{\pi}_{t+1}+\frac{-R^{e}\tau^{T}\hat{\tau}_{t+1}^{T}+R^{e}\left(1-\tau^{T}\right)\hat{R}_{t+1}^{e}}{1+\left(1-\tau^{T}\right)R^{e}}\right].$$

Imposing the steady state conditions:

$$(**)\ E\left\{-\hat{\lambda}_{z,t}+\hat{\lambda}_{z,t+1}-\hat{\mu}_{z,t+1}-\hat{\pi}_{t+1}-\frac{R^{e}\tau^{T}}{1+\left(1-\tau^{T}\right)R^{e}}\hat{\tau}_{t+1}^{T}+\frac{R^{e}\left(1-\tau^{T}\right)}{1+\left(1-\tau^{T}\right)R^{e}}\hat{R}_{t+1}^{e}|\Omega_{t}^{\mu}\right\}=0.$$

2.6.2. The $\bar{K}_{t+1}$ First Order Condition

Although the capital decision is made by the entrepreneur in the benchmark model, we also explore a more standard formulation in which that decision is made by the household. In this formulation, we drop the variables, $\bar{\omega}_t$ and N_t, and the three equations pertaining to the CSV contract. This leaves us short one equation. Replace this by the following household euler equation:

$$E\left\{-\lambda_{t}+\beta\lambda_{t+1}\left[1+R_{t+1}^{k}\right]|\Omega_{t}\right\}=0.$$

To scale this, we multiply by z_tP_t :

$$E\left\{-\lambda_{zt}+\beta\frac{z_{t}P_{t}}{z_{t+1}P_{t+1}}\lambda_{zt+1}\left[1+R_{t+1}^{k}\right]|\Omega_{t}\right\}=0,$$

or,

$$E\left\{-\lambda_{zt}+\frac{\beta}{\pi_{t+1}\mu_{z,t+1}}\lambda_{zt+1}\left[1+R_{t+1}^{k}\right]|\Omega_{t}\right\}=0.$$

We now linearize the expression in braces:

$$-\lambda_{z}\hat{\lambda}_{zt}+\frac{\beta}{\pi\mu_{z}}\lambda_{z}\left[1+R^{k}\right]\left[\frac{R^{k}}{1+R^{k}}\hat{R}_{t+1}^{k}+\hat{\lambda}_{z,t+1}-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}\right],$$

or, after division by λ_z, setting $\beta\left[1+R^k\right]=\pi\mu_z$, and reinserting the expectation operator:

$$(**)\ E\left\{-\hat{\lambda}_{zt}+\left[\frac{R^{k}}{1+R^{k}}\hat{R}_{t+1}^{k}+\hat{\lambda}_{z,t+1}-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}\right]|\Omega_{t}\right\}$$

2.6.3. The M_t First Order Condition

The first order condition with respect to M_t is:

$$\upsilon_t \left[\left(\frac{P_t C_t}{M_t} \right)^{\theta_t} \left(\frac{P_t C_t}{M_t^b - M_t + X_t} \right)^{1-\theta_t} \right]^{1-\sigma_q} \left[\frac{\theta_t}{M_t} - \frac{(1-\theta_t)}{M_t^b - M_t + X_t} \right] - \lambda_t \left(1 - \tau_t^D\right) R_{at} = 0 \tag{2.43}$$

We can compute a money demand elasticity from this expression. This is the elasticity of demand for M_t with respect to R_{at}. We obtain this by totally differentiating (2.43) with respect to M_t and R_{at}. Rewrite (2.43) a little first:

$$\begin{aligned} & \upsilon_t \left(P_t C_t\right)^{(1-\sigma_q)} M_t^{-\theta_t(1-\sigma_q)} \left(M_t^b - M_t + X_t\right)^{-(1-\theta_t)(1-\sigma_q)} \left[\frac{\theta_t}{M_t} - \frac{(1-\theta_t)}{M_t^b - M_t + X_t} \right] \\ = & \ \lambda_t \left(1 - \tau_t^D\right) R_{at}. \end{aligned}$$

Now, differentiate:

$$\begin{aligned} & \{ -\theta_t \left(1-\sigma_q\right) \frac{\lambda_t \left(1-\tau_t^D\right) R_{at}}{M_t} + \left(1-\theta_t\right)\left(1-\sigma_q\right) \frac{\lambda_t \left(1-\tau_t^D\right) R_{at}}{\left(M_t^b - M_t + X_t\right)} \\ & + \frac{\lambda_t \left(1-\tau_t^D\right) R_{at}}{\frac{\theta_t}{M_t} - \frac{(1-\theta_t)}{M_t^b - M_t + X_t}} \left[-\frac{\theta_t}{M_t^2} - \frac{(1-\theta_t)}{\left(M_t^b - M_t + X_t\right)^2} \right] \} dM_t \\ = & \ \lambda_t \left(1-\tau_t^D\right) dR_{at} \end{aligned}$$

or

$$\begin{aligned} & \lambda_t \left(1-\tau_t^D\right) R_{at} \{ \left(1-\sigma_q\right) \left[-\frac{\theta_t}{M_t} + \frac{1-\theta_t}{M_t^b - M_t + X_t} \right] \\ & + \frac{1}{\frac{\theta_t}{M_t} - \frac{(1-\theta_t)}{M_t^b - M_t + X_t}} \left[-\frac{\theta_t}{M_t^2} - \frac{(1-\theta_t)}{\left(M_t^b - M_t + X_t\right)^2} \right] \} dM_t \\ = & \ \lambda_t \left(1-\tau_t^D\right) dR_{at} \end{aligned}$$

so that the elasticity is:

$$\frac{R_{at}}{M_t} \frac{dM_t}{dR_{at}} = \frac{1/M_t}{\left(1-\sigma_q\right) \left[-\frac{\theta_t}{M_t} + \frac{1-\theta_t}{M_t^b - M_t + X_t} \right] + \frac{1}{\frac{\theta_t}{M_t} - \frac{(1-\theta_t)}{M_t^b - M_t + X_t}} \left[-\frac{\theta_t}{M_t^2} - \frac{(1-\theta_t)}{\left(M_t^b - M_t + X_t\right)^2} \right]}.$$

Simplifying:

$$\begin{aligned}
\frac{R_{at}}{M_t}\frac{dM_t}{dR_{at}} &= \frac{1}{(1-\sigma_q)\left[-\theta_t + M_t\frac{1-\theta_t}{M_t^b - M_t + X_t}\right] + \frac{M_t^2}{\theta_t - M_t\frac{(1-\theta_t)}{M_t^b - M_t + X_t}}\left[-\theta_t - \frac{M_t^2(1-\theta_t)}{\left(M_t^b - M_t + X_t\right)^2}\right]} \\
&= \frac{1}{(1-\sigma_q)\left[-\theta_t + m_t\frac{1-\theta_t}{1-m_t+x_t}\right] + \frac{1}{\theta_t - m_t\frac{(1-\theta_t)}{1-m_t+x_t}}\left[-\theta_t - \frac{m_t^2(1-\theta_t)}{(1-m_t+x_t)^2}\right]} \\
&= \frac{\theta_t - m_t\frac{(1-\theta_t)}{1-m_t+x_t}}{-(1-\sigma_q)\left[\theta_t - m_t\frac{1-\theta_t}{1-m_t+x_t}\right]^2 + \left[-\theta_t - \frac{m_t^2(1-\theta_t)}{(1-m_t+x_t)^2}\right]}
\end{aligned}$$

Multiply (2.43) by $z_t P_t$:

$$\upsilon_t\left[c_t\left(\frac{1}{m_t}\right)^{\theta_t}\left(\frac{1}{1-m_t+x_t}\right)^{1-\theta_t}\right]^{1-\sigma_q}\left[\frac{\theta_t}{m_t} - \frac{1-\theta_t}{1-m_t+x_t}\right]\left(\frac{1}{m_t^b}\right)^{2-\sigma_q} -\lambda_{zt}\left(1-\tau_t^D\right)R_{at} = 0 \tag{2.44}$$

In the ACEL specification, simply replace c_t by unity in the previous expression. Now, we linearize this expression. Consider the first piece:

$$\begin{aligned}
&\upsilon\left[c\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{1-\theta}\right]^{1-\sigma_q}\left[\frac{\theta}{m} - \frac{1-\theta}{1-m+x}\right]\left(\frac{1}{m^b}\right)^{2-\sigma_q} \qquad (2.45)\\
&\times\left\{\hat{\upsilon}_t + \left[c_t\left(\frac{1}{m_t}\right)^{\theta_t}\widehat{\left(\frac{1}{1-m_t+x_t}\right)}^{1-\theta_t}\right]^{1-\sigma_q} + \left[\frac{\theta_t}{m_t} - \frac{\widehat{1-\theta_t}}{1-m_t+x_t}\right] + \widehat{\left(\frac{1}{m_t^b}\right)^{2-\sigma_q}}\right\}
\end{aligned}$$

The first hat in this expression is:

$$\begin{aligned}
&(1-\sigma_q)\left[\hat{c}_t - \theta\hat{m}_t - (1-\theta)(1-\widehat{m_t+}x_t) - \log(m)\,\theta\hat{\theta}_t + \log(1-m+x)\,\theta\hat{\theta}_t\right] \\
=\;&(1-\sigma_q)\left[\hat{c}_t - \theta\hat{m}_t - (1-\theta)\frac{-m\hat{m}_t + x\hat{x}_t}{1-m_t+x_t} - \log(m)\,\theta\hat{\theta}_t + \log(1-m+x)\,\theta\hat{\theta}_t\right]
\end{aligned}$$

The second hat in the expression is:

$$\left[\frac{\theta_t}{m_t} - \frac{\widehat{1-\theta_t}}{1-m_t+x_t}\right] = \frac{\frac{\theta}{m}\left(\hat{\theta}_t - \hat{m}_t\right) + \frac{\theta\hat{\theta}_t}{1-m+x} + \frac{1-\theta}{(1-m+x)^2}\left[-m\hat{m}_t + x\hat{x}_t\right]}{\frac{\theta}{m} - \frac{1-\theta}{1-m+x}}$$

The third hat is:

$$\widehat{\left(\frac{1}{m_t^b}\right)^{2-\sigma_q}} = \frac{-\left(2-\sigma_q\right)\left(\frac{1}{m_t^b}\right)^{1-\sigma_q}\left(\frac{1}{m_t^b}\right)^2 m^b \hat{m}^b}{\left(\frac{1}{m_t^b}\right)^{2-\sigma_q}} = -\left(2-\sigma_q\right)\hat{m}_t^b$$

Substituting these three pieces into (2.45):

$$\begin{aligned}
&\upsilon\left[c\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{1-\theta}\right]^{1-\sigma_q}\left[\frac{\theta}{m}-\frac{1-\theta}{1-m+x}\right]\left(\frac{1}{m^b}\right)^{2-\sigma_q} \\
&\times\{\hat{\upsilon}_t+\left(1-\sigma_q\right)[\hat{c}_t-\theta\hat{m}_t-(1-\theta)\frac{-m\hat{m}_t+x\hat{x}_t}{1-m_t+x_t} \\
&-\log\left(m\right)\theta\hat{\theta}_t+\log\left(1-m+x\right)\theta\hat{\theta}_t] \\
&+\frac{\frac{\theta}{m}\left(\hat{\theta}_t-\hat{m}_t\right)+\frac{\theta\hat{\theta}_t}{1-m+x}+\frac{1-\theta}{(1-m+x)^2}\left[-m\hat{m}_t+x\hat{x}_t\right]}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}-\left(2-\sigma_q\right)\hat{m}_t^b\}
\end{aligned} \tag{2.46}$$

Linearizing the second part of (2.44):

$$\begin{aligned}
&\lambda_z\left(1-\tau^D\right)R_a\left[\hat{\lambda}_{z,t}+\widehat{(1-\tau_t^D)}+\hat{R}_{a,t}\right] \\
=\ &\lambda_z\left(1-\tau^D\right)R_a\left[\hat{\lambda}_{z,t}+\frac{-\tau^D}{1-\tau^D}\hat{\tau}_t^D+\hat{R}_{a,t}\right]
\end{aligned} \tag{2.47}$$

Substituting (2.46) and (2.47) into the linearized version of (2.44):

$$\begin{aligned}
\hat{\upsilon}_t+\left(1-\sigma_q\right)&\left[\hat{c}_t-\theta\hat{m}_t-(1-\theta)\frac{-m\hat{m}_t+x\hat{x}_t}{1-m+x}-\log\left(m\right)\theta\hat{\theta}_t+\log\left(1-m+x\right)\theta\hat{\theta}_t\right] \\
&+\frac{\frac{\theta}{m}\left(\hat{\theta}_t-\hat{m}_t\right)+\frac{\theta\hat{\theta}_t}{1-m+x}+\frac{1-\theta}{(1-m+x)^2}\left[-m\hat{m}_t+x\hat{x}_t\right]}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}-\left(2-\sigma_q\right)\hat{m}_t^b \\
&-\left[\hat{\lambda}_{z,t}+\frac{-\tau^D}{1-\tau^D}\hat{\tau}_t^D+\hat{R}_{a,t}\right]=0
\end{aligned}$$

Collecting terms:

$$
\begin{aligned}
(**)\ \hat{\upsilon}_t + (1-\sigma_q)\,\hat{c}_t + \left[-(1-\sigma_q)\left(\theta-(1-\theta)\frac{m}{1-m+x}\right) - \frac{\frac{\theta}{m}+\frac{1-\theta}{(1-m+x)^2}m}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}\right]\hat{m}_t & \\
-\left[\frac{(1-\sigma_q)(1-\theta)\,x}{1-m+x} - \frac{\frac{1-\theta}{(1-m+x)^2}x}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}\right]\hat{x}_t & \\
+\left[-(1-\sigma_q)\left(\log(m)-\log(1-m+x)\right)+\frac{1+x}{\theta(1+x)-m}\right]\theta\hat{\theta}_t & \\
-(2-\sigma_q)\,\hat{m}_t^b - \left[\hat{\lambda}_{z,t}+\frac{-\tau^D}{1-\tau^D}\hat{\tau}_t^D+\hat{R}_{a,t}\right] & = 0
\end{aligned}
$$

For the ACEL specification, simply set $\hat{c}_t = 0$.

2.6.4. The M_{t+1}^b First Order Condition

The first order condition with respect to M_{t+1}^b is:

$$
\begin{aligned}
E\{\beta\upsilon_{t+1}(1-\theta_{t+1})\left[P_{t+1}C_{t+1}\left(\frac{1}{M_{t+1}}\right)^{\theta_{t+1}}\left(\frac{1}{M_{t+1}^b-M_{t+1}+X_{t+1}}\right)^{(1-\theta_{t+1})}\right]^{1-\sigma_q}\frac{1}{M_{t+1}^b-M_{t+1}+X_{t+1}} & \\
+\beta\lambda_{t+1}\left[1+\left(1-\tau_{t+1}^D\right)R_{a,t+1}\right]-\lambda_t|\Omega_t^\mu\} = 0 &
\end{aligned}
$$

The first two terms on the left of the equality capture the discounted value of an extra unit of currency in base in the next period. The last term captures the cost, which is the multiplier on the current period budget constraint. We now scale this expression. Multiply by $P_t z_t$:

$$
\begin{aligned}
E\{\beta\upsilon_{t+1}(1-\theta_{t+1})\left[c_{t+1}\left(\frac{1}{m_{t+1}}\right)^{\theta_{t+1}}\left(\frac{1}{1-m_{t+1}+x_{t+1}}\right)^{(1-\theta_{t+1})}\right]^{1-\sigma_q} & \\
\times\left(\frac{1}{m_{t+1}^b}\right)^{2-\sigma_q}\frac{1}{\pi_{t+1}\mu_{z,t+1}}\frac{1}{1-m_{t+1}+x_{t+1}} & \\
+\beta\frac{1}{\pi_{t+1}\mu_{z,t+1}}\lambda_{z,t+1}\left[1+\left(1-\tau_{t+1}^D\right)R_{a,t+1}\right]-\lambda_{z,t}|\Omega_t^\mu\} = 0 &
\end{aligned}
$$

For the ACEL specification, simply replace c_{t+1} by unity. We now linearize the above expression. Isolating the object in braces, and rearranging a little:

$$\beta \upsilon_{t+1} (1-\theta_{t+1}) \left[c_{t+1} \left(\frac{1}{m_{t+1}} \right)^{\theta_{t+1}} \right]^{1-\sigma_q} \left(\frac{1}{1-m_{t+1}+x_{t+1}} \right)^{(1-\theta_{t+1})(1-\sigma_q)+1} \left(\frac{1}{m_{t+1}^b} \right)^{2-\sigma_q} \frac{1}{\pi_{t+1}\mu_{z,t+1}} \tag{2.48}$$
$$+\beta \frac{1}{\pi_{t+1}\mu_{z,t+1}} \lambda_{z,t+1} \left[1 + \left(1-\tau_{t+1}^D\right) R_{a,t+1} \right] - \lambda_{z,t}$$

Linearizing the first element in the sum:

$$\beta \upsilon (1-\theta) \left[c \left(\frac{1}{m} \right)^{\theta} \right]^{1-\sigma_q} \left(\frac{1}{1-m+x} \right)^{(1-\theta)(1-\sigma_q)+1} \left(\frac{1}{m^b} \right)^{2-\sigma_q} \frac{1}{\pi \mu_z} \tag{2.49}$$
$$\times \{ \hat{\upsilon}_{t+1} - \frac{\theta \hat{\theta}_{t+1}}{1-\theta} + (1-\sigma_q)\hat{c}_{t+1}$$
$$+ \widehat{\left[\left(\frac{1}{m_{t+1}} \right)^{\theta_{t+1}} \right]^{1-\sigma_q}} + \widehat{\left(\frac{1}{1-m_{t+1}+x_{t+1}} \right)^{(1-\theta_{t+1})(1-\sigma_q)+1}} - (2-\sigma_q)\, \hat{m}_{t+1}^b - \hat{\pi}_{t+1} - \hat{\mu}_{z,t+1} \}.$$

Now, consider the fourth term in square brackets. This involves dm_{t+1} and $d\theta_{t+1}$. Let

$$f(\theta_{t+1}, m_{t+1}) = \left[\left(\frac{1}{m_{t+1}} \right)^{\theta_{t+1}} \right]^{1-\sigma_q}.$$

Then,

$$\hat{f} = \frac{df}{f} = \frac{f_1(\theta, m)\theta\hat{\theta}_{t+1} + f_2(\theta, m) m \hat{m}_{t+1}}{f(\theta, m)}.$$

Now,

$$\begin{aligned} f_1(\theta, m) &= \frac{d \left[\left(\frac{1}{m_{t+1}} \right)^{\theta_{t+1}} \right]^{1-\sigma_q}}{d\theta_{t+1}} = \frac{d \exp\left[-\theta_{t+1}(1-\sigma_q) \log(m_{t+1}) \right]}{d\theta_{t+1}} \\ &= -(1-\sigma_q) \log(m_{t+1}) \exp\left[-\theta_{t+1}(1-\sigma_q) \log(m_{t+1}) \right] \\ &= -(1-\sigma_q) \log(m)\, f(\theta, m). \end{aligned}$$

Also,

$$f_2(\theta, m) = \frac{d\left[\left(\frac{1}{m_{t+1}}\right)^{\theta_{t+1}}\right]^{1-\sigma_q}}{dm_{t+1}} = -\theta(1-\sigma_q)\left(\frac{1}{m}\right)^{\theta(1-\sigma_q)-1}\left(\frac{1}{m}\right)^2$$
$$= -\theta(1-\sigma_q)\left(\frac{1}{m}\right)^{\theta(1-\sigma_q)}\left(\frac{1}{m}\right) = -\theta(1-\sigma_q)f(\theta, m)\left(\frac{1}{m}\right)$$

Then,

$$\hat{f} = \frac{df}{f} = -\frac{(1-\sigma_q)\log(m)\, f(\theta, m)\theta\hat{\theta}_{t+1} + \theta(1-\sigma_q)f(\theta, m)\left(\frac{1}{m}\right)m\hat{m}_{t+1}}{f(\theta, m)}$$
$$= -(1-\sigma_q)\log(m)\,\theta\hat{\theta}_{t+1} - \theta(1-\sigma_q)\hat{m}_{t+1}$$

Substituting this into (2.49):

$$\beta\upsilon(1-\theta)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}\frac{1}{\pi\mu_z} \tag{2.50}$$
$$\times[\hat{\upsilon}_{t+1} - \frac{\theta\hat{\theta}_{t+1}}{1-\theta} + (1-\sigma_q)\hat{c}_{t+1} - (1-\sigma_q)\log(m)\,\theta\hat{\theta}_{t+1} - \theta(1-\sigma_q)\hat{m}_{t+1}$$
$$+\widehat{\left(\frac{1}{1-m_{t+1}+x_{t+1}}\right)^{(1-\theta_{t+1})(1-\sigma_q)+1}} - (2-\sigma_q)\,\hat{m}^b_{t+1} - \hat{\pi}_{t+1} - \hat{\mu}_{z,t+1}].$$

Now consider the term with a hat over it. Let

$$f(m_{t+1}, x_{t+1}, \theta_{t+1}) = \left(\frac{1}{1-m_{t+1}+x_{t+1}}\right)^{(1-\theta_{t+1})(1-\sigma_q)+1}.$$

Then,

$$\hat{f}_t = \frac{df_t}{f} = \frac{f_1(m, x, \theta)m\hat{m}_{t+1} + f_2(m, x, \theta)x\hat{x}_{t+1} + f_3(m, x, \theta)\theta\hat{\theta}_{t+1}}{f}.$$

Now

$$f_1(m, x, \theta) = [(1-\theta)(1-\sigma_q)+1]\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)}\left(\frac{1}{1-m+x}\right)^2 m\hat{m}_{t+1}$$
$$= [(1-\theta)(1-\sigma_q)+1]\,f \times \left(\frac{1}{1-m+x}\right)m\hat{m}_{t+1}.$$

Also,

$$\begin{aligned} f_2(m,x,\theta) &= -\left[(1-\theta)(1-\sigma_q)+1\right]\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)}\left(\frac{1}{1-m+x}\right)^2 x\hat{x}_{t+1} \\ &= -\left[(1-\theta)(1-\sigma_q)+1\right] f \times \left(\frac{1}{1-m+x}\right) x\hat{x}_{t+1}. \end{aligned}$$

Finally,

$$\begin{aligned} f_3(m,x,\theta) &= \frac{d\left(\frac{1}{1-m_{t+1}+x_{t+1}}\right)^{(1-\theta_{t+1})(1-\sigma_q)+1}}{d\theta_{t+1}} \\ &= \frac{d\exp\left[-\left((1-\theta_{t+1})(1-\sigma_q)+1\right)\log\left(1-m_{t+1}+x_{t+1}\right)\right]}{d\theta_{t+1}} \\ &= (1-\sigma_q)\log(1-m+x)\, f \end{aligned}$$

Substituting these into the expression for $\hat{f}_t$:

$$\hat{f}_t = -\left[(1-\theta)(1-\sigma_q)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1} - m\hat{m}_{t+1}\right] + (1-\sigma_q)\log(1-m+x)\,\theta\hat{\theta}_{t+1}.$$

Substituting this into (2.50):

$$\begin{aligned} &\beta\upsilon(1-\theta)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}\frac{1}{\pi\mu_z} \qquad (2.51) \\ &\times\{\hat{\upsilon}_{t+1} - \frac{\theta\hat{\theta}_{t+1}}{1-\theta} + (1-\sigma_q)\hat{c}_{t+1} - (1-\sigma_q)\log(m)\,\theta\hat{\theta}_{t+1} - \theta(1-\sigma_q)\hat{m}_{t+1} \\ &-\left[(1-\theta)(1-\sigma_q)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1} - m\hat{m}_{t+1}\right] \\ &+(1-\sigma_q)\log(1-m+x)\,\theta\hat{\theta}_{t+1} - (2-\sigma_q)\,\hat{m}^b_{t+1} - \hat{\pi}_{t+1} - \hat{\mu}_{z,t+1}\}. \end{aligned}$$

Now consider the second and third elements in (2.48):

$$
\begin{aligned}
& \frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\left[-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}+\hat{\lambda}_{z,t+1}+\left[1+\widehat{\left(1-\tau^D_{t+1}\right)R_{a,t+1}}\right]\right]-\lambda_z\hat{\lambda}_{z,t} \\
= \; & \frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\left[-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}+\hat{\lambda}_{z,t+1}+\frac{\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^D R_a\hat{\tau}^D_{t+1}}{1+\left(1-\tau^D\right)R_a}\right]-\lambda_z\hat{\lambda} \\
= \; & \frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\left[-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}+\hat{\lambda}_{z,t+1}\right] \\
& +\frac{1}{\pi\mu_z}\lambda_z\left[\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^D R_a\hat{\tau}^D_{t+1}\right]-\lambda_z\hat{\lambda}
\end{aligned}
$$

Using this and (2.51), we have our linearized version of (2.48):

$$
\begin{aligned}
& \beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}\frac{1}{\pi\mu_z} \\
& \times\{\hat{\upsilon}_{t+1}-\frac{\theta\hat{\theta}_{t+1}}{1-\theta}+(1-\sigma_q)\hat{c}_{t+1}-\left(1-\sigma_q\right)\log\left(m\right)\theta\hat{\theta}_{t+1}-\theta(1-\sigma_q)\hat{m}_{t+1} \\
& -\left[\left(1-\theta\right)\left(1-\sigma_q\right)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1}-m\hat{m}_{t+1}\right] \\
& +\left(1-\sigma_q\right)\log\left(1-m+x\right)\theta\hat{\theta}_{t+1}-\left(2-\sigma_q\right)\hat{m}^b_{t+1}-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}\} \\
& +\frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\left[-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}+\hat{\lambda}_{z,t+1}\right] \\
& +\frac{\beta}{\pi\mu_z}\lambda_z\left[\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^D R_a\hat{\tau}^D_{t+1}\right]-\lambda_z\hat{\lambda}
\end{aligned}
$$

Multiply by $\pi\mu_z$ and collect terms in $\hat{\pi}_{t+1}+\hat{\mu}_{z,t+1}$:

$$\beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}$$
$$\times\{\hat{\upsilon}_{t+1}-\frac{\theta\hat{\theta}_{t+1}}{1-\theta}+(1-\sigma_q)\hat{c}_{t+1}-(1-\sigma_q)\log\left(m\right)\theta\hat{\theta}_{t+1}-\theta(1-\sigma_q)\hat{m}_{t+1}$$
$$-\left[\left(1-\theta\right)\left(1-\sigma_q\right)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1}-m\hat{m}_{t+1}\right]$$
$$+\left(1-\sigma_q\right)\log\left(1-m+x\right)\theta\hat{\theta}_{t+1}-\left(2-\sigma_q\right)\hat{m}^b_{t+1}\}$$
$$+\beta\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\hat{\lambda}_{z,t+1}+\beta\lambda_z\left[\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^DR_a\hat{\tau}^D_{t+1}\right]-\pi\mu_z\lambda_z\hat{\lambda}$$
$$-\{\beta\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]$$
$$+\beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}\}\left(\hat{\pi}_{t+1}+\hat{\mu}_{z,t+1}\right)$$

The object in braces can be simplified using the steady state result:

$$\beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{(1-\theta)}\right]^{1-\sigma_q}\left(\frac{1}{m^b}\right)^{2-\sigma_q}\frac{1}{1-m+x}$$
$$+\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]-\pi\mu_z\lambda_z=0\quad,$$

so,

$$(**)\qquad\beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}$$
$$\times\{\hat{\upsilon}_{t+1}-\frac{\theta\hat{\theta}_{t+1}}{1-\theta}+(1-\sigma_q)\hat{c}_{t+1}-(1-\sigma_q)\log\left(m\right)\theta\hat{\theta}_{t+1}-\theta(1-\sigma_q)\hat{m}_{t+1}$$
$$-\left[\left(1-\theta\right)\left(1-\sigma_q\right)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1}-m\hat{m}_{t+1}\right]$$
$$+\left(1-\sigma_q\right)\log\left(1-m+x\right)\theta\hat{\theta}_{t+1}-\left(2-\sigma_q\right)\hat{m}^b_{t+1}\}$$
$$+\beta\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\hat{\lambda}_{z,t+1}$$
$$+\beta\lambda_z\left[\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^DR_a\hat{\tau}^D_{t+1}\right]-\pi\mu_z\lambda_z\left[\hat{\lambda}_t+\hat{\pi}_{t+1}+\hat{\mu}_{z,t+1}\right].$$

For the ACEL specification, replaced c by unity and set $\hat{c}_t=0$.

2.6.5. The C_t First Order Condition

We now consider C_t. It is useful to define $u_{c,t}$ as the derivative of the present discounted value of utility with respect to C_t :

$$E\left\{u_{c,t}-u'(C_t-bC_{t-1})+b\beta u'(C_{t+1}-bC_t)|\Omega_t^\mu\right\}=0.$$

It is useful to obtain a linearization of the expression in braces, after multiplication by z_t. First, scale:

$$\begin{aligned}
& u_{c,t}^z-\frac{z_t}{C_t-bC_{t-1}}+b\beta\frac{z_{t+1}}{C_{t+1}-bC_t}\frac{1}{\mu_{z,t+1}}\\
= & u_{c,t}^z-\frac{1}{\frac{C_t}{z_t}-b\frac{C_{t-1}}{z_{t-1}\mu_{z,t}}}+b\beta\frac{1}{\frac{C_{t+1}}{z_{t+1}}-b\frac{C_t}{z_t\mu_{z,t+1}}}\frac{1}{\mu_{z,t+1}}\\
= & u_{c,t}^z-\frac{\mu_{z,t}}{c_t\mu_{z,t}-bc_{t-1}}+b\beta\frac{1}{c_{t+1}\mu_{z,t+1}-bc_t}
\end{aligned}$$

One way to obtain the linearization, works like this:

$$\begin{aligned}
f(c_t) &= u^z_{c,t} - \frac{\mu_{z,t}}{c_t\mu_{z,t} - bc_{t-1}} + b\beta\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t} \\
f'(c_t)c\hat{c}_t &= \left[\left(\frac{1}{c_t\mu_{z,t} - bc_{t-1}}\right)^2 \mu^2_{zt} + b^2\beta\left(\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t}\right)^2\right] c\hat{c}_t \\
&= \left[\left(\frac{\mu_z}{c\mu_z - bc}\right)^2 \mu^2_z + b^2\beta\left(\frac{1}{c\mu_z - bc}\right)^2\right] c\hat{c}_t \\
&= \left(\frac{1}{c\mu_z - bc}\right)^2 \left[\mu^2_z + b^2\beta\right] c\hat{c}_t \\
&= \left(\frac{1}{c\left(\mu_z - b\right)}\right)^2 \left[\mu^2_z + b^2\beta\right] c \times \hat{c}_t \\
f(c_{t+1}) &= u^z_{c,t} - \frac{\mu_{z,t}}{c_t\mu_{z,t} - bc_{t-1}} + b\beta\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t} \\
f'(c_{t+1})c\hat{c}_{t+1} &= -b\beta\left(\frac{1}{c\left(\mu_z - b\right)}\right)^2 \mu_z c\hat{c}_{t+1} \\
f(c_{t-1}) &= u^z_{c,t} - \frac{\mu_{z,t}}{c_t\mu_{z,t} - bc_{t-1}} + b\beta\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t} \\
f'(c_{t-1})c\hat{c}_{t-1} &= \left(\frac{1}{c\left(\mu_z - b\right)}\right)^2 \mu_z c\hat{c}_{t-1} \\
f(\mu_{z,t}) &= u^z_{c,t} - \frac{\mu_{z,t}}{c_t\mu_{z,t} - bc_{t-1}} + b\beta\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t} \\
f'(\mu_{z,t})\mu_z\hat{\mu}_{z,t} &= \left[-\frac{\mu_z}{c\left(\mu_z - b\right)} + \left(\frac{1}{c\left(\mu_z - b\right)}\right)^2 c\mu^2_z\right] \hat{\mu}_{z,t} \\
f(\mu_{z,t+1}) &= u^z_{c,t} - \frac{\mu_{z,t}}{c_t\mu_{z,t} - bc_{t-1}} + b\beta\frac{1}{c_{t+1}\mu_{z,t+1} - bc_t} \\
f'(\mu_{z,t+1})\mu_z\hat{\mu}_{z,t+1} &= b\beta\left(\frac{1}{c\left(\mu_z - b\right)}\right)^2 c\mu_z\hat{\mu}_{z,t+1}
\end{aligned}$$

The coefficients in the canonical form are:

$$
\begin{aligned}
\alpha_1(14,18) &= \left(\frac{1}{c\left(\mu_z-b\right)}\right)^2\left[\mu_z^2+b^2\beta\right]c,:\ \hat{c}_t \\
\alpha_0(14,18) &= -b\beta\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2\mu_z c:\ \hat{c}_{t+1} \\
\alpha_2(14,18) &= -\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 b\mu_z c:\ \hat{c}_{t-1} \\
\beta_0(14,46) &= b\beta\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 c\mu_z^2:\mu_{z,t+1} \\
\beta_1(14,46) &= \left[-\frac{\mu_z}{c\left(\mu_z-b\right)}+\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 c\mu_z\right]:\mu_{z,t}
\end{aligned}
$$

Another way to obtain the linearization is:

$$
u_c^z\hat{u}_{c,t}^z-\frac{\mu_z}{c\mu_z-bc}\frac{\widehat{\mu_{z,t}}}{c_t\mu_{z,t}-bc_{t-1}}+b\beta\frac{1}{c\mu_z-bc}\frac{\widehat{1}}{c_{t+1}\mu_{z,t+1}-bc_t}
$$

But,

$$
\begin{aligned}
\frac{\widehat{\mu_{z,t}}}{c_t\mu_{z,t}-bc_{t-1}} &= \hat{\mu}_{z,t}-\left[c_t\widehat{\mu_{z,t}-b}c_{t-1}\right] \\
&= \hat{\mu}_{z,t}-\frac{c\mu_z\left(\hat{c}_t+\hat{\mu}_{z,t}\right)-bc\hat{c}_{t-1}}{c\left(\mu_z-b\right)} \\
&= \hat{\mu}_{z,t}-\frac{\mu_z\left(\hat{c}_t+\hat{\mu}_{z,t}\right)-b\hat{c}_{t-1}}{\mu_z-b}
\end{aligned}
$$

$$
\frac{\widehat{1}}{c_{t+1}\mu_{z,t+1}-bc_t}=-\frac{\mu_z\left(\hat{c}_{t+1}+\hat{\mu}_{z,t+1}\right)-b\hat{c}_t}{\mu_z-b}
$$

Then,

$$
\begin{aligned}
& u_c^z \hat{u}_{c,t}^z - \frac{\mu_z}{c(\mu_z - b)} \left\{ \hat{\mu}_{z,t} - \frac{\mu_z (\hat{c}_t + \hat{\mu}_{z,t}) - b\hat{c}_{t-1}}{\mu_z - b} \right\} - b\beta \frac{1}{c\mu_z - bc} \frac{\mu_z (\hat{c}_{t+1} + \hat{\mu}_{z,t+1}) - b\hat{c}_t}{\mu_z - b} \\
= \; & u_c^z \hat{u}_{c,t}^z - \left[\frac{\mu_z}{c(\mu_z - b)} + \frac{\mu_z}{c(\mu_z - b)} \frac{\mu_z}{\mu_z - b} \right] \hat{\mu}_{z,t} - b\beta \frac{1}{c\mu_z - bc} \frac{\mu_z \hat{\mu}_{z,t+1}}{\mu_z - b} \\
& + \left[\frac{\mu_z}{c(\mu_z - b)} \frac{\mu_z}{\mu_z - b} + b\beta \frac{1}{c\mu_z - bc} \frac{b}{\mu_z - b} \right] \hat{c}_t \\
& - b\beta \frac{1}{c\mu_z - bc} \frac{\mu_z}{\mu_z - b} \hat{c}_{t+1} - \frac{\mu_z}{c(\mu_z - b)} \frac{b}{\mu_z - b} \hat{c}_{t-1} \\
= \; & u_c^z \hat{u}_{c,t}^z - \left[\frac{\mu_z}{c(\mu_z - b)} - \frac{\mu_z^2 c}{c^2 (\mu_z - b)^2} \right] \hat{\mu}_{z,t} - b\beta \frac{\mu_z c}{c^2 (\mu_z - b)^2} \hat{\mu}_{z,t+1} \\
& + \frac{\mu_z^2 + \beta b^2}{c^2 (\mu_z - b)^2} c\hat{c}_t - \frac{b\beta\mu_z}{c^2 (\mu_z - b)^2} c\hat{c}_{t+1} - \frac{b\mu_z}{c^2 (\mu_z - b)^2} c\hat{c}_{t-1}
\end{aligned}
$$

Finally, reintroduce the expectation operator:

$$
\begin{aligned}
(**) \; E\{ u_c^z \hat{u}_{c,t}^z - \left[\frac{\mu_z}{c(\mu_z - b)} - \frac{\mu_z^2 c}{c^2 (\mu_z - b)^2} \right] \hat{\mu}_{z,t} - b\beta \frac{\mu_z c}{c^2 (\mu_z - b)^2} \hat{\mu}_{z,t+1} & \\
+ \frac{\mu_z^2 + \beta b^2}{c^2 (\mu_z - b)^2} c\hat{c}_t - \frac{b\beta\mu_z}{c^2 (\mu_z - b)^2} c\hat{c}_{t+1} - \frac{b\mu_z}{c^2 (\mu_z - b)^2} c\hat{c}_{t-1} | \Omega_t^\mu \} &= 0.
\end{aligned}
$$

The first order condition associated with C_t is:

$$
E_t \left\{ u_{c,t} - \upsilon_t C_t^{-\sigma_q} \left[\left(\frac{P_t}{M_t} \right)^{\theta_t} \left(\frac{P_t}{M_t^b - M_t + X_t} \right)^{1-\theta_t} \right]^{1-\sigma_q} - (1 + \tau_t^c) P_t \lambda_t \right\} = 0.
$$

Multiply by z_t :

$$
E_t \left\{ u_{c,t} z_t - \upsilon_t c_t^{-\sigma_q} \left[\left(\frac{z_t P_t / M_t^b}{M_t / M_t^b} \right)^{\theta_t} \left(\frac{z_t P_t / M_t^b}{1 - M_t / M_t^b + X_t / M_t^b} \right)^{1-\theta_t} \right]^{1-\sigma_q} - (1 + \tau_t^c) \lambda_{z,t} \right\} = 0,
$$

or,

$$
E \left\{ u_{c,t}^z - \upsilon_t c_t^{-\sigma_q} \left[\frac{1}{m_t^b} \left(\frac{1}{m_t} \right)^{\theta_t} \left(\frac{1}{1 - m_t + x_t} \right)^{1-\theta_t} \right]^{1-\sigma_q} - (1 + \tau_t^c) \lambda_{z,t} | \Omega_t \right\} = 0,
$$

where

$$m_t = \frac{M_t}{M_t^b},\ x_t = \frac{X_t}{M_t^b},\ m_t^b = \frac{M_t^b}{z_t P_t},\ u_{c,t}^z = u_{c,t} z_t$$

For the ACEL specification, the second term inside the braces is replaced by zero.

Now we linearize the object in braces:

$$u_c^z \hat{u}_{c,t}^z - \upsilon c^{-\sigma_q} \left[\frac{1}{m^b} \left(\frac{1}{m} \right)^{\theta} \left(\frac{1}{1-m+x} \right)^{1-\theta} \right]^{1-\sigma_q}$$
$$\times \{ \hat{\upsilon}_t - \sigma_q \hat{c}_t + (1-\sigma_q) \left(-\hat{m}_t^b - \theta_t \hat{m}_t - (1-\theta) \left(\widehat{1 - m_t + x_t} \right) \right)$$
$$(1-\sigma_q) \left[\log \left(\frac{1}{m} \right) - \log \left(\frac{1}{1-m+x} \right) \right] \theta \hat{\theta}_t \}$$
$$- (1+\tau^c) \lambda_z \left[(\widehat{1+\tau_t^c}) + \hat{\lambda}_{z,t} \right]$$

But,

$$\begin{aligned} \widehat{1 - m_t + x_t} &= \frac{-m\hat{m}_t + x\hat{x}_t}{1-m+x} = \frac{-m}{1-m+x}\hat{m}_t + \frac{x}{1-m+x}\hat{x}_t \\ (\widehat{1+\tau_t^c}) &= \frac{\tau^c \hat{\tau}_t^c}{1+\tau^c} \end{aligned}$$

so,

$$(**)\ E\{ u_c^z \hat{u}_{c,t}^z - \upsilon c^{-\sigma_q} \left[\frac{1}{m^b} \left(\frac{1}{m} \right)^{\theta_t} \left(\frac{1}{1-m+x} \right)^{1-\theta_t} \right]^{1-\sigma_q}$$
$$\times [\hat{\upsilon}_t - \sigma_q \hat{c}_t + (1-\sigma_q) \left(-\hat{m}_t^b - \theta_t \hat{m}_t - (1-\theta_t) \left(\frac{-m}{1-m+x}\hat{m}_t + \frac{x}{1-m+x}\hat{x}_t \right) \right)$$
$$+ (1-\sigma_q) \left[\log \left(\frac{1}{m} \right) - \log \left(\frac{1}{1-m+x} \right) \right] \theta \hat{\theta}_t]$$
$$- (1+\tau^c) \lambda_z \left[\frac{\tau^c}{1+\tau^c} \hat{\tau}_t^c + \hat{\lambda}_{z,t} \right] |\Omega_t \} = 0$$

For the ACEL specification the second term in the first line of (**) is replaced by zero.

2.6.6. The Wage Decision

The wage rate set by the household that gets to reoptimize today is $\tilde{W}_t$. The household takes into account that if it does not get to reoptimize next period, it's wage rate then is

$$W_{t+1} = \pi_t \mu_{z,t+1} \tilde{W}_t,$$

where

$$\mu_{z,t+1} = \frac{z_{t+1}}{z_t}.$$

In period $t+2$ it is

$$W_{t+2} = \pi_t \pi_{t+1} \mu_{z,t+1} \mu_{z,t+2} \tilde{W}_t,$$

and

$$W_{t+l} = \pi_t \times \cdots \times \pi_{t+l-1} \mu_{z,t+1} \times \cdots \times \mu_{z,t+l} \tilde{W}_t.$$

Technically, it is useful to note the slight difference in timing between inflation and the technology shock. The former reflects that indexing is lagged. The latter reflects that indexing to the technology shock is contemporaneous.

The demand curve that the individual household faces is:

$$h_{t+j} = \left(\frac{\tilde{W}_{t+j}}{W_{t+j}} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+j} = \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+j} z_{t+j} P_t} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+j},$$

where $\tilde{W}_t$ denotes the nominal wage set by households that reoptimize in period t, and W_t denotes the nominal wage rate associated with aggregate, homogeneous labor, X_t (don't confuse this with the different object, $X_{t,j} = \pi_t / \pi_{t+j}$). Also,

$$X_{t,l} = \frac{\pi_t \times \pi_{t+1} \times \cdots \times \pi_{t+l-1}}{\pi_{t+1} \times \cdots \times \pi_{t+l}} = \frac{\pi_t}{\pi_{t+l}}.$$

The homogeneous labor is related to household labor by:

$$l = \left[\int_0^1 (h_j)^{\frac{1}{\lambda_w}} dj \right]^{\lambda_w}, \ 1 \leq \lambda_w < \infty.$$

The j^{th} household that reoptimizes its wage, $\tilde{W}_t$, does so to optimize (neglecting irrelevant terms in the household objective):

$$E_t \sum_{l=0}^{\infty} (\beta \xi_w)^{l-t} \{ -\zeta_{t+l} z(h_{j,t+l}) + \lambda_{t+l} (1 - \tau_{t+l}^l) W_{j,t+l} h_{j,t+l} \},$$

where

$$z(h) = \psi_L \frac{h_t^{1+\sigma_L}}{1+\sigma_L}$$

The presence of ξ_w by the discount factor reflects that the household is only concerned with the future states of the world in which it cannot reoptimize its wage.

Substituting out for $h_{j,t+1}$ using the demand curve:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l})$$
$$+ (1 - \tau^l_{t+l}) \lambda_{t+l} W_{j,t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}\}.$$

Now, make use of the fact, $\lambda_{z,t+l} = \lambda_{t+l} P_{t+l} z_{t+l}$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l})$$
$$+ \frac{(1 - \tau^l_{t+l}) \lambda_{z,t+l}}{z_{t+l}} \frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} X_{t,l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}\},$$

or, after rearranging:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l})$$
$$+ \frac{(1 - \tau^l_{t+l}) \lambda_{z,t+l}}{z_{t+l}} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} \left(\frac{1}{z_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}\},$$

or, $(\psi_{z,t+l} = z_{t+l} \psi_{t+l})$:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l})$$
$$+ (1 - \tau^l_{t+l}) \lambda_{z,t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{z_{t+l} P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}\}.$$

But, note that:

$$\begin{aligned} z_{t+1} &= \mu_{z,t+1} z_t \\ z_{t+2} &= \mu_{z,t+2} \mu_{z,t+1} z_t, \end{aligned}$$

etc., so that

$$\frac{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l}}{z_{t+l}}=\frac{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l}}{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l}z_t}=\frac{1}{z_t}.$$

Then,

$$E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\{-\zeta_{t+l}z(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l})+(1-\tau_{t+l}^l)\lambda_{z,t+l}\left(\frac{\tilde{W}_t}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\}.$$

Differentiate with respect to $\tilde{W}_t$:

$$\begin{aligned}E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\{-\zeta_{t+l}z'(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l})\frac{\lambda_w}{1-\lambda_w}\tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}-1}\left(\frac{X_{t,l}}{w_{t+l}z_tP_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\\
+(1-\tau_{t+l}^l)\lambda_{z,t+l}\left(\frac{1}{1-\lambda_w}\right)\tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}}\left(\frac{1}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\} &= 0\end{aligned}$$

Multiply by $\tilde{W}_t^{-\frac{\lambda_w}{1-\lambda_w}+1}(1-\lambda_w)/\lambda_w$:

$$\begin{aligned}E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\{-\zeta_{t+l}z'(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l})\left(\frac{X_{t,l}}{w_{t+l}z_tP_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l}\\
+(1-\tau_{t+l}^l)\lambda_{z,t+l}\left(\frac{1}{\lambda_w}\right)\tilde{W}_t\left(\frac{1}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\} &= 0.\end{aligned}$$

Multiply by $P_t^{\frac{\lambda_w}{1-\lambda_w}}$:

$$\begin{aligned}E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\{-\zeta_{t+l}z'(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l})\left(\frac{X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\\
+(1-\tau_{t+l}^l)\lambda_{z,t+l}\frac{1}{\lambda_w}\frac{\tilde{W}_t}{P_t}\left(\frac{1}{z_t}\right)^{\frac{1}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\} &= 0.\end{aligned}$$

Now get this in terms of stationary variables using $\tilde{w}_t=\tilde{W}_t/W_t$, $w_t=W_t/(z_tP_t)$:

$$\begin{aligned}E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\{-\zeta_{t+l}z'(\left(\frac{\tilde{w}_tW_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l})\left(\frac{X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\\
+(1-\tau_{t+l}^l)\lambda_{z,t+l}\frac{1}{\lambda_w}\frac{\tilde{w}_tW_t}{P_t}\left(\frac{1}{z_t}\right)^{\frac{1}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}l_{t+l}\} &= 0,\end{aligned}$$

and

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-\zeta_{t+l} z'(\left(\frac{\tilde{w}_t z_t w_t}{w_{t+l} z_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}) \left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}$$

$$+(1-\tau_{t+l}^l)\lambda_{z,t+l} \frac{1}{\lambda_w} \tilde{w}_t z_t w_t z_t^{\frac{\lambda_w}{1-\lambda_w} - \frac{1}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{z_t w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}\} = 0.$$

or,

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l} \{-\zeta_{t+l} z'(\left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l}) + (1-\tau_{t+l}^l)\lambda_{z,t+l} \frac{1}{\lambda_w} \tilde{w}_t w_t X_{t,l}\} = 0.$$

or,

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l} (1-\tau_{t+l}^l)\lambda_{z,t+l} \frac{1}{\lambda_w} \{\tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{(1-\tau_{t+l}^l)\lambda_{z,t+l}}\} = 0.$$

Finally, multiply both sides of this expression by

$$(\tilde{w}_t w_t z_t)^{\frac{\lambda_w}{1-\lambda_w}},$$

so that

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{\tilde{w}_t w_t X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l} (1-\tau_{t+l}^l)\lambda_{z,t+l} \frac{1}{\lambda_w} \{\tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{(1-\tau_{t+l}^l)\lambda_{z,t+l}}\} = 0,$$

and

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l h_{t+l} \{\frac{(1-\tau_{t+l}^l)\lambda_{z,t+l}}{\lambda_w} \tilde{w}_t w_t X_{t,l} - \zeta_{t+l} z'_{t+l}\} = 0,$$

using the demand curve. Rewriting this

$$E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{(1-\tau_{t+l}^l)\lambda_{z,t+l}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L(h_{j,t+l})\right] = 0,$$

where

$$f_L(h_{j,t+l}) = -z'_{t+l} = -\psi_L h_{j,t+l}^{\sigma_L}$$

or,

$$E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{(1-\tau^l_{t+l})\lambda_{z,t+l}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+l} \right) \right] = 0.$$

(Here, the functions f and z refer to the same thing. This is inconvenient for reading, but should not cause confusion.) Writing this out, term by term:

$$\begin{aligned} 0 &= h_{j,t} \left[\tilde{w}_t w_t \frac{\lambda_{z,t}(1-\tau^l_t)}{\lambda_w} + \zeta_t f_L \left(\tilde{w}_t^{\frac{\lambda_w}{1-\lambda_w}} l_t \right) \right] \\ &\quad + \beta \xi_w h_{j,t+1} \left[\tilde{w}_t w_t X_{t,1} \frac{(1-\tau^l_{t+1})\lambda_{z,t+1}}{\lambda_w} + \zeta_{t+1} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+1}} X_{t,1} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+1} \right) \right] \\ &\quad + (\beta \xi_w)^2 h_{j,t+2} \left[\tilde{w}_t w_t X_{t,2} \frac{\lambda_{z,t+2}(1-\tau^l_{t+2})}{\lambda_w} + \zeta_{t+2} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+2}} X_{t,2} \right)^{\frac{\lambda_w}{1-\lambda_w}} l_{t+2} \right) \right] \\ &\quad + ... \end{aligned}$$

In steady state, $\tilde{w} = 1, \pi_t = \bar{\pi}$, $w \frac{\psi_z}{\lambda_w} + f_L = 0$. The derivative of this expression with respect to ζ_{t+l}, evaluated in steady state, is:

$$l (\beta \xi_w)^l f_L = -L (\beta \xi_w)^l w \frac{\lambda_z (1-\tau^l)}{\lambda_w}.$$

Derivatives with respect to the other variables can be found in newfile2.tex. Using the latter and the result just obtained, we find:

$$\begin{aligned} 0 &= \frac{1}{1-\beta\xi_w} L \left[w \frac{(1-\tau^l)\lambda_z}{\lambda_w} + f_{LL} \frac{\lambda_w}{1-\lambda_w} L \right] (\hat{\tilde{w}}_t + \hat{w}_t) - L f_{LL} \frac{\lambda_w}{1-\lambda_w} L \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{w}_{t+j} \\ &\quad - L \left[w \frac{(1-\tau^l)\lambda_z}{\lambda_w} + f_{LL} \frac{\lambda_w}{1-\lambda_w} L \right] \sum_{j=1}^{\infty} (\beta\xi_w)^j \hat{\pi}_{t+j} \\ &\quad + \frac{\beta\xi_w}{1-\beta\xi_w} L \left[w \frac{(1-\tau^l)\lambda_z}{\lambda_w} + f_{LL} \frac{\lambda_w}{1-\lambda_w} L \right] \hat{\pi}_t \\ &\quad + L^2 f_{LL} \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{L}_{t+j} + \frac{Lw(1-\tau^l)\lambda_z}{\lambda_w} \sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\hat{\lambda}_{z,t+j} - \frac{\tau^l}{1-\tau^l} \hat{\tau}^l_{t+j} \right] \\ &\quad - Lw \frac{(1-\tau^l)\lambda_z}{\lambda_w} \sum_{j=0}^{\infty} (\beta\xi_w)^l \hat{\zeta}_{t+j}, \end{aligned}$$

where there has been a little switch in notation, replacing X_t by L_t and X by L. This expression corresponds to (wagelin) with $\hat{u}_{c,t}$ replaced by $\hat{\psi}_{z,t}$, and with the understanding that w_t is now the real wage *scaled by* z_t. After dividing by L, this expression is written:

$$\begin{aligned} 0 &= \frac{1}{1-\beta\xi_w}\left[w\frac{(1-\tau^l)\lambda_z}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}l\right][\widehat{\tilde{w}}_t+\hat{w}_t]-\left[w\frac{(1-\tau^l)\lambda_z}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}l\right]\sum_{l=1}^{\infty}(\beta\xi_w)^l\hat{\pi}_{t+l} \\ &\quad +w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^j\left[\hat{\lambda}_{z,t+j}-\frac{\tau^l}{1-\tau^l}\hat{\tau}^l_{t+j}\right]-f_{LL}\frac{\lambda_w}{1-\lambda_w}l\sum_{l=0}^{\infty}(\beta\xi_w)^l\hat{w}_{t+l} \\ &\quad +f_{LL}l\sum_{j=0}^{\infty}(\beta\xi_w)^j\hat{l}_{t+j}+\frac{\beta\xi_w}{1-\beta\xi_w}\left[w\frac{(1-\tau^l)\lambda_z}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}l\right]\hat{\pi}_t-w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^l\hat{\zeta}_{t+j}\end{aligned}$$

Denoting $\left[w\frac{(1-\tau^l)\lambda_z}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}l\right]=\left[-f_L+f_{LL}\frac{\lambda_w}{1-\lambda_w}l\right]\equiv\tilde{\sigma}_L$,$w\frac{(1-\tau^l)\lambda_z}{\lambda_w}=-f_L$ we obtain:

$$\begin{aligned} 0 &= \frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_t+\hat{w}_t]-\tilde{\sigma}_L\sum_{l=1}^{\infty}(\beta\xi_w)^l\hat{\pi}_{t+l}-f_L\sum_{j=0}^{\infty}(\beta\xi_w)^j\left[\hat{\lambda}_{z,t+j}-\frac{\tau^l}{1-\tau^l}\hat{\tau}^l_{t+j}\right] \\ &\quad -f_{LL}l\frac{\lambda_w}{1-\lambda_w}\sum_{l=0}^{\infty}(\beta\xi_w)^l\hat{w}_{t+l}+f_{LL}l\sum_{j=0}^{\infty}(\beta\xi_w)^j\hat{l}_{t+j} \\ &\quad +\frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\hat{\pi}_t-w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^l\hat{\zeta}_{t+j}.\end{aligned}$$

This can be written:

$$
\begin{aligned}
\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_t + \hat{w}_t] &= \tilde{\sigma}_L \sum_{l=1}^{\infty}(\beta\xi_w)^l \hat{\pi}_{t+l} + f_L \sum_{j=0}^{\infty}(\beta\xi_w)^j \left[\hat{\lambda}_{z,t+j} - \frac{\tau^l}{1-\tau^l}\hat{\tau}^l_{t+j}\right] \\
&\quad + f_{LL} l \frac{\lambda_w}{1-\lambda_w}\sum_{l=0}^{\infty}(\beta\xi_w)^l \hat{w}_{t+l} - f_{LL} l \sum_{j=0}^{\infty}(\beta\xi_w)^j \hat{l}_{t+j} - \frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\hat{\pi}_t \\
&\quad + w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^l \hat{\zeta}_{t+j} \\
&= \tilde{\sigma}_L \sum_{l=0}^{\infty}(\beta\xi_w)^l \hat{\pi}_{t+l} + f_L \sum_{j=0}^{\infty}(\beta\xi_w)^j \left[\hat{\lambda}_{z,t+j} - \frac{\tau^l}{1-\tau^l}\hat{\tau}^l_{t+j}\right] \\
&\quad + f_{LL} l \frac{\lambda_w}{1-\lambda_w}\sum_{l=0}^{\infty}(\beta\xi_w)^l \hat{w}_{t+l} \\
&\quad - f_{LL} l \sum_{j=0}^{\infty}(\beta\xi_w)^j \hat{l}_{t+j} + w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^l \hat{\zeta}_{t+j} - \left[\tilde{\sigma}_L + \frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\right]\hat{\pi}_t \\
&= \sum_{l=0}^{\infty}(\beta\xi_w)^l[\tilde{\sigma}_L\hat{\pi}_{t+l} + f_L\left[\hat{\lambda}_{z,t+j} - \frac{\tau^l}{1-\tau^l}\hat{\tau}^l_{t+j}\right] \\
&\quad + f_{LL} l \frac{\lambda_w}{1-\lambda_w}\hat{w}_{t+l} - f_{LL} l \hat{l}_{t+j} + w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_{t+j}] \\
&\quad - \frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_t.
\end{aligned}
$$

Consider the following example:

$$x_t + z_t = \sum_{l=0}^{\infty}(\beta\xi_w)^l y_{t+l}.$$

Also,

$$\beta\xi_w\left(x_{t+1} + z_{t+1}\right) = \sum_{l=1}^{\infty}(\beta\xi_w)^l y_{t+l}$$

Differencing these expressions:

$$x_t + z_t - \beta\xi_w\left(x_{t+1} + z_{t+1}\right) = y_t.$$

Applying this to the previous expression:

$$\begin{aligned}
&\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_t+\hat{w}_t]+\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_t-\beta\xi_w\left\{\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_{t+1}+\hat{w}_{t+1}]+\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1}\right\} \\
= \;& \tilde{\sigma}_L\hat{\pi}_t+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_t+\hat{w}_t]+\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_t \\
= \;& \beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1}+\tilde{\sigma}_L\hat{\pi}_t \\
&+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_t+\hat{w}_t] \\
= \;& \beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1} \\
&+\tilde{\sigma}_L\left[1-\frac{1}{1-\beta\xi_w}\right]\hat{\pi}_t+f_L\hat{\psi}_{z,t}+f_{LL}L\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}L\hat{L}_t+w\frac{\psi_z}{\lambda_w}\hat{\zeta}_t
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_t+\hat{w}_t] \\
= \;& \beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1} \\
&-\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\hat{\pi}_t+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_t+\hat{w}_t] &= \beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\widetilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1} \\
-\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\hat{\pi}_t&+\left[\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\beta\xi_w\hat{\pi}_{t+1}-\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\beta\xi_w\hat{\pi}_{t+1}\right]+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right] \\
&+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t
\end{aligned}$$

or

$$\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_t+\hat{w}_t]=\beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_{t+1}$$
$$+\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\left[\beta\xi_w\hat{\pi}_{t+1}-\hat{\pi}_t\right]-\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\beta\xi_w\hat{\pi}_{t+1}+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t$$
$$-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t$$

or

$$\begin{aligned}\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_t+\hat{w}_t] &= \beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\left(1-\beta\xi_w\right)\hat{\pi}_{t+1}\\ &+\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\left[\beta\xi_w\hat{\pi}_{t+1}-\hat{\pi}_t\right]+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t\\ &-f_{LL}l\hat{l}_t+w\frac{(1-\tau^l)\lambda_z}{\lambda_w}\hat{\zeta}_t\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_t+\hat{w}_t]\\ =\ &\beta\xi_w\frac{1}{1-\beta\xi_w}\tilde{\sigma}_L[\widehat{\tilde{w}}_{t+1}+\hat{w}_{t+1}]+\beta\xi_w\tilde{\sigma}_L\hat{\pi}_{t+1}+\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\left[\beta\xi_w\hat{\pi}_{t+1}-\hat{\pi}_t\right]\\ &+f_L\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+f_{LL}l\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}l\hat{l}_t-f_L\hat{\zeta}_t\end{aligned}$$

Define

$$\sigma_L=\frac{f_{LL}L}{f_L}.$$

Note, that $\tilde{\sigma}_L=-f_L+f_{LL}\frac{\lambda_w}{1-\lambda_w}l=f_L\left[\frac{\lambda_w}{1-\lambda_w}\frac{f_{LL}l}{f_L}-1\right]=f_L\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L-1\right]$, so dividing the expression above by f_L

$$
\begin{aligned}
& \frac{1}{1-\beta\xi_w}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\left[\widehat{\tilde{w}}_t + \hat{w}_t\right] \qquad (2.52) \\
= & \frac{\beta\xi_w}{1-\beta\xi_w}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\left[\widehat{\tilde{w}}_{t+1} + \hat{w}_{t+1}\right] \\
& +\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\beta\xi_w\hat{\pi}_{t+1} + \left[\hat{\lambda}_{z,t} - \frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right] \\
& +\sigma_L\frac{\lambda_w}{1-\lambda_w}\hat{w}_t - \sigma_L\hat{l}_t \\
& -\frac{\beta\xi_w}{1-\beta\xi_w}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\left[\hat{\pi}_t - \beta\xi_w\hat{\pi}_{t+1}\right] - \hat{\zeta}_t
\end{aligned}
$$

We now turn to the aggregate wage equation.

$$
W_t = \left[(1-\xi_w)\left(\tilde{W}_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\pi_{t-1}\mu_{z,t}W_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}
$$

Dividing this by $z_t P_t$, we get:

$$
w_t = \left[(1-\xi_w)\left(\tilde{w}_t w_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\frac{\pi_{t-1}}{\pi_t}w_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.
$$

(It is easy to see from this that the steady state value of $\tilde{w}_t$ is unity.) In newfile2, it shown that linearizing this leads to:

$$
\hat{w}_t = (1-\xi_w)\left(\hat{w}_t + \widehat{\tilde{w}}_t\right) + \xi_w\left(\hat{w}_{t-1} - (\hat{\pi}_t - \hat{\pi}_{t-1})\right)
$$

or

$$
(1-\xi_w)\left(\hat{w}_t + \widehat{\tilde{w}}_t\right)\hat{w}_t = \hat{w}_t - \xi_w\left(\hat{w}_{t-1} - (\hat{\pi}_t - \hat{\pi}_{t-1})\right)
$$

Subsituting this into (2.52), and multiply by $1-\lambda_w$:

$$\begin{aligned}
& \frac{1}{(1-\beta\xi_w)(1-\xi_w)}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left[\hat{w}_t-\xi_w\left(\hat{w}_{t-1}-(\hat{\pi}_t-\hat{\pi}_{t-1})\right)\right] \\
= & \frac{\beta\xi_w}{(1-\beta\xi_w)(1-\xi_w)}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left[\hat{w}_{t+1}-\xi_w\left(\hat{w}_t-(\hat{\pi}_{t+1}-\hat{\pi}_t)\right)\right] \\
& +\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\beta\xi_w\hat{\pi}_{t+1}+(1-\lambda_w)\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right] \\
& +\sigma_L\lambda_w\hat{w}_t-(1-\lambda_w)\,\sigma_L\hat{l}_t \\
& -\frac{\beta\xi_w}{1-\beta\xi_w}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left[\hat{\pi}_t-\beta\xi_w\hat{\pi}_{t+1}\right]-(1-\lambda_w)\,\hat{\zeta}_t
\end{aligned}$$

Letting $b_w=\left[\lambda_w\sigma_L-(1-\lambda_w)\right]/\left[(1-\xi_w)(1-\beta\xi_w)\right]$, we obtain

$$\begin{aligned}
& b_w[\hat{w}_t-\xi_w\left(\hat{w}_{t-1}-(\hat{\pi}_t-\hat{\pi}_{t-1})\right)] \\
= & \beta\xi_w b_w[\hat{w}_{t+1}-\xi_w\left(\hat{w}_t-(\hat{\pi}_{t+1}-\hat{\pi}_t)\right)] \\
& +b_w(1-\xi_w)(1-\beta\xi_w)\,\beta\xi_w\hat{\pi}_{t+1}+(1-\lambda_w)\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right] \\
& +\sigma_L\lambda_w\hat{w}_t-(1-\lambda_w)\,\sigma_L\hat{l}_t \\
& -\beta\xi_w b_w(1-\xi_w)\left[\hat{\pi}_t-\beta\xi_w\hat{\pi}_{t+1}\right]-(1-\lambda_w)\,\hat{\zeta}_t
\end{aligned}$$

So, the linearized expression for the real wage is what it was before, with a (minor) adjustment to reflect the presence of a preference shock:

$$(**)\ E_t\left\{\eta_0\hat{w}_{t-1}+\eta_1\hat{w}_t+\eta_2\hat{w}_{t+1}+\eta_{\bar{3}}\hat{\pi}_{t-1}+\eta_3\hat{\pi}_t+\eta_4\hat{\pi}_{t+1}+\eta_5\hat{l}_t+\eta_6\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+\eta_7\hat{\zeta}_t\right\}=0$$

where

$$\eta=\begin{pmatrix} b_w\xi_w \\ -b_w\left(1+\beta\xi_w^2\right)+\sigma_L\lambda_w \\ \beta\xi_w b_w \\ b_w\xi_w \\ -\xi_w b_w\,(1+\beta) \\ b_w\beta\xi_w \\ -\sigma_L\,(1-\lambda_w) \\ 1-\lambda_w \\ -\left(1-\lambda_w\right) \end{pmatrix}=\begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_{\bar{3}} \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \end{pmatrix}.$$

The above expression is the linearized Euler equation for the wage rate, scaled by technology. One thing to notice here is the presence of l_t. This corresponds to homogeneous labor, not the sum of households' differentiated labor, L_t. Usually, we will work in settings where it is safe to ignore the distinction. However, we may also consider settings where the distinction matters a lot.

2.7. Monetary and Fiscal Policy

We plan to investigate various alternative monetary policies. In particular we plan to investigate varients of the Taylor rule, including several which make use of monetary aggregates, as in Christiano and Rostagno (2001).

For now, we consider a representation of monetary policy in which base growth feeds back on the shocks. The law of motion for the base is:

$$M_{t+1}^b = M_t^b(1 + x_t),$$

where x_t is the net growth rate of the monetary base. (Above, we have also used the notation, X_t, where $x_t = X_t/M_t^b$.) We have adopted the scaling, $m_t^b = M_t^b/(P_t z_t)$. So, the law of motion of m_t^b is:

$$\frac{M_{t+1}^b}{P_{t+1}z_{t+1}} = \frac{P_t z_t}{P_{t+1}z_{t+1}} \frac{M_t^b}{P_t z_t}(1 + x_t),$$

or,

$$m_{t+1}^b = \frac{1}{\pi_{t+1}\mu_{z,t+1}} m_t^b(1 + x_t).$$

Then,

$$(**)\ \hat{m}_{t+1}^b = \hat{m}_t^b + \frac{x}{1+x}\hat{x}_t - \hat{\pi}_{t+1} - \hat{\mu}_{z,t+1}$$

Monetary policy is characterized by a feedback from $\hat{x}_t$ to an innovation in monetary policy and to the innovation in all the other shocks in the economy. Let the $p-$ dimensional vector summarizing these innovations be denoted $\hat{\varphi}_t$, and suppose that the first element in $\hat{\varphi}_t$ is the innovation to monetary policy. Then, monetary policy has the following representation:

$$(**)\ \hat{x}_t = \sum_{i=1}^{p} x_{it},$$

where x_{it} is the component of money growth reflecting the i^{th} element in $\hat{\varphi}_t$. Also,

$$x_{it} = \rho_i x_{i,t-1} + \theta_i^0 \hat{\varphi}_{it} + \theta_i^1 \hat{\varphi}_{i,t-1}, \tag{2.53}$$

for $i = 1, ..., p$, with $\theta_1^0 \equiv 1$.

2.8. Goods Market Clearing

The demand for goods arises from several sources: the fraction, $1-\gamma$, of dying entrepreneurs who consume a fraction, Θ_t, of their net worth

$$\frac{\Theta(1-\gamma)V_t}{P_t},$$

purchases by entrepreneurs associated with utilization costs of capital,

$$a(u_t)\bar{K}_t,$$

purchases by households of consumption goods

$$C_t,$$

purchases by capital goods producers of investment goods,

$$I_t,$$

monitoring costs of banks,

$$\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R^k\right) Q_{\bar{K}',t-1}\bar{K}_t.$$

The household budget constraint is:

$$\begin{aligned} Q_{t+1} &= (1+R_{at})(A_t+X_t) - T_t - P_tC_t + (1-\Theta)(1-\gamma)V_t - W_t^e \\ &+(1+R_t^e)T_{t-1} + W_{j,t}h_{j,t} + M_t + \Pi_t^b + \Pi_t^k + \int \Pi_t^f df + A_{j,t}. \end{aligned}$$

Profits of the capital producers and the banks are:

$$\Pi_t^k = Q_{\bar{K}',t}\left[x + F(I_t, I_{t-1})\right] - Q_{\bar{K}',t}x - P_tI_t.$$

$$\begin{aligned} \Pi_t^b &= (A_t+X_t) + (1+R_t+R_{at})S_t^w - (1+R_{at})D_t - \left[(1+\psi_{k,t}R_t)\, P_t r_t^k K_t^b\right] \\ &+ \left[1 + R_t^e + \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega)\left(1+R_t^k\right) Q_{\bar{K}',t-1}\bar{K}_t}{Q_{\bar{K}',t-1}\bar{K}_t - N_t}\right] B_t \\ &-\mu \int_0^{\bar{\omega}_t} \omega dF(\omega)\left(1+R_t^k\right) Q_{\bar{K}',t-1}\bar{K}_t - \left(1+R_t^e\right) T_{t-1} \\ &+T_t - B_{t+1} \end{aligned}$$

so,

$$\Pi_t^k + \Pi_t^b = (A_t + X_t) + R_t S_t^w - (1 + R_{at})(A_t + X_t) - \left[(1 + \psi_{k,t} R_t) P_t r_t^k K_t^b\right]$$

where we have used

$$D_t = A_t + X_t + S_t^w, T_{t-1} = B_t, T_t = B_{t+1}, B_t = Q_{\bar{K}',t-1}\bar{K}_t - N_t$$

2.9. Resource Constraint

We now develop the aggregate resource constraint for this economy. For this, we make use of the tricks of Tak Yun. Thus, define Y^* at the unweighted sum of output of the intermediate good producers:

$$Y^* = \int_0^1 Y(f)df = \int_0^1 F(\epsilon, z, K(f), l(f))df$$

and, assuming production is positive everywhere,

$$F(\epsilon, z, K(f), l(f)) = \epsilon z^{1-\alpha} K(f)^{\alpha} l(f)^{1-\alpha} - z\phi.$$

Here, by $l(f)$, we mean homogeneous labor hired by the f^{th} intermediate good firm, $f \in (0,1)$. Recall that all firms confront the same wage rate and rental rate on capital. As a result, they all have the same capital-labor ratio, $K(f)/l(f)$. Moreover, this ratio coincides with the ratio of the aggregate inputs:

$$\frac{K^f}{l^f},\ K^f = \int_0^1 K(f)df,\ l^f = \int_0^1 l(f)df,$$

where K^f and l^f are aggregate capital and labor used in the goods producing sector, respectively. Then,

$$\begin{aligned} Y^* &= \int_0^1 \left[z^{1-\alpha}\epsilon K(f)^{\alpha} l(f)^{1-\alpha} - z\phi\right] df \\ &= \int_0^1 \left[\epsilon z^{1-\alpha}\left(\frac{K^f}{l^f}\right)^{\alpha} l(f) - z\phi\right] df \\ &= \left[\epsilon z^{1-\alpha}\left(\frac{K^f}{l^f}\right)^{\alpha} l^f - z\phi\right] \\ &= F(\epsilon, z, K^f, l^f) \end{aligned}$$

The demand curve for $Y(f)$ is

$$\left(\frac{P}{P(f)}\right)^{\frac{\lambda_f}{\lambda_f - 1}} = \frac{Y(f)}{Y},$$

so that

$$\begin{aligned} Y^* &= \int_0^1 Y(f)df = \int_0^1 Y\left(\frac{P}{P(f)}\right)^{\frac{\lambda_f}{\lambda_f-1}} df \\ &= YP^{\frac{\lambda_f}{\lambda_f-1}} \int_0^1 P(f)^{\frac{\lambda_f}{1-\lambda_f}} df \\ &= YP^{\frac{\lambda_f}{\lambda_f-1}} (P^*)^{\frac{\lambda_f}{1-\lambda_f}}, \end{aligned}$$

where

$$P^* = \left[\int_0^1 P(f)^{\frac{\lambda_f}{1-\lambda_f}} df\right]^{\frac{1-\lambda_f}{\lambda_f}}$$

Then,

$$Y^* = Y\left(\frac{P}{P^*}\right)^{\frac{\lambda_f}{\lambda_f-1}}.$$

Then, putting these things together,

$$Y = \left(\frac{P^*}{P}\right)^{\frac{\lambda_f}{\lambda_f-1}} \left[z^{1-\alpha}\epsilon\left(K^f\right)^{\alpha}\left(l^f\right)^{1-\alpha} - z\phi\right],$$

or,

$$\begin{aligned} &\mu\int_0^{\bar{\omega}_t} \omega dF(\omega)\left(1+R_t^k\right) Q_{\bar{K}',t-1}\bar{K}_t + a(u)\bar{K}_t + \frac{\Theta(1-\gamma)V_t}{P_t} + G_t + C_t + I_t \\ = &\left(\frac{P^*}{P}\right)^{\frac{\lambda_f}{\lambda_f-1}} \left[z^{1-\alpha}\epsilon(\nu_t^k K_t)^{\alpha}\left(\nu_t^l l_t\right)^{1-\alpha} - z_t\phi\right], \end{aligned}$$

where

$$K_t^f = \nu^k K_t,\ l_t^f = \nu^l l_t.$$

The left side of the goods market clearing condition displays the uses of goods: payment for monitoring costs, utilization costs of capital, last meals of entrepreneurs slated for death, consumption and investment. The right side displays the source of goods. Here, G_t denotes government consumption:

$$G_t = z_t g_t.$$

Note that l is the sum of all employment of the labor 'produced' by the representative labor contractor. It is not necessarily the simple sum over all the labor supplied by households. We now relate l to the sum of differentiated labor supplied by households. Recall,

$$l = \left[\int_0^1 (h_j)^{\frac{1}{\lambda_w}} dj \right]^{\lambda_w}, \ 1 \leq \lambda_w < \infty.$$

Let the sum of the differentiated labor supplied by households be denoted by l^* :

$$L = \int_0^1 h^j dj$$

Substituting out using the labor demand curve:

$$\begin{aligned} L &= \int_0^1 \left[\frac{W_j}{W} \right]^{\frac{\lambda_w}{1-\lambda_w}} l dj \\ &= l W^{\frac{\lambda_w}{\lambda_w - 1}} (W^*)^{\frac{\lambda_w}{1-\lambda_w}} \\ &= l \left(\frac{W}{W^*} \right)^{\frac{\lambda_w}{\lambda_w - 1}}, \end{aligned}$$

where

$$W^* = \left[\int_0^1 W_j^{\frac{\lambda_w}{1-\lambda_w}} dj \right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

Finally,

$$l = \left(\frac{W^*}{W} \right)^{\frac{\lambda_w}{\lambda_w - 1}} L$$

Using an argument like the one used for prices, we find:

$$\begin{aligned} &\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1 + R^k\right) Q_{\bar{K}',t-1} \bar{K} + a(u) \bar{K} + \Theta(1-\gamma) v_t z_t + G_t + C_t + I_t \quad (2.54) \\ \leq \ & (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left\{ z_t^{1-\alpha} \epsilon \left(\nu_t^k K_t \right)^{\alpha} \left[\nu_t^l (w_t^*)^{\frac{\lambda_w}{\lambda_w - 1}} L_t \right]^{1-\alpha} - z_t \phi \right\}, \end{aligned}$$

where

$$v_t = \frac{V_t}{z_t P_t}, \ p^* = \frac{P^*}{P}, \ w^* = \frac{W^*}{W}.$$

Both p_t^* and w_t^* represent 'efficiency gaps'. It turns out that when non-optimized prices and wages are indexed, either to steady state inflation, or to actual lagged inflation, these efficiency gaps are roughly constant. This was shown by Tak Yun, and is demonstrated in the first subsection below.

The pricing equation is:

$$P_t^* = \left[(1-\xi_p) \left(\tilde{P}_t\right)^{\frac{\lambda_f}{1-\lambda_f}} + \xi_p \left(\pi_{t-1} P_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}} \right]^{\frac{1-\lambda_f}{\lambda_f}} .$$

Dividing by P_t :

$$p_t^* = \left[(1-\xi_p) \left(\tilde{p}_t\right)^{\frac{\lambda_f}{1-\lambda_f}} + \xi_p \left(\frac{\pi_{t-1}}{\pi_t} p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}} \right]^{\frac{1-\lambda_f}{\lambda_f}} . \tag{2.55}$$

Linearizing:

$$\begin{aligned} p^* \hat{p}_t^* \;\; = \;\; & \frac{1-\lambda_f}{\lambda_f} \left(p^*\right)^{\frac{\lambda_f}{1-\lambda_f}\left[\frac{1-\lambda_f}{\lambda_f}-1\right]} [(1-\xi_p) \frac{\lambda_f}{1-\lambda_f} \left(\tilde{p}_t\right)^{\frac{\lambda_f}{1-\lambda_f}-1} \tilde{p}\widehat{\tilde{p}}_t + \xi_p \frac{\lambda_f}{1-\lambda_f} \left(\frac{\pi_{t-1}}{\pi_t} p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}-1} \\ & \times (\frac{1}{\pi_t} p_{t-1}^* \pi \hat{\pi}_{t-1} - \frac{\pi_{t-1}}{\pi_t^2} p_{t-1}^* \pi \hat{\pi}_t + \frac{\pi_{t-1}}{\pi_t} p^* \hat{p}_{t-1}^*)] \end{aligned}$$

Evaluating in steady state when - from (C.2) and (2.55) respectively - we see $\tilde{p} = p^* = 1$:

$$\hat{p}_t^* = (1-\xi_p) \widehat{\tilde{p}}_t + \xi_p \left(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{p}_{t-1}^*\right). \tag{2.56}$$

We find that in principle p_t^* could move around with other variables. If a shock were to move p_t^*, it would have to do so via its impact on one of the other variables on the right of (2.56). However, in practice those other variables must comove in such a way that their impact on p_t^* cancels out, so that p_t^* is only a function of p_{t-1}^*. Thus, movements in other variables cannot get into p_t^*, at least to a first approximation.

To establish these observations requires linearizing $\widehat{\tilde{p}}_t$ in the expression for the aggregate price level:

$$P_t = \left[(1-\xi_p) \left(\tilde{P}_t\right)^{\frac{1}{1-\lambda_f}} + \xi_p \left(\pi_{t-1} P_{t-1}\right)^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f} .$$

Divide on both sides by P_t :

$$1 = \left[(1-\xi_p) \left(\tilde{p}_t\right)^{\frac{1}{1-\lambda_f}} + \xi_p \left(\frac{\pi_{t-1}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}} \right]^{1-\lambda_f} .$$

Linearizing this:

$$0 = (1-\xi_p)\frac{1}{1-\lambda_f}\left(\tilde{p}_t\right)^{\frac{1}{1-\lambda_f}-1}\tilde{p}\widehat{\tilde{p}}_t + \xi_p\frac{1}{1-\lambda_f}\left(\frac{\pi_{t-1}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}-1}\left[\frac{1}{\pi_t}\pi\hat{\pi}_{t-1} - \frac{\pi_{t-1}}{\pi_t^2}\pi\hat{\pi}_t\right],$$

or, in steady state:

$$0 = (1-\xi_p)\widehat{\tilde{p}}_t + \xi_p\left[\hat{\pi}_{t-1} - \hat{\pi}_t\right]$$

Substituting this back into (2.56), we obtain:

$$\hat{p}_t^* = -\xi_p\left[\hat{\pi}_{t-1} - \hat{\pi}_t\right] + \xi_p\left(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{p}_{t-1}^*\right),$$

which implies

$$\hat{p}_t^* = \xi_p\hat{p}_{t-1}^*.$$

this establishes the result sought. Namely, p_t^* can be treated as being a function only of p_{t-1}^*. Since the linear relationship is damped, it is fair to suppose that $p_t^* = 1$ always, and that $\hat{p}_t^* = 0$.

We now turn to evaluating the evolution of w_t^*:

$$W_t^* = \left[(1-\xi_w)\left(\tilde{W}_t\right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w\left(\mu_{zt}\pi_{t-1}W_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

Divide both sides by W:

$$\frac{W_t^*}{W_t} = \left[(1-\xi_w)\left(\frac{\tilde{W}_t}{W_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w\left(\mu_{zt}\pi_{t-1}\frac{W_{t-1}}{W_t}\frac{W_{t-1}^*}{W_{t-1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}$$

or,

$$w_t^* = \left[(1-\xi_w)\left(\tilde{w}_t\right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w\left(\mu_{zt}\pi_{t-1}\frac{z_{t-1}P_{t-1}w_{t-1}}{z_tP_tw_t}\frac{W_{t-1}^*}{W_{t-1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}},$$

where

$$\tilde{w}_t = \frac{\tilde{W}_t}{W_t},\ w_t^* = \frac{W_t^*}{W_t},\ w_t = \frac{W_t}{z_tP_t}.$$

Then,

$$w_t^* = \left[(1-\xi_w)\left(\tilde{w}_t\right)^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w\left(\frac{\pi_{t-1}}{\pi_t}\frac{w_{t-1}}{w_t}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

$$
\begin{aligned}
w^* \hat{w}_t^* &= \frac{1-\lambda_w}{\lambda_w} \left(w_t^*\right)^{\frac{\lambda_w}{1-\lambda_w}\left[\frac{1-\lambda_w}{\lambda_w}-1\right]} \times \\
& [(1-\xi_w)\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_t + \xi_w \frac{\lambda_w}{1-\lambda_w}\left(\frac{\pi_{t-1}}{\pi_t}\frac{w_{t-1}}{w_t}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}-1} \\
& \times (\frac{1}{\pi_t}\frac{w_{t-1}}{w_t}w_{t-1}^*\pi\hat{\pi}_{t-1} - \frac{\pi_{t-1}}{\pi_t^2}\frac{w_{t-1}}{w_t}w_{t-1}^*\pi\hat{\pi}_t \\
& + \frac{\pi_{t-1}}{\pi_t}\frac{1}{w_t}w_{t-1}^* w\hat{w}_{t-1} - \frac{\pi_{t-1}}{\pi_t}\frac{w_{t-1}}{w_t^2}w_{t-1}^* w\hat{w}_t + \frac{\pi_{t-1}}{\pi_t}\frac{w_{t-1}}{w_t}w^*\hat{w}_{t-1}^*)]
\end{aligned}
$$

Evaluating this in steady state:

$$
\hat{w}_t^* = (1-\xi_w)\left(w_t^*\right)^{-\frac{\lambda_w}{1-\lambda_w}}\widehat{\tilde{w}}_t + \xi_w(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} - \hat{w}_t + \hat{w}_{t-1}^*)
$$

We now obtain w^* in steady state from the w_t^* equation above:

$$
w^* = \left[(1-\xi_w) + \xi_w \left(w^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}.
$$

From this it is obvious that $w^* = 1$. So,

$$
\hat{w}_t^* = (1-\xi_w)\widehat{\tilde{w}}_t + \xi_w(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} - \hat{w}_t + \hat{w}_{t-1}^*), \tag{2.57}
$$

and in principle w_t^* moves around. However, it turns out that the comovements between $\tilde{w}_t$ and π_{t-1}, π_t induced by the aggregate wage equation are such that, to a first approximation, shocks which make the latter variables move around, have no impact on w_t^*. This is why in the end we can just set w_t^* to its steady state value. This is why this variable does not, to a first approximation, enter as a kind of 'Solow residual' in the aggregate resource constraint.

To figure out $\hat{w}_t^*$, we need an expression for $\widehat{\tilde{w}}_t$. For this, we work with the expression for the aggregate wage:

$$
W_t = \left[(1-\xi_w)\tilde{W}_t^{\frac{1}{1-\lambda_w}} + \xi_w\left(\mu_{z,t}\pi_{t-1}W_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.
$$

Divide both sides by $z_t P_t$:

$$
w_t = \left[(1-\xi_w)\left(\frac{\tilde{W}_t}{W_t}\frac{W_t}{z_t P_t}\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\frac{z_t}{z_{t-1}}\frac{\pi_{t-1}W_{t-1}}{z_t P_{t-1}}\frac{P_{t-1}}{P_t}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w},
$$

or,

$$w_t = \left[(1-\xi_w) \left(\tilde{w}_t w_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1}}{\pi_t} w_{t-1} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}.$$

Totally differentiating this:

$$\begin{aligned} w\hat{w}_t &= (1-\lambda_w)\left(w_t\right)^{\frac{-\lambda_w}{1-\lambda_w}} [(1-\xi_w) \frac{1}{1-\lambda_w} \left(\tilde{w}_t w_t\right)^{\frac{1}{1-\lambda_w}-1} \left(\tilde{w}_t w \hat{w}_t + \tilde{w} w_t \hat{\tilde{w}}_t \right) \\ &\quad + \xi_w \frac{1}{1-\lambda_w} \left(\frac{\pi_{t-1}}{\pi_t} w_{t-1} \right)^{\frac{1}{1-\lambda_w}-1} \left(\frac{1}{\pi_t} w_{t-1} \pi \hat{\pi}_{t-1} - \frac{\pi_{t-1}}{\pi_t^2} w_{t-1} \pi \hat{\pi}_t + \frac{\pi_{t-1}}{\pi_t} w \hat{w}_{t-1} \right)] \end{aligned}$$

or, taking into account $\tilde{w} = 1$,

$$\hat{w}_t = (1-\xi_w)\left(\hat{w}_t + \hat{\tilde{w}}_t \right) + \xi_w \left(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} \right)$$

From the earlier expression, (2.57),

$$(1-\xi_w)\hat{\tilde{w}}_t = \hat{w}_t^* - \xi_w(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} - \hat{w}_t + \hat{w}_{t-1}^*)$$

Combining the previous two expressions:

$$\hat{w}_t = (1-\xi_w)\hat{w}_t + \hat{w}_t^* - \xi_w(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} - \hat{w}_t + \hat{w}_{t-1}^*) + \xi_w \left(\hat{\pi}_{t-1} - \hat{\pi}_t + \hat{w}_{t-1} \right)$$

or,

$$\hat{w}_t^* = \xi_w \hat{w}_{t-1}^*.$$

According to this, if $\hat{w}_t^*$ starts in steady state, it will stay there, even if there are shocks. So, we can just assume $\hat{w}_t^* = 0$ for all t.

To summarize what we have so far, write (2.54) neglecting terms of p_t^* and w_t^* :

$$\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t + a(u_t)\bar{K}_t + G_t + C_t + I_t \leq \left\{ z_t^{1-\alpha} \epsilon \left(\nu_t^k K^f \right)^\alpha \left(\nu_t^l L_t \right)^{1-\alpha} - z_t \phi \right\}.$$

Scaling this by z_t:

$$d_t + a(u_t)\frac{\bar{K}_t}{z_t} + g_t + \frac{C_t}{z_t} + \frac{I_t}{z_t} + \Theta(1-\gamma)v_t \leq \epsilon_t \left(u_t \frac{\nu_t^k \bar{K}_t}{z_t} \right)^\alpha \left(\nu_t^l L_t \right)^{1-\alpha} - \phi,$$

where

$$
\begin{aligned}
d_t &= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) Q_{\bar{K}',t-1}\bar{K}_t}{z_t P_t} \\
&= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) P_{t-1}q_{t-1}\bar{K}_t}{z_t P_t} \\
&= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) q_{t-1}\bar{K}_t}{z_t}\frac{1}{\pi_t}
\end{aligned}
$$

or,

$$
d_t + a(u_t)\frac{z_{t-1}}{z_t}\frac{\bar{K}_t}{z_{t-1}} + g_t + \frac{C_t}{z_t} + \frac{I_t}{z_t} + \Theta(1-\gamma)v_t \leq \epsilon_t \left(u_t \frac{z_{t-1}}{z_t}\nu_t^k \frac{\bar{K}_t}{z_{t-1}}\right)^{\alpha} \left(\nu_t^l L_t\right)^{1-\alpha} - \phi,
$$

or,

$$
d_t + a(u_t)\frac{1}{\mu_{z,t}}\bar{k}_t + g_t + c_t + i_t + \Theta(1-\gamma)v_t \leq \epsilon_t \left(u_t \frac{1}{\mu_{z,t}}\nu_t^k \bar{k}_t\right)^{\alpha} \left(\nu_t^l L_t\right)^{1-\alpha} - \phi, \tag{2.58}
$$

where

$$
\bar{k}_t = \frac{\bar{K}_t}{z_{t-1}}.
$$

The reason $\bar{K}_t$ has to be scaled by lagged z_{t-1} and not z_t is that it is $\bar{K}_{t+1}$ that is chosen in period t. If we scaled $\bar{K}_t$ by z_t, then think of the implication that $\bar{k}_{t+1}$ is chosen in period t. Since z_{t+1} is not known in period t, this would be tantamount to assuming we are choosing $\bar{K}_{t+1}$ as the outcome of a random mechanism. We also adopt the notation:

$$
c_t = \frac{C_t}{z_t},\ i_t = \frac{I_t}{z_t}.
$$

It is useful to have a linearization of the d_t equation. Rewriting this in a slightly more convenient form:

$$
d_t = \mu G(\bar{\omega}_t)\left(1+R_t^k\right) q_{t-1}\bar{k}_t \frac{1}{\mu_{z,t}\pi_t}.
$$

Then,

$$
\begin{aligned}
\hat{d}_t &= \widehat{G(\bar{\omega}_t)} + \frac{R^k}{1+R^k}\hat{R}_t^k + \hat{q}_{t-1} + \hat{\bar{k}}_t - \hat{\mu}_{z,t} - \hat{\pi}_t \\
&= \frac{G'(\bar{\omega})}{G(\bar{\omega})}\bar{\omega}\widehat{\bar{\omega}}_t + \frac{R^k}{1+R^k}\hat{R}_t^k + \hat{q}_{t-1} + \hat{\bar{k}}_t - \hat{\mu}_{z,t} - \hat{\pi}_t
\end{aligned}
$$

Linearize the scaled resource constraint:

$$
\begin{aligned}
& d\hat{d}_t + a'(u_t)\frac{1}{\mu_{z,t}}\bar{k}_t u\hat{u}_t + a(u_t)\frac{1}{\mu_{z,t}}\bar{k}\widehat{\bar{k}}_t - a(u_t)\frac{1}{\mu_{z,t}^2}\bar{k}_t\mu_z\hat{\mu}_{z,t} + g\hat{g}_t + c\hat{c}_t + i\hat{\imath}_t \\
+\Theta(1-\gamma)v\hat{v}_t \quad = \quad & \alpha\left(u_t\frac{\nu_t^k}{\mu_{z,t}}\bar{k}_t\right)^{\alpha-1}\left(\nu_t^l L_t\right)^{1-\alpha}[\frac{\nu_t^k}{\mu_{z,t}}\bar{k}_t u\hat{u}_t - u_t\frac{\nu_t^k}{\mu_{z,t}^2}\bar{k}_t\mu_z\hat{\mu}_{z,t} + u_t\frac{\nu_t^k}{\mu_{z,t}}\bar{k}\widehat{\bar{k}}_t + u_t\frac{1}{\mu_{z,t}}\bar{k}_t\nu^k\hat{\nu}_t^k] \\
& +\epsilon_t(1-\alpha)\left(u_t\frac{\nu_t^k}{\mu_{z,t}}\bar{k}_t\right)^{\alpha}(\nu_t^l L_t)^{-\alpha}\left[\nu_t^l L\hat{L}_t + \nu^l L\hat{\nu}_t^l\right] + \left(u_t\frac{\nu_t^k}{\mu_{z,t}}\bar{k}_t\right)^{\alpha}\left(\nu_t^l L_t\right)^{1-\alpha}\epsilon\hat{\epsilon}_t \\
& +\left(\frac{1}{\mu_z}\nu^k\bar{k}\right)^{\alpha}\left(\nu^l L\right)^{1-\alpha}\widehat{(p_t^*)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}}}
\end{aligned}
$$

$$
\begin{aligned}
f(\lambda_{t,t}) &= (p_t^*)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \\
&= \exp\left[\frac{\lambda_{f,t}}{\lambda_{f,t}-1}\log\left(p_t^*\right)\right] \\
f'(\lambda_{f,t}) &= \log\left(p_t^*\right)\left[\frac{\lambda_t}{\lambda_f-1} - \frac{\lambda_f}{\left(\lambda_f-1\right)^2}\right]\hat{\lambda}_{f,t} \\
&= 0,
\end{aligned}
$$

since $p_t^* = 1$ in steady state.

Evaluating in steady state, when $u_t = 1$ and $a(1) = 0$ and $a'(1) = a'$:

$$
\begin{aligned}
& d\hat{d}_t + a'\frac{1}{\mu_z}\bar{k}\hat{u}_t + g\hat{g}_t + c\hat{c}_t + i\hat{\imath}_t + \Theta(1-\gamma)v\hat{v}_t \\
= \quad & \alpha\left(\frac{\nu^k}{\mu_z}\bar{k}\right)^{\alpha}\left(\nu^l L\right)^{1-\alpha}[\hat{u}_t - \hat{\mu}_{z,t} + \widehat{\bar{k}}_t + \hat{\nu}_t^k] \\
& +(1-\alpha)\left(\frac{\nu^k}{\mu_z}\bar{k}\right)^{\alpha}(\nu^l L)^{1-\alpha}\left[\hat{L}_t + \hat{\nu}_t^l\right] + \left(\frac{\nu^k}{\mu_z}\bar{k}\right)^{\alpha}\left(\nu^l L\right)^{1-\alpha}\hat{\epsilon}_t
\end{aligned}
$$

Output is the sum of consumption and investment. Thus,

$$
g_t + c_t + i_t + \Theta(1-\gamma)v_t = y_t
$$

so that:

$$
y_t = \epsilon_t\left(u_t\frac{1}{\mu_{z,t}}\nu_t^k\bar{k}_t\right)^{\alpha}\left(\nu_t^l L_t\right)^{1-\alpha} - \phi - a(u_t)\frac{1}{\mu_{z,t}}\bar{k}_t - d_t,
$$

and, in steady state:

$$y=\left(\frac{\nu^k}{\mu_z}\bar{k}\right)^{\alpha}\left(\nu^l L\right)^{1-\alpha}-\phi-d.$$

The variable, y_t, is our measure of final output of goods, scaled by z_t. This does not correspond exactly to gnp as it is measured in the data, because that includes services from the banking sector. We should be able to figure out a *right* measure of final banking services to add to y_t, to get a full measure of final goods and services production. Presumably, the services from the demand deposits held by firms should not be included in final output. They should be thought of as intermediate goods, already included in y_t. But, the services of the demand deposits held by households seem not to be included in y_t. How to measure those? One possibility would be to measure them as $(R_t-R_{at})D_t/P_t$.

Continuing with the linearization of the resource constraint

$$\begin{aligned} & d\hat{d}_t+a'\frac{1}{\mu_z}\bar{k}\hat{u}_t+g\hat{g}_t+c\hat{c}_t+i\hat{\imath}_t+\Theta(1-\gamma)v\hat{v}_t \\ = & \ [y+\phi+d]\,[\alpha\left(\hat{u}_t-\hat{\mu}_{z,t}+\hat{\bar{k}}_t+\hat{\nu}_t^k\right)+(1-\alpha)\left(\hat{L}_t+\hat{\nu}_t^l\right)+\hat{\epsilon}_t], \end{aligned} \tag{2.59}$$

or

$$d_y\hat{d}_t+u_y\hat{u}_t+g_y\hat{g}_t+c_y\hat{c}_t+\frac{i}{y+\phi+d}\hat{\imath}_t+\Theta(1-\gamma)v_y\hat{v}_t=\alpha\left(\hat{u}_t-\hat{\mu}_{z,t}+\hat{\bar{k}}_t+\hat{\nu}_t^k\right)+(1-\alpha)\left(\hat{L}_t+\hat{\nu}_t^l\right)+\hat{\epsilon}_t,$$

where

$$\begin{aligned} c_y &= \frac{c}{y+\phi+d},\ g_y=\frac{g}{y+\phi+d},\ d_y=\frac{d}{y+\phi+d} \\ v_y &= \frac{v}{y+\phi+d},\ u_y=\frac{a'\frac{1}{\mu_z}\bar{k}}{y+\phi+d}. \end{aligned}$$

The next step is to develop an expression for $\hat{\imath}_t$ in terms of physical capital. The capital accumulation technology that we assume is:

$$\bar{K}_{t+1}=(1-\delta)\bar{K}_t+F(I_t,I_{t-1}).$$

Divide both sides by z_t :

$$\bar{k}_{t+1}=(1-\delta)\frac{1}{\mu_{z,t}}\bar{k}_t+\frac{F(I_t,I_{t-1})}{z_t}.$$

The adjustment cost function is:

$$F(I,I_{-1})=\left[1-S(I/I_{-1})\right]I,$$

where I_{-1} denotes last period's level of investment. We suppose that $S = S' = 0$ and $S'' < 0$ in non-stochastic steady state. The growth rate of investment is $\mu_z > 1$ in non-stochastic steady state:

$$\frac{I}{I_{-1}} = \frac{i \times z}{i_{-1} \times z_{-1}} = \mu_z.$$

Evaluating the capital accumulation technology in steady state:

$$\bar{k} = (1-\delta)\frac{1}{\mu_z}\bar{k} + i,$$

so that:

$$i = \bar{k}\left[1 - \frac{1-\delta}{\mu_z}\right] = \bar{k}\frac{\mu_z - 1 + \delta}{\mu_z}.$$

Writing out the adjustment costs in terms of the scaled variables:

$$\frac{F(I_t, I_{t-1})}{z_t} = \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t = f\left(i_t, i_{t-1}, \mu_{z,t}\right) = f_t,$$

say. Linearizing f:

$$f\hat{f}_t = \left[-S'\frac{\mu_{z,t}}{i_{t-1}}i_t + \left(1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right)\right] i\hat{\imath}_t - S'\frac{i_t^2 \mu_z}{i_{t-1}}\hat{\mu}_{zt} + S'\frac{i_t \mu_{z,t}}{i_{t-1}^2} i_t i \hat{\imath}_{t-1}.$$

Evaluating this in steady state:

$$f\hat{f}_t = i\hat{\imath}_t$$

The capital accumulation technology is:

$$\bar{k}_{t+1} = (1-\delta)\frac{1}{\mu_{z,t}}\bar{k}_t + \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t.$$

Linearizing this, and evaluating the result in steady state:

$$\bar{k}\hat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_z}\bar{k}\hat{\bar{k}}_t - (1-\delta)\frac{1}{\mu_z}\bar{k}\hat{\mu}_{z,t} + i\hat{\imath}_t,$$

so that

$$(**)\ \hat{\bar{k}}_{t+1} = \frac{1-\delta}{\mu_z}\left(\hat{\bar{k}}_t - \hat{\mu}_{z,t}\right) + \frac{i}{\bar{k}}\hat{\imath}_t.$$

Substituting this into (2.59):

$$\begin{aligned} & d_y\hat{d}_t + u_y\hat{u}_t + g_y\hat{g}_t + c_y\hat{c}_t + \bar{k}_y\frac{i}{\bar{k}}\hat{\imath}_t + \Theta(1-\gamma)v_y\hat{v}_t \\ = & \ \alpha\left(\hat{u}_t - \hat{\mu}_{z,t} + \hat{\bar{k}}_t + \hat{\nu}_t^k\right) + (1-\alpha)\left(\hat{L}_t + \hat{\nu}_t^l\right) + \hat{\epsilon}_t, \end{aligned}$$

where

$$\bar{k}_y = \frac{\bar{k}}{y+\phi+d}.$$

It is convenient to substitute out for $\hat{d}_t$ from (2.59):

$$\begin{aligned} & (**)\ d_y\left[\frac{G'(\bar{\omega})}{G(\bar{\omega})}\bar{\omega}\widehat{\bar{\omega}}_t + \frac{R^k}{1+R^k}\hat{R}_t^k + \hat{q}_{t-1} + \hat{\bar{k}}_t - \hat{\mu}_{z,t} - \hat{\pi}_t\right] \\ & +u_y\hat{u}_t + g_y\hat{g}_t + c_y\hat{c}_t + \bar{k}_y\frac{i}{\bar{k}}\hat{\imath}_t + \Theta(1-\gamma)v_y\hat{v}_t \\ = & \ \alpha\left(\hat{u}_t - \hat{\mu}_{z,t} + \hat{\bar{k}}_t + \hat{\nu}_t^k\right) + (1-\alpha)\left(\hat{L}_t + \hat{\nu}_t^l\right) + \hat{\epsilon}_t, \end{aligned}$$

3. Solving the Model

To solve the model, we to do three things: determine the steady state, collect the linearized equations that characterize the equilibrium, and combine the latter into matrix format.

3.1. Steady State

We now develop equations for the steady state of our benchmark model. For purposes of these calculations, the exogenously set variables are:

$$\begin{aligned} & \tau^l,\ \tau^c,\ \beta,\ F(\bar{\omega}),\ \mu,\ x,\ \mu_z,\ \lambda_f,\ \lambda_w,\ \alpha,\ \psi_k,\ \psi_l,\ \delta,\ \upsilon, \\ & \tau^k,\ \gamma,\ \tau,\ \tau^T,\ \tau^D,\ \sigma_L,\ \zeta,\ \sigma_q,\ \theta,\ \upsilon,\ w^e,\ \nu^l,\ \nu^k,\ m,\ \eta_g,\ r^k \end{aligned}$$

The variables to be solved for are

$$q,\ \pi,\ R^e,\ R_a,\ h_{e^r},\ R,\ R^k,\ \bar{\omega},\ k,\ n,\ i,\ w,\ l,\ c,\ u_c^z,\ m^b,\ \lambda_z,\ \psi_L,\ e_z^r,\ e_v,\ a^b x^b,\ \xi,\ h_{K^b},\ y,\ g$$

The equations available for solving for these unknowns are summarized below. The first three variables are trivial functions of the structural parameters, and from here on we treat them as known. There remain 22 unknowns. Below, we have 22 equations that can be used to solve for them.

The algorithm proceeds as follows. Solve for R_a using (3.16); h_{e^r} using (3.12).

We now compute R to enforce (3.8). This equation is a nonlinear function of R. For a given R, evaluate (3.8) as follows. Solve for R^k using (3.4); solve for $\bar{\omega}$ using (3.5); solve for k and n using (3.6) and (3.7); solve for i using (3.3); solve for w using (3.1); solve for l using (3.2); solve for c using (3.18); solve (3.20) and (3.21) for g and y; solve for u_c^z using (3.16); solve for m^b and λ_z using (3.15) and (3.15); solve (3.17) for ψ_L; solve for e_z^r using (3.14); solve ξ from (3.13); solve e_v from (3.11); solve $a^b x^b$ from (3.10); h_{K^b} from (3.9). Vary R until (3.8) is satisfied. In these calculations, all variables must be positive, and:

$$0 \leq m \leq 1 + x,\ 0 \leq \xi \leq 1,\ \lambda_z > 0,\ k > n > 0.$$

We are interested in solving the steady state when the 'exogenous variables' are the economically exogenous ones, and the 'endogenous variables' are the economically endogenous ones. In particular, consider the situation in which the exogenous variables are:

$$\tau^l,\ \tau^c,\ \beta,\ F(\bar{\omega}),\ \mu,\ x,\ \mu_z,\ \lambda_f,\ \lambda_w,\ \alpha,\ \psi_k,\ \psi_l,\ \delta,\ \upsilon,\ a^b x^b,\ \xi,$$
$$\tau^k,\ \gamma,\ \tau,\ \tau^T,\ \tau^D,\ \sigma_L,\ \zeta,\ \sigma_q,\ \theta,\ \upsilon,\ w^e,\ \nu^l,\ \eta_g,\ \psi_L,$$

and the variables to be solved for are:

$$q,\ \pi,\ R^e,\ R_a,\ h_{e^r},\ r^k,\ R^k,\ \bar{\omega},\ k,\ n,\ i,\ w,\ l,\ c,\ u_c^z,\ m^b,\ R,\ \lambda_z,\ e_z^r,\ e_v,\ h_{K^b},\ y,\ g,\ \nu^k,\ m$$

We solve for the above 25 variables as follows. The first three are solved in the same way as before. The remainder are solved by solving three equations, (3.8), (3.10) and (3.12), in the three unknowns, r^k, ν^k and R. Ideally, we start in a neighborhood of the solution obtained in the previous calculations. Fix a set of values for r^k, ν^k and R. The basic sequence of calculations is the same as above. Solve for R^k using (3.4), and then $\bar{\omega}$ using (3.5), and then for k, n, and i using (3.6), (3.7) and (3.3) in that order. Then, we obtain w from (3.1) and l from (3.2). The resource constraint, (3.18) can be used to obtain c, and (3.20), (3.21) can be used to compute y, g. Then, obtain λ_z and u_c^z from (3.16) and (3.17). Solve (3.15), (3.15) and (3.16) for R_a, m, and m^b. This can be made into a one-dimension search in m. In particular, for a given m, solve for R_a from (3.16) and for m^b from (3.15). Vary m until (3.15) is satisfied. Compute h_{e^r}, h_{K^b}, e_ν, e_z^r, using (3.9), (3.11), (3.13), and (3.14). We can now evaluate (3.8), (3.10) and (3.12). Vary r^k, ν^k and R until these equations are satisfied.

We are also interested in a version of our model in which the entrepreneurial sector has been removed. In practical terms, this means dropping the entrepreneur sector equations (equations (3.5), (3.6) and (3.7) below) and replacing them by:

$$R^k = R^e.$$

Our strategy for solving the steady state adapts the first strategy described above. In particular, for purposes of the calculations, the exogenous variables are

$$\tau^l,\ \tau^c,\ \beta,\ \mu,\ x,\ \mu_z,\ \lambda_f,\ \lambda_w,\ \alpha,\ \psi_k,\ \psi_l,\ \delta,\ \upsilon,\ l,$$
$$\tau^k,\ \gamma,\ \tau,\ \tau^T,\ \tau^D,\ \sigma_L,\ \zeta,\ \sigma_q,\ \theta,\ \upsilon,\ w^e,\ \nu^l,\ \nu^k,\ m,\ \eta_g,\ r^k$$

The variables to be solved for are

$$q,\ \pi,\ R^e,\ r^k,\ R_a,\ h_{e^r},\ R,\ R^k,\ k,\ i,\ w,\ c,\ u_c^z,\ m^b,\ \lambda_z,\ \psi_L,\ e_z^r,\ e_v,\ a^b x^b,\ \xi,\ h_{K^b},\ y,\ g$$

The variables, $\bar{\omega}$ and n, have been removed and l has been shifted to the list of exogenous variables. Among the variables to be solved, q, π, R^e, R_a, r^k and R^k are now trivial. There remain 17 variables to be solved for. For this, there are 17 equations. The algorithm proceeds as follows. We view equation (3.8) as a nonlinear equation in R. We now discuss how to evaluate this equation. Fix a value for R. Solve (3.1)-(3.3) for l/k, w and i/k. Solve (3.18) with $\mu = 0$ for c/k. Solve for cu_c^z using (3.16). Multiplying each of (3.15) and (3.15) by c, those equations become two equations in unknowns, c/m^b and $c\lambda_z$. These can be found by doing a one-dimensional search in c/m^b. Solve (3.11) for e_ν and (3.12) for h_{e^r}. Solve (3.13) for ξ. Solve (3.10) for x^b. Solve for h_{K^b} using (3.9). Evaluate (3.8). Vary R until (3.8) is satisfied.

We are also concerned with a version of the model in which we drop both the entrepreneurial and banking sectors (see section 3.3 below). The exogenous variables are like the ones for the version of the model without the entrepreneurs, except that now we move m from the list of exogenous variables to the list of variables to be solved for (the variable, l, stays on the list of exogenous variables). As before, solving for q, π, R^e, r^k and R^k is trivial. The remaining variables are found by solving a single non-linear equation, (3.15), in $m \in [0, 1+x]$.[13] To evaluate (3.15), fix m. Solve for R_a using (3.16). Zero profits and zero costs in the financial intermediary then implies $R = R_a$. Solve for k, w, i, c and u_c^z as in the version of the problem with no entrepreneurs. Solve for m^b using the steady state version of the loan market clearing condition:

$$\psi_l w l + \frac{\psi_k r^k}{\mu_z}\bar{k} = m^b\left(1 - m + x\right).$$

Solve for λ_z using (3.15). Now evaluate (3.15).

Whether we adopt our benchmark preferences, (2.40), or the ACEL preferences does not substantially change the algorithm for finding the steady state. The relevant changes are indicated below.

[13] Actually, to ensure $R_a \geq 0$, $m \leq \theta(1+x)$.

3.1.1. Firm Sector

From the firm sector, and the assumption that there are no price distortions in a steady state, we have

$$s = \frac{1}{\lambda_f}.$$

Also, evaluating (2.3) in steady state:

$$\frac{1}{\lambda_f} = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(r^k\left[1+\psi_k R\right]\right)^{\alpha}\left(w\left[1+\psi_l R\right]\right)^{1-\alpha}, \tag{3.1}$$

Combining (2.3) and (2.4):

$$\frac{r^k\left[1+\psi_k R\right]}{w\left[1+\psi_l R\right]} = \frac{\alpha}{1-\alpha}\frac{\mu_z l}{\bar{k}} \tag{3.2}$$

3.1.2. Capital Producers

From the capital producers,

$$\lambda_{zt}q_t F_{1,t} - \lambda_{z,t} + \frac{\beta}{\mu_{z,t+1}}\lambda_{z,t+1}q_{t+1}F_{2,t+1} = 0$$

or, since $F_{1,t} = 1$ and $F_{2,t} = 0$,

$$q = 1.$$

Also,

$$\bar{k}_{t+1} = (1-\delta)\frac{1}{\mu_{z,t}}\bar{k}_t + \left[1 - S\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)\right]i_t,$$

so that in steady state, when $S = 0$,

$$\frac{i}{\bar{k}} = 1 - \frac{1-\delta}{\mu_z}. \tag{3.3}$$

3.1.3. Entrepreneurs

From the entrepreneurs:

$$r^k = a'.$$

Also,

$$u = 1.$$

The after tax rate of return on capital, in steady state, is:

$$R^k = \left[(1-\tau^k)r^k + (1-\delta)\right]\pi + \tau^k\delta - 1 \tag{3.4}$$

Conditional on a value for R^k, R^e, the steady state value for $\bar{\omega}$ may be found using the following equation:

$$[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e} + \frac{1}{1-\mu\bar{\omega}h(\bar{\omega})}\left[\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)-1\right] = 0, \tag{3.5}$$

where the hazard rate, h, is defined as follows:

$$h(\omega) = \frac{F'(\omega)}{1-F(\omega)}.$$

This equation has two additional parameters, the two parameters of the lognormal distribution, F. These two parameters, however, are pinned down by the assumption, $E\omega = 1$, and the fact that we specify $F(\bar{\omega})$ exogenously. With these conditions, the above equation forms a basis for computing $\bar{\omega}$. Note here that when $\mu = 0$,(3.5) reduces to $R^k = R^e$. Then, combining (3.4) with the first order condition for time deposits, we end up with the conclusion that r^k is determined as it is in the neoclassical growth model.

Conditional on $F(\bar{\omega})$ and $\bar{\omega}$, we may solve for k using (2.21):

$$\frac{\bar{k}}{n} = \frac{1}{1-\frac{1+R^k}{1+R^e}\left(\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right)}. \tag{3.6}$$

The law of motion for net worth implies the following relation in steady state:

$$n = \frac{\frac{\gamma}{\pi\mu_z}\left[R^k - R^e - \mu G(\bar{\omega})\left(1+R^k\right)\right]\bar{k} + w^e}{1-\gamma\left(\frac{1+R^e}{\pi}\right)\frac{1}{\mu_z}}. \tag{3.7}$$

3.1.4. Banks

The first order condition associated with the bank's capital decision is:

$$\left(1+\psi_k R\right)r^k = \frac{R h_{K^b}}{1+\tau h_{e^r}}. \tag{3.8}$$

The first order condition for labor is redundant given (3.1), (3.2), and (3.8), and so we do not list it here. In the preceding equations,

$$h_{K^b} = \alpha\xi a^b x^b \left(e_v\right)^{1-\xi}\left(\frac{\mu_z l}{k}\right)^{1-\alpha}, \tag{3.9}$$

$$h_{e^r} = (1-\xi)\, a^b x^b \left(e_v\right)^{-\xi}, \tag{3.10}$$

and

$$e_v = \frac{(1-\tau)\, m^b \,(1-m+x) - \tau\left(\psi_l wl + \frac{1}{\mu_z}\psi_k r^k \bar{k}\right)}{\left(\frac{1}{\mu_z}(1-\nu^k)\bar{k}\right)^{\alpha} \left((1-\nu^l)l\right)^{1-\alpha}}. \tag{3.11}$$

Another efficiency condition for the banks is (2.34). Rewriting that expression, we obtain:

$$1 + \frac{R}{R_a} = h_{e^r}\left[(1-\tau)\frac{R}{R_a} - \tau\right] \tag{3.12}$$

Substituting out for $a^b x^b \left(e_v\right)^{-\xi}$ from (3.10) into the scaled production function, we obtain:

$$\frac{h_{e^r}}{(1-\xi)} e_z^r = m^b \,(1-m+x) + \psi_l wl + \psi_k r^k \frac{\bar{k}}{\mu_z}, \tag{3.13}$$

where

$$e_z^r = (1-\tau) m^b \,(1-m+x) - \tau\left(\psi_l wl + \psi_k r^k \frac{\bar{k}}{\mu_z}\right). \tag{3.14}$$

3.1.5. Households

The first order condition for T :

$$1 + \left(1-\tau^T\right) R^e = \frac{\mu_z \pi}{\beta}$$

The first order condition for M :

$$\upsilon\left[c\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{1-\theta}\right]^{1-\sigma_q} [\frac{\theta}{m} - \frac{1-\theta}{1-m+x}] \left(m^b\right)^{\sigma_q - 2}$$
$$-\lambda_z \left(1-\tau^D\right) R_a = 0$$

The first order condition for M^b

$$\upsilon\,(1-\theta)\left[c\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{1-\theta}\right]^{1-\sigma_q} \left(\frac{1}{m^b}\right)^{2-\sigma_q}\left(\frac{1}{1-m+x}\right)$$
$$= \pi\lambda_z\left\{\frac{\mu_z}{\beta} - \frac{\left[1+\left(1-\tau_t^D\right) R_a\right]}{\pi}\right\}$$

Under the ACEL specification of preferences, c in the previous two expressions are replaced by unity. The first order condition for consumption corresponds to:

$$u_c^z - (1+\tau^c)\,\lambda_z = \upsilon c^{-1} \left(m^b\right)^{\sigma_q - 1} \left[c \left(\frac{1}{m}\right)^{\theta} \left(\frac{1}{1-m+x}\right)^{1-\theta} \right]^{1-\sigma_q}, \tag{3.15}$$

Under the ACEL specification, the expression to the right of the equality in (3.15) is replaced by zero.

Taking the ratio of (3.15) and the first order conditions for m^b, and rearranging, we obtain:

$$\begin{aligned} R_a &= \frac{\frac{(1-m+x)}{m}\theta - (1-\theta)}{\frac{(1-m+x)}{m}\theta} \frac{\left(\frac{\pi\mu_z}{\beta} - 1\right)}{(1-\tau^D)} \\ &= \left[1 - \frac{m}{1-m+x}\frac{(1-\theta)}{\theta}\right] \frac{1-\tau^T}{1-\tau^D} R^e \end{aligned}$$

The marginal utility of consumption is:

$$cu_c^z = \frac{\mu_{z,}}{\mu_z - b} - b\beta\frac{1}{\mu_z - b} = \frac{\mu_z - b\beta}{\mu_z - b} \tag{3.16}$$

The first order condition for households setting wages is:

$$w\frac{\lambda_z(1-\tau^l)}{\lambda_w} = \zeta\psi_L l^{\sigma_L} \tag{3.17}$$

3.1.6. Monetary Authority

$$\pi = \frac{(1+x)}{\mu_z}.$$

3.1.7. Resource Constraint and Zero Profits

After substituting out for the fixed cost in the resource constraint using the restriction that firm profits are zero in steady state, and using $g = \eta_g y$, we obtain:

$$c = (1-\eta_g)\left[\frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\nu^k \bar{k}\right)^{\alpha} \left(\nu^l l\right)^{1-\alpha} - \mu G(\bar{\omega})(1+R^k)\frac{k}{\mu_z \pi}\right] - i. \tag{3.18}$$

Here, we have made use of the facts,

$$y=\frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\nu^k\bar{k}\right)^{\alpha}\left(\nu^l l\right)^{1-\alpha}-\mu G(\bar{\omega})(1+R^k)\frac{k}{\mu_z\pi},$$

and $g=\eta_g y$, so that $c=(1-\eta_g)y-i$.

We now develop the condition on ϕ to assure that intermediate good firm profits in steady state are zero. If we loosely write their production function as $F-\phi z$, then the total cost of labor and capital inputs to the firm are sF, where s is real marginal cost, or the (reciprocal of the) markup (at least, in steady state when aggregate price and the individual intermediate good firm prices coincide). We want sF to exhaust total revenues, $F-\phi z$, i.e., we want $sF=F-\phi z$, or, $\phi=F(1-s)/z=(F/z)(1-1/\lambda_f)$, or

$$\phi=\left(\frac{z_{t-1}\nu^k K_t}{z_{t-1}z_t}\right)^{\alpha}\left(\nu^l l\right)^{1-\alpha}(1-\frac{1}{\lambda_f})=\left(\frac{\nu^k k}{\mu_z}\right)^{\alpha}\left(\nu^l l\right)^{1-\alpha}(1-\frac{1}{\lambda_f}) \tag{3.19}$$

We obtain (3.18) by substituting from the last equation into the resource constraint:

$$y=\left(\frac{1}{\mu_z}\nu^k\bar{k}\right)^{\alpha}\left(\nu^l l\right)^{1-\alpha}-\phi-\mu G(\bar{\omega})(1+R^k)\frac{k}{\mu_z\pi}, \tag{3.20}$$

We obtain g from output from:

$$g=\eta_g y. \tag{3.21}$$

3.1.8. Other Variables

Other variables of interest include measures of the external finance premium. For each dollar borrowed by entrepreneurs, the bank receives:

$$1+R_t^e+\frac{\mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R_t^k\right)Q_{\bar{K}',t-1}\bar{K}_t}{Q_{\bar{K}',t-1}\bar{K}_t-\bar{N}_t}.$$

This is a return across both entrepreneurs who are bankrupt and entrepreneurs who can pay the interest rate specified in the CSV. In steady state, this reduces to:

$$1+R^e+\frac{\mu G(\bar{\omega})\left(1+R_t^k\right)k}{k-n}.$$

We refer to the excess of this over $1+R^e$ as the average external finance premium. The marginal external finance premium is the interest rate that non-bankrupt entrepreneurs pay the bank over $1+R^e$. The gross rate of return, Z, is the value that solves the following expression:

$$\bar{\omega}\left(1+R^k\right)P_t z_t k = Z\left(P_t z_t k - P_t z_t n\right),$$

or,

$$Z = \frac{\bar{\omega}\left(1+R^k\right)k}{k-n}.$$

Monitoring costs, expressed as a ratio to quarterly GDP is:

$$\frac{\mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R_t^k\right)Q_{\bar{K}',t-1}\bar{K}_t}{P_t Y_t},$$

where Y_t is gross output before it has been scaled. Expressing this in scaled terms:

$$\frac{\mu G(\bar{\omega})\left(1+R^k\right)P_{t-1}z_{t-1}k}{P_t z_t y} = \frac{\mu G(\bar{\omega})\left(1+R^k\right)k}{\pi\mu_z y}$$

The currency to total deposit ratio is given by:

$$\frac{M_t}{A_t + X_t + S_t^w},$$

where:

$$S_t^w = \psi_{l,t}W_t l_t + \psi_{k,t}P_t r_t^k K_t,$$

or, substituting and using (2.42):

$$\begin{aligned}
& \frac{M_t}{M_t^b - M_t + X_t + \psi_{l,t}W_t l_t + \psi_{k,t}P_t r_t^k K_t} \\
= & \frac{m_t}{1 - m_t + x_t + \psi_{l,t}\frac{W_t}{M_t^b}l_t + \psi_{k,t}\frac{P_t r_t^k K_t}{M_t^b}} \\
= & \frac{m}{1 - m + x + \psi_l\frac{w}{m^b}l + \psi_k\frac{P_t r^k z_{t-1}k}{P_t z_t m^b}} \\
= & \frac{m}{1 - m + x + \psi_l\frac{w}{m^b}l + \psi_k\frac{r^k k}{\mu_z m^b}}
\end{aligned}$$

We're interested in velocity. The velocity of the base is:

$$\frac{P_t Y_t}{M_t^b} = \frac{P_t z_t y}{P_t z_t m^b} = \frac{y}{m^b}.$$

The velocity of $M1$ is:

$$\begin{aligned}
& \frac{P_t Y_t}{M_t + M_t^b - M_t + X_t + \psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t} \\
= & \frac{P_t z_t y}{M_t^b \left[1 + x + \left(\psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t\right)/M_t^b\right]} \\
= & \frac{y}{(1+x)\, m^b + \psi_l w l + \psi_k r^k k/\mu_z}
\end{aligned}$$

We define period t $M3$ as $M1$ plus the time deposits that mature in period t, $Q_{\bar{K}',t-1}\bar{K}_t^N - N_t$. So, the velocity of $M3$ is:

$$\frac{P_t Y_t}{M_t + M_t^b - M_t + X_t + \psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t + P_{t-1}\bar{K}_t^N - N_t},$$

using the fact that, along a steady state growth path, $Q_{\bar{K}',t-1} = P_{t-1}$. Taking into account scaling,

$$\begin{aligned}
& \frac{P_t z_t y}{M_t^b \left[1 + x_t + \left(\psi_{l,t} P_t z_t w l_t + \psi_{k,t} P_t z_t \left(z_{t-1}/z_t\right) r_t^k k + P_{t-1} z_{t-1} k - P_{t-1} z_{t-1} n\right)/M_t^b\right]} \\
= & \frac{P_t z_t y}{M_t^b \left[1 + x_t + \left(\psi_{l,t} P_t z_t w l_t + \psi_{k,t} P_t z_t r^k k/\mu_z + P_t z_t k/(\pi\mu_z) - P_t z_t n/(\pi\mu_z)\right)/M_t^b\right]} \\
= & \frac{y}{m^b (1+x) + \psi_l w l + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)}
\end{aligned}$$

We now develop expressions for the balance sheet of the banks. On the asset side are the working capital loans (labor, as well as capital rental), loans to entrepreneurs (the ones maturing in the current period) and reserves. On the liability side there is the demand deposits and time deposits. We express the components of the balance sheet as a fraction of the bank's total assets:

$$\begin{aligned}
& M_t^b - M_t + X_t + \psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t + P_{t-1}\bar{K}_t^N - N_t \\
= & M_t^b \left[1 - m + x + \left(\psi_l w l + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]
\end{aligned}$$

Working capital loans for wages are:

$$\begin{aligned}
& \frac{\psi_{l,t} W_t l_t}{M_t^b - M_t + X_t + \psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t + P_{t-1}\bar{K}_t^N - N_t} \\
= & \frac{P_t z_t \psi_l w l}{M_t^b \left[1 - m + x + \left(\psi_l w l + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]} \\
= & \frac{\psi_l w l}{m^b (1 - m + x) + \left(\psi_l w l + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)}
\end{aligned}$$

Working capital loans for capital rental are:

$$\begin{aligned} & \frac{\psi_{k,t} P_t r_t^k K_t}{M_t^b \left[1 - m + x + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]} \\ = & \frac{\psi_k r^k k/\mu_z}{m^b (1 - m + x) + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)} \end{aligned}$$

Loans to entrepreneurs:

$$\frac{(k-n)/(\mu_z \pi)}{m^b (1 - m + x) + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)}$$

Total reserves are:

$$\begin{aligned} & \frac{M_t^b - M_t + X_t}{M_t^b \left[1 - m + x + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]} \\ = & \frac{(1 - m + x) m^b}{m^b (1 - m + x) + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)} \end{aligned}$$

Required reserves are:

$$\frac{\tau \left((1 - m + x) m^b + \left(\psi_l wl + \psi_k r^k k/\mu_z\right)\right)}{m^b (1 - m + x) + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)}$$

Excess reserves are:

$$\frac{(1 - m + x) m^b - \tau \left((1 - m + x) m^b + \left(\psi_l wl + \psi_k r^k k/\mu_z\right)\right)}{m^b (1 - m + x) + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)}$$

The ratio of firm demand deposits to total assets are:

$$\begin{aligned} & \frac{\psi_{l,t} W_t l_t + \psi_{k,t} P_t r_t^k K_t}{M_t^b \left[1 - m + x + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]} \\ = & \frac{\psi_l wl + \psi_k r^k k/\mu_z}{m^b \left[1 - m + x + \left(\psi_l wl + \psi_k r^k k/\mu_z + (k-n)/(\pi\mu_z)\right)/m^b\right]} \end{aligned}$$

3.2. Linearization

There are 24 endogenous variables whose values are determined at time t. We load them into a vector, z_t. The elements in this vector are reported in the following table. In addition, there is an indication about which shocks the variable depends on. If it depends on the realization of all period t shocks, then we indicate a, for 'all'. If it depends only on the realization of the current period non-financial shocks, then we indicate p, for 'partial'. The table also indicates the information associated with each of the 24 equations used to solve the model. These equations are collected below from the preceding discussion. Note that the number of equations and elements in z_t is the same. Note also, in each case, the third and fourth columns always have the same entry. In several cases, z_t contains variables dated $t+1$. In the case of $\hat{\bar{k}}_{t+1}$, for example, the presence of a p in the third column indicates that $\hat{\bar{k}}_{t+1}$ is a function of the realization of the period t non-financial shocks, and is not a function of the realization of period t financial shocks, or later period shocks. In the case of $\hat{R}^e_{t+1}$, the presence of an a indicates that this variable is a function of all period t shocks, but not of any period $t+1$ shocks.

	z_t	information, z	information, equation
1	$\hat{\pi}_t$	p	p
2	$\hat{s}_t$	a	a
3	$\hat{r}^k_t$	a	a
4	$\hat{\imath}_t$	p	p
5	$\hat{u}_t$	p	p
6	$\hat{\omega}_t$	a	a
7	$\hat{R}^k_t$	a	a
8	$\hat{n}_{t+1}$	a	a
9	$\hat{q}_t$	a	a
10	$\hat{\nu}^l_t$	a	a
11	$\hat{e}_{\nu,t}$	a	a
12	$\hat{m}^b_t$	a	a
13	$\hat{R}_t$	a	a
14	$\hat{u}^z_{c,t}$	a	a
15	$\hat{\lambda}_{z,t}$	a	a
16	$\hat{m}_t$	a	a
17	$\hat{R}_{a,t}$	a	a
18	$\hat{c}_t$	p	p
19	$\hat{w}_t$	p	p
20	$\hat{l}_t$	a	a
21	$\hat{\bar{k}}_{t+1}$	p	p
22	$\hat{R}^e_{t+1}$	a	a
23	$\hat{x}_t$	a	a

(3.22)

The last of these variables is money growth, $\hat{x}_t$. As we show below, this is simply a trivial function of the underlying shocks. In addition, recall (2.35), in which the 10^{th} and 11^{th} variables are the same. A combination of the efficiency conditions for labor and capital in the firm sector, equations (1) and (2) below, are redundant with the efficiency conditions for labor and capital in the banking sector, (11) and (12). We deleted equation (11) below from our system.

In fact, we have 25 equations and unknowns in our model. The system we work with is one dimension less because we set $\Theta \equiv 0$, so that $\hat{v}_t$ disappears from the system. When we want $\Theta > 0$, we can get our 25^{th} equation by linearizing (2.24), and $\hat{v}_t$ is then our 25^{th} variable.

3.2.1. Firms

The inflation equation, when there is indexing to lagged inflation, is:

$$(1)\ E\left[\hat{\pi}_t - \frac{1}{1+\beta}\hat{\pi}_{t-1} - \frac{\beta}{1+\beta}\hat{\pi}_{t+1} - \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta)\,\xi_p}\left(\hat{s}_t + \hat{\lambda}_{f,t}\right)|\Omega_t\right] = 0$$

The linearized expression for marginal cost is:

$$(2)\ \alpha\hat{r}_t^k + \frac{\alpha\psi_k R}{1+\psi_k R}\hat{\psi}_{k,t} + (1-\alpha)\,\hat{w}_t + \frac{(1-\alpha)\,\psi_l R}{1+\psi_l R}\hat{\psi}_{l,t}$$
$$+\left[\frac{\alpha\psi_k R}{1+\psi_k R} + \frac{(1-\alpha)\,\psi_l R}{1+\psi_l R}\right]\hat{R}_t - \hat{\epsilon}_t - \hat{s}_t = 0$$

Another condition that marginal cost must satisfy is that it is equal to the marginal cost of one unit of capital services, divided by the marginal product of one unit of services. After linearization, this implies:

$$(3)\ \hat{r}_t^k + \frac{\psi_k R\left(\hat{\psi}_{k,t} + \hat{R}_t\right)}{1+\psi_k R} - \hat{\epsilon}_t - (1-\alpha)\left(\hat{\mu}_{z,t} + \hat{l}_t - \left[\hat{\bar{k}}_t + \hat{u}_t\right]\right) - \hat{s}_t = 0$$

3.2.2. Capital Producers

The 'Tobin's q' relation is:

$$(4)\ E\left\{\hat{q}_t - S''\mu_z^2(1+\beta)\hat{\imath}_t - S''\mu_z^2\hat{\mu}_{z,t} + S''\mu_z^2\hat{\imath}_{t-1} + \beta S''\mu_z^2\hat{\imath}_{t+1} + \beta S''\mu_z^2\hat{\mu}_{z,t+1}|\Omega_t\right\} = 0$$

The coefficients in the canonical form are:

$$\begin{aligned}
\alpha_1(4,9) &= 1 \\
\alpha_1(4,4) &= -S''\mu_z^2(1+\beta) \\
\alpha_2(4,4) &= S''\mu_z^2 \\
\alpha_0(4,4) &= \beta S''\mu_z^2 \\
\beta_0(4,46) &= \beta S''\mu_z^2 \\
\beta_1(4,46) &= -S''\mu_z^2
\end{aligned}$$

3.2.3. Entrepreneurs

The variable utilization equation is

$$(5)\ E\left[\hat{r}_t^k - \sigma_a \hat{u}_t | \Omega_t\right] = 0,$$

where $\hat{r}_t^k$ denotes the rental rate on capital. The date t standard debt contract has two parameters, the amount borrowed and $\widehat{\bar{\omega}}_{t+1}$. The former is not a function of the period $t+1$ state of nature, and the latter is not. Two equations characterize the efficient contract. The first order condition associated with the quantity loaned by banks in period t in the optimal contract is:

$$\begin{aligned}
&(6)\ E\{\lambda\left(\frac{R^k \hat{R}_{t+1}^k}{1+R^k} - \frac{R^e \hat{R}_{t+1}^e}{1+R^e}\right) \\
&-[1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R^e}\left[\frac{\Gamma''(\bar{\omega})\bar{\omega}}{\Gamma'(\bar{\omega})} - \frac{\lambda\left[\Gamma''(\bar{\omega}) - \mu G''(\bar{\omega})\right]\bar{\omega}}{\Gamma'(\bar{\omega})}\right]\widehat{\bar{\omega}}_{t+1}|\Omega_t^\mu\} = 0.
\end{aligned}$$

Note that this is not a function of the period $t+1$ uncertainty. Also, note that when $\mu = 0$, so that $\lambda = 1$, then this equation simply reduces to $E\left[\hat{R}_{t+1}^k | \Omega_t^\mu\right] = \hat{R}_{t+1}^e$. The linearized zero profit condition is:

$$\begin{aligned}
(7)\quad &\left(\tfrac{\bar{k}}{n}-1\right)\tfrac{R^k}{1+R^k}\hat{R}_t^k - \left(\tfrac{\bar{k}}{n}-1\right)\tfrac{R^e}{1+R^e}\hat{R}_t^e + \left(\tfrac{\bar{k}}{n}-1\right)\tfrac{(\Gamma'(\bar{\omega})-\mu G'(\bar{\omega}))}{(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))}\bar{\omega}\widehat{\bar{\omega}}_t \\
&-\left(\hat{q}_{t-1} + \widehat{\bar{k}}_t - \hat{n}_t\right) = 0.
\end{aligned}$$

The law of motion for net worth is:

$$(8)\ -\hat{n}_{t+1} + a_0\hat{R}_t^k + a_1\hat{R}_t^e + a_2\widehat{\bar{k}}_t + a_3\hat{w}_t^e + a_4\hat{\gamma}_t + a_5\hat{\pi}_t + a_6\hat{\mu}_{z,t} + a_7\hat{q}_{t-1} + a_8\widehat{\bar{\omega}}_t + a_9\hat{n}_t = 0$$

The definition of the rate of return on capital is:

$$(9)\ \hat{R}^k_{t+1} - \frac{(1-\tau^k)r^k + (1-\delta)q}{R^k q}\pi\left[\frac{\left(1-\tau^k\right)r^k\hat{r}^k_{t+1} - \tau^k r^k\hat{\tau}^k_t + (1-\delta)q\hat{q}_{t+1}}{(1-\tau^k)r^k + (1-\delta)q} + \hat{\pi}_{t+1} - \hat{q}_t\right] - \frac{\delta\tau^k\hat{\tau}^k_t}{R^k}$$

These are the coefficients corresponding to this equation, in the canonical representation of the model:

$$\begin{aligned}
\alpha_1(9,7) &= 1:\ \hat{R}^k_t,\ \alpha_1(9,3) = -\frac{\pi\left(1-\tau^k\right)r^k}{R^k q} : \hat{r}^k_t \\
\beta_1(9,57) &= \left(r^k\pi - \delta\right)\frac{\tau^k}{R^k} : \hat{\tau}^k_{t-1} \\
\alpha_1(9,1) &= -\frac{(1-\tau^k)r^k + (1-\delta)}{R^k}\pi : \hat{\pi}_t \\
\alpha_1(9,9) &= -\frac{(1-\delta)\pi}{R^k} : \hat{q}_t \\
\alpha_2(9,9) &= \frac{(1-\tau^k)r^k + (1-\delta)}{R^k}\pi :\ \hat{q}_{t-1}
\end{aligned}$$

3.2.4. Banking Sector

In the equations for the banking sector, it is capital services, k_t, which appears, not the physical stock of capital, $\bar{k}_t$. The link between them is:

$$\hat{k}_t = \hat{\bar{k}}_t + \hat{u}_t.$$

An expression for the ratio of excess reserves to value added in the banking sector is:

$$\begin{aligned}
&(10)\ -\hat{e}_{v,t} + n_\tau\hat{\tau}_t + n_{m^b}\hat{m}^b_t + n_m\hat{m}_t + n_x\hat{x}_t + n_{\psi_l}\hat{\psi}_{l,t} \\
&+n_{\psi k}\hat{\psi}_{k,t} + (n_k - d_k)\left[\hat{\bar{k}}_t + \hat{u}_t\right] + n_{r^k}\hat{r}^k_t + n_w\hat{w}_t \\
&+(n_l - d_l)\,\hat{l}_t + \left(n_{\mu_z} - d_{\mu_z}\right)\hat{\mu}_{z,t} - d_{\nu^k}\hat{\nu}^k_t - d_{\nu l}\hat{\nu}^l_t = 0
\end{aligned}$$

where m_t^b is the scaled monetary base, m_t is the currency-to-base ratio, x_t is the growth rate of the base

$$\begin{aligned}
n_\tau &= \frac{-\tau m^b\left(1-m+x\right)-\tau\left(\psi_l wl+\frac{1}{\mu_z}\psi_k r^k k\right)-\tau\frac{1}{\mu_z}\psi_k r^k k}{n}, \\
n &= \left(1-\tau\right)m^b\left(1-m+x\right)-\tau\left(\psi_l wl+\frac{1}{\mu_z}\psi_k r^k k\right), \\
n_{m^b} &= \left(1-\tau\right)m^b\left(1-m+x\right)/n \\
n_m &= -\left(1-\tau\right)m^b m/n \\
n_x &= \left(1-\tau\right)m^b x/n \\
n_{\psi_l} &= n_w = n_l = -\tau\psi_l wl/n \\
n_{\psi_k} &= n_{r^k} = n_k = -\tau\frac{1}{\mu_z}\psi_k r^k k/n \\
n_{\mu_z} &= \tau\frac{1}{\mu_z}\psi_k r^k k/n
\end{aligned}$$

and

$$\begin{aligned}
d &= \left(\frac{1}{\mu_z}(1-\nu^k)k\right)^\alpha\left((1-\nu^l)l\right)^{1-\alpha} \\
d_{\mu_z} &= \frac{-\alpha\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^\alpha\left((1-\nu^l)l\right)^{1-\alpha}}{\left(\frac{1}{\mu_z}(1-\nu^k)k\right)^\alpha\left((1-\nu^l)l\right)^{1-\alpha}} = -\alpha \\
d_k &= \alpha \\
d_{\nu^k} &= -\alpha\frac{\nu^k}{1-\nu^k} \\
d_l &= 1-\alpha \\
d_{\nu^l} &= -(1-\alpha)\frac{\nu^l}{1-\nu^l}
\end{aligned}$$

The first order condition for capital in the banking sector is:

$$\begin{aligned}
0 &= k_R\hat{R}_t + k_\xi\hat{\xi}_t - \hat{r}_t^k + k_x\hat{x}_t^b + k_e\hat{e}_{v,t} + k_\mu\hat{\mu}_{z,t} \\
&\quad + k_{\nu^l}\hat{\nu}_t^l + k_{\nu^k}\hat{\nu}_t^k + k_l\hat{l}_t + k_k\left[\hat{\bar{k}}_t + \hat{u}_t\right] + k_\tau\hat{\tau}_t + k_{\psi_k}\hat{\psi}_{k,t}
\end{aligned}$$

$$
\begin{aligned}
k_R &= \left[1-\frac{\psi_k R}{1+\psi_k R}\right],\ k_\xi = 1-\log\left(e_v\right)\xi + \frac{\tau h_{e^r}\left[\frac{1}{1-\xi}+\log\left(e_v\right)\right]\xi}{1+\tau h_{e^r}} \\
k_x &= \frac{1}{1+\tau h_{e^r}},\ k_e = 1-\xi+\frac{\tau h_{e^r}\xi}{1+\tau h_{e^r}},\ k_\mu = (1-\alpha) \\
k_{\nu^l} &= -(1-\alpha)\frac{\nu^l}{1-\nu^l},\ k_{\nu^k} = (1-\alpha)\frac{\nu^k}{1-\nu^k},\ k_l = (1-\alpha),\ k_k = -(1-\alpha) \\
k_\tau &= -\frac{\tau h_{e^r}}{1+\tau h_{e^r}},\ k_{\psi_k} = -\frac{\psi_k R}{1+\psi_k R}.
\end{aligned}
$$

The latter equation was deleted from our system, because it is redundant given the two firm Euler equations and the following equation.

The first order condition for labor in the banking sector is:

$$
\begin{aligned}
(11)\ 0 &= l_R\hat{R}_t + l_\xi\hat{\xi}_t - \hat{w}_t + l_x\hat{x}_t^b + l_e\hat{e}_{v,t} + l_\mu\hat{\mu}_{z,t} \\
&\quad + l_{\nu^l}\hat{\nu}_t^l + l_{\nu^k}\hat{\nu}_t^k + l_l\hat{l}_t + l_k\left[\hat{\bar{k}}_t + \hat{u}_t\right] + l_\tau\hat{\tau}_t + l_{\psi_l}\hat{\psi}_{l,t},
\end{aligned}
$$

where

$$
\begin{aligned}
l_i &= k_i \text{ for all } i, \text{ except} \\
l_R &= \left[1-\frac{\psi_l R}{1+\psi_l R}\right],\ l_{\psi l} = -\frac{\psi_l R}{1+\psi_l R} \\
l_\mu &= k_\mu - 1,\ l_{\nu^l} = k_{\nu^l} + \frac{\nu^l}{1-\nu^l},\ l_l = k_l - 1, \\
l_{\nu^k} &= k_{\nu^k} - \frac{\nu^k}{1-\nu^k},\ l_k = k_k + 1.
\end{aligned}
$$

The production function for deposits is:

$$
\begin{aligned}
(12)\qquad & \hat{x}_t^b - \xi\hat{e}_{v,t} - \log\left(e_{v,t}\right)\xi\hat{\xi}_t - \frac{\tau\left(m_1+m_2\right)}{(1-\tau)m_1-\tau m_2}\hat{\tau}_t \\
= & \left[\frac{m_1}{m_1+m_2} - \frac{(1-\tau)m_1}{(1-\tau)m_1-\tau m_2}\right]\left[\hat{m}_t^b + \frac{-m\hat{m}_t + x\hat{x}_t}{1-m+x}\right] \\
& + \left[\frac{m_2}{m_1+m_2} + \frac{\tau m_2}{(1-\tau)m_1-\tau m_2}\right] \\
& \times[\frac{\psi_l wl}{\psi_l wl+\psi_k r^k k/\mu_z}\left(\hat{\psi}_{l,t}+\hat{w}_t+\hat{l}_t\right) + \frac{\psi_k r^k k/\mu_z}{\psi_l wl+\psi_k r^k k/\mu_z}\left(\hat{\psi}_{k,t}+\hat{r}_t^k+\hat{k}_t-\hat{\mu}_{zt}\right)].
\end{aligned}
$$

The coefficients in the canonical form are:

$$
\begin{aligned}
\beta_1(12,16) &= -1\text{: } \hat{x}_t^b, \\
\beta_1(12,4) &= \frac{\tau\left(m_1+m_2\right)}{m_1-\tau\left(m_1+m_2\right)} : \ \hat{\tau}_t \text{ `ratio, required to excess reserves'} \\
\beta_1(12,13) &= \log\left(e_v\right)\xi : \ \hat{\xi}_t \\
\alpha_1(12,11) &= \xi : \ \hat{e}_{v,t}, \ \alpha_1(12,12) = m_1\bar{\mathcal{F}} : \ \hat{m}_t^b \\
\alpha_1(12,19) &= \alpha_1(12,20) = \beta_1(12,7) = m_2\mathcal{F}w_l : \ \hat{w}_t, \hat{l}_t, \hat{\psi}_{l,t} \\
\alpha_1(12,3) &= \alpha_1(12,5) = \alpha_2(12,21) = \beta_1(12,10) = -\beta_1(12,46) \\
&= m_2\mathcal{F}w_k : \ \hat{r}_t^k, \ \hat{u}_t, \ \hat{\bar{\bar{k}}}_t, \ \hat{\psi}_{k,t}, \ \hat{\mu}_{zt} \\
\alpha_1(12,23) &= m_1\bar{\mathcal{F}}\frac{x}{1-m+x} \ : \hat{x}_t \\
\alpha_1(12,16) &= -m_1\bar{\mathcal{F}}\frac{m}{1-m+x} \ : \hat{m}_t,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\mathcal{F}} &= \frac{1}{m_1+m_2} - \frac{(1-\tau)}{(1-\tau)m_1-\tau m_2} = \frac{(1-\tau)m_1-\tau m_2-(1-\tau)\left(m_1+m_2\right)}{\left(m_1+m_2\right)\left[m_1-\tau(m_1+m_2)\right]} \\
&= -\frac{m_2}{\left(m_1+m_2\right)\left[m_1-\tau(m_1+m_2)\right]} \\
\mathcal{F} &= \frac{1}{m_1+m_2} + \frac{\tau}{(1-\tau)m_1-\tau m_2} = \frac{(1-\tau)m_1-\tau m_2+\tau\left(m_1+m_2\right)}{\left(m_1+m_2\right)\left[m_1-\tau(m_1+m_2)\right]} \\
&= \frac{m_1}{\left(m_1+m_2\right)\left[m_1-\tau(m_1+m_2)\right]}, \\
w_l &= \frac{\psi_l wl}{\psi_l wl+\psi_k r^k k/\mu_z}, \text{ labor component of working capital loans} \\
w_k &= \frac{\psi_k r^k k/\mu_z}{\psi_l wl+\psi_k r^k k/\mu_z}, \text{ capital component in working capital loans,}
\end{aligned}
$$

where m_1+m_2 is total deposits and $m_1-\tau(m_1+m_2)$ is excess reserves.

$$
(13)\ \hat{R}_{at} - \left[\frac{h_{e^r}-\tau h_{e^r}}{(1-\tau)\,h_{e^r}-1} - \frac{\tau h_{e^r}}{\tau h_{e^r}+1}\right]\left[-\left(\frac{1}{1-\xi}+\log\left(e_v\right)\right)\xi\hat{\xi}_t+\hat{x}_t^b-\xi\hat{e}_{v,t}\right]
$$
$$
+\left[\frac{\tau h_{e^r}}{(1-\tau)\,h_{e^r}-1}+\frac{\tau h_{e^r}}{\tau h_{e^r}+1}\right]\hat{\tau}_t-\hat{R}_t=0
$$

The coefficients in the canonical form are:

$$
\begin{aligned}
\alpha_1(13,17) &= 1 \ : \ \hat{R}_{at} \\
\alpha_1(13,11) &= \xi h_{e^r}\left[\frac{1-\tau}{(1-\tau)\,h_{e^r}-1}-\frac{\tau}{\tau h_{e^r}+1}\right] : \hat{e}_{v,t} \\
\alpha_1(13,13) &= -1 : \hat{R}_t \\
\beta_1(13,13) &= h_{e^r}\left[\frac{1-\tau}{(1-\tau)\,h_{e^r}-1}-\frac{\tau}{\tau h_{e^r}+1}\right]\left(\frac{1}{1-\xi}+\log\left(e_v\right)\right)\xi : \ \hat{\xi}_t \\
\beta_1(13,16) &= -h_{e^r}\left[\frac{1-\tau}{(1-\tau)\,h_{e^r}-1}-\frac{\tau}{\tau h_{e^r}+1}\right] : \ \hat{x}_t^b \\
\beta_1(13,4) &= \tau h_{e^r}\left[\frac{1}{(1-\tau)\,h_{e^r}-1}+\frac{1}{\tau h_{e^r}+1}\right] \\
&= \frac{\tau h_{e^r}^2}{\left[(1-\tau)\,h_{e^r}-1\right](\tau h_{e^r}+1)} \ : \ \hat{\tau}_t
\end{aligned}
$$

In the version of the model in which the banking sector is dropped, we must nevertheless have a loan market clearing condition:

$$\psi_{l,t}W_tl_t+\psi_{k,t}P_tr_t^kK_t=M_t^b-M_t+X_t.$$

The right side of this equation is the supply of base for the purpose of lending. The left hand side is the corresponding demand. Scale this by dividing by P_tz_t :

$$\psi_{l,t}w_tl_t+\psi_{k,t}r_t^ku_t\frac{\bar{k}_t}{\mu_{z,t}}=m_t^b\left(1-m_t+x_t\right). \tag{3.23}$$

Linearizing this:

$$
\begin{aligned}
(25)\ \psi_l wl\left[\hat{\psi}_{l,t}+\hat{w}_t+\hat{l}_t\right]+\psi_k r^k\tfrac{\bar{k}}{\mu_z}\left[\hat{\psi}_{k,t}+\hat{r}_t^k+\hat{u}_t+\hat{\bar{k}}_t-\hat{\mu}_{z,t}\right] & \\
-m^b\left(1-m+x\right)\left[\hat{m}_t^b+\tfrac{-m\hat{m}_t+x\hat{x}_t}{1-m+x}\right] = & \quad 0.
\end{aligned}
$$

The parameters of the reduced form are:

$$\begin{aligned}
\alpha_1(25,19) &= \psi_l wl : \hat{w}_t,\ \alpha_1(25,20) = \psi_l wl : \hat{l}_t \\
\alpha_1(25,3) &= \psi_k r^k \frac{\bar{k}}{\mu_z} : \hat{r}_t^k,\ \alpha_1(25,5) = \psi_k r^k \frac{\bar{k}}{\mu_z} : \hat{u}_t \\
\alpha_1(25,12) &= -m^b\left(1-m+x\right) : \hat{m}_t^b,\ \alpha_1(25,16) = m^b m : \hat{m}_t \\
\alpha_2(25,21) &= \psi_k r^k \frac{\bar{k}}{\mu_z} : \widehat{\bar{k}}_t,\ \alpha_1(25,23) = -m^b x : \hat{x}_t \\
\beta_1(25,7) &= \psi_l wl : \hat{\psi}_{l,t},\ \beta_1(25,10) = \psi_k r^k \frac{\bar{k}}{\mu_z} : \hat{\psi}_{k,t} \\
\beta_1(25,46) &= -\psi_k r^k \frac{\bar{k}}{\mu_z} : \hat{\mu}_{z,t}.
\end{aligned}$$

3.2.5. Household Sector

The definition of u_c^z is:

$$(14)\ E\{u_c^z \hat{u}_{c,t}^z - \left[\frac{\mu_z}{c\left(\mu_z-b\right)} - \frac{\mu_z^2 c}{c^2\left(\mu_z-b\right)^2}\right]\hat{\mu}_{z,t} - b\beta\frac{\mu_z c}{c^2\left(\mu_z-b\right)^2}\hat{\mu}_{z,t+1}$$
$$+\frac{\mu_z^2+\beta b^2}{c^2\left(\mu_z-b\right)^2}c\hat{c}_t - \frac{b\beta\mu_z}{c^2\left(\mu_z-b\right)^2}c\hat{c}_{t+1} - \frac{b\mu_z}{c^2\left(\mu_z-b\right)^2}c\hat{c}_{t-1}|\Omega_t^\mu\} \quad = \quad 0.$$

The coefficients in the canonical form are:

$$\begin{aligned}
\alpha_1(14,18) &= \left(\frac{1}{c\left(\mu_z-b\right)}\right)^2\left[\mu_z^2+b^2\beta\right]c,:\ \hat{c}_t \\
\alpha_0(14,18) &= -b\beta\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2\mu_z c:\ \hat{c}_{t+1} \\
\alpha_2(14,18) &= -\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 b\mu_z c:\ \hat{c}_{t-1} \\
\beta_0(14,46) &= b\beta\left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 c\mu_z^2 : \mu_{z,t+1} \\
\beta_1(14,46) &= \left[-\frac{\mu_z}{c\left(\mu_z-b\right)} + \left(\frac{1}{c\left(\mu_z-b\right)}\right)^2 c\mu_z\right] : \mu_{z,t}
\end{aligned}$$

The household's first order condition for time deposits is:

$$(15)\ E\left\{-\hat{\lambda}_{z,t}+\hat{\lambda}_{z,t+1}-\hat{\mu}_{z,t+1}-\hat{\pi}_{t+1}-\frac{R^e\tau^T}{1+\left(1-\tau^T\right)R^e}\hat{\tau}_{t+1}^T+\frac{R^e\left(1-\tau^T\right)}{1+\left(1-\tau^T\right)R^e}\hat{R}_{t+1}^e|\Omega_t^\mu\right\}=0.$$

The household's first order condition for capital is:

$$(24)\ E\left\{-\hat{\lambda}_{zt}+\left[\frac{R^k}{1+R^k}\hat{R}_{t+1}^k+\hat{\lambda}_{z,t+1}-\hat{\pi}_{t+1}-\hat{\mu}_{z,t+1}\right]|\Omega_t\right\}$$

The coefficients in the canonical form are:

$$\begin{aligned}
\alpha_0(24,15) &= 1:\ \hat{\lambda}_{z,t+1}\\
\alpha_0(24,7) &= \frac{R^k}{1+R^k}:\ \hat{R}_{t+1}^k\\
\alpha_0(24,1) &= -1:\ \hat{\pi}_{t+1}\\
\alpha_1(24,15) &= -1:\ \hat{\lambda}_{z,t}\\
\beta_0(24,46) &= -1:\hat{\mu}_{z,t+1}
\end{aligned}$$

The first order condition for currency, M_t :

$$\begin{aligned}
(16)\ \hat{\upsilon}_t+\left(1-\sigma_q\right)\hat{c}_t+\left[-\left(1-\sigma_q\right)\left(\theta-(1-\theta)\frac{m}{1-m+x}\right)-\frac{\frac{\theta}{m}+\frac{1-\theta}{(1-m+x)^2}m}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}\right]\hat{m}_t&\\
-\left[\frac{\left(1-\sigma_q\right)(1-\theta)x}{1-m+x}-\frac{\frac{1-\theta}{(1-m+x)^2}x}{\frac{\theta}{m}-\frac{1-\theta}{1-m+x}}\right]\hat{x}_t&\\
+\left[-\left(1-\sigma_q\right)\left(\log(m)-\log(1-m+x)\right)+\frac{1+x}{\theta(1+x)-m}\right]\theta\hat{\theta}_t&\\
-\left(2-\sigma_q\right)\hat{m}_t^b-\left[\hat{\lambda}_{z,t}+\frac{-\tau^D}{1-\tau^D}\hat{\tau}_t^D+\hat{R}_{a,t}\right]&=0
\end{aligned}$$

With ACEL preferences, set the coefficient on $\hat{c}_t$ to zero here. The household's first order condition for currency, M_{t+1}^b, is:

$$(17)\ E\{\frac{\beta}{\pi\mu_z}\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}$$

$$\times\{\hat{\upsilon}_{t+1}-\frac{\theta\hat{\theta}_{t+1}}{1-\theta}+(1-\sigma_q)\hat{c}_{t+1}-(1-\sigma_q)\log\left(m\right)\theta\hat{\theta}_{t+1}-\theta(1-\sigma_q)\hat{m}_{t+1}$$

$$-\left[\left(1-\theta\right)\left(1-\sigma_q\right)+1\right]\left(\frac{1}{1-m+x}\right)\left[x\hat{x}_{t+1}-m\hat{m}_{t+1}\right]$$

$$+\left(1-\sigma_q\right)\log\left(1-m+x\right)\theta\hat{\theta}_{t+1}-\left(2-\sigma_q\right)\hat{m}_{t+1}^b\}$$

$$+\frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right]\hat{\lambda}_{z,t+1}$$

$$+\frac{\beta}{\pi\mu_z}\lambda_z\left[\left(1-\tau^D\right)R_a\hat{R}_{a,t+1}-\tau^D R_a\hat{\tau}_{t+1}^D\right]-\lambda_z\left[\hat{\lambda}_{zt}+\hat{\pi}_{t+1}+\hat{\mu}_{z,t+1}\right]|\Omega_t^{\mu}\}$$

$$=0.$$

With ACEL preferences, replace c by unity and set $\hat{c}_{t+1}$ to zero. We now derive the coefficients in the canonical form are. Let

$$\Upsilon=\beta\upsilon\left(1-\theta\right)\left[c\left(\frac{1}{m}\right)^{\theta}\right]^{1-\sigma_q}\left(\frac{1}{1-m+x}\right)^{(1-\theta)(1-\sigma_q)+1}\left(\frac{1}{m^b}\right)^{2-\sigma_q}$$

$$
\begin{aligned}
\alpha_0(17,18) &= \frac{\Upsilon}{\pi\mu_z}(1-\sigma_q) \\
\alpha_0(17,16) &= -\frac{\Upsilon}{\pi\mu_z}\left[\theta(1-\sigma_q)+[(1-\theta)(1-\sigma_q)+1]\left(\frac{m}{1-m+x}\right)\right] \\
\alpha_0(17,12) &= -\frac{\Upsilon}{\pi\mu_z}(2-\sigma_q) \\
\alpha_0(17,15) &= \frac{\beta}{\pi\mu_z}\lambda_z\left[1+\left(1-\tau^D\right)R_a\right] \\
\alpha_0(17,17) &= \frac{\beta}{\pi\mu_z}\lambda_z\left(1-\tau^D\right)R_a \\
\alpha_0(17,1) &= \lambda_z \\
\alpha_1(17,15) &= -\lambda_z : \hat{\lambda}_{z,t} \\
\beta_0(17,40) &= \frac{\Upsilon}{\pi\mu_z} : \hat{\upsilon}_{t+1} \\
\beta_0(17,22) &= -\frac{\Upsilon}{\pi\mu_z}\left[\frac{\theta}{1-\theta}+(1-\sigma_q)\log(m)\theta-(1-\sigma_q)\log(1-m+x)\theta\right] : \hat{\theta}_{t+1} \\
\alpha_0(17,23) &= -\frac{\Upsilon}{\pi\mu_z}[(1-\theta)(1-\sigma_q)+1]\left(\frac{x}{1-m+x}\right) : \hat{x}_{t+1} \\
\beta_0(17,25) &= -\frac{\beta}{\pi\mu_z}\lambda_z\tau^D R_a : \hat{\tau}^D_{t+1} \\
\beta_0(17,46) &= -\lambda_z : \hat{\mu}_{z,t+1}
\end{aligned}
$$

With ACEL preferences, replace c with unity in Υ. Also, $\alpha_0(17,18)$ should be zero. The first order condition for consumption is:

$$
\begin{aligned}
&(18)\ E\{u_c^z\hat{u}_{c,t}^z-\upsilon c^{-\sigma_q}\left[\frac{1}{m^b}\left(\frac{1}{m}\right)^{\theta}\left(\frac{1}{1-m+x}\right)^{1-\theta}\right]^{1-\sigma_q} \\
&\times[\hat{\upsilon}_t-\sigma_q\hat{c}_t+(1-\sigma_q)\left(-\hat{m}_t^b-\theta_t\hat{m}_t-(1-\theta_t)\left(\frac{-m}{1-m+x}\hat{m}_t+\frac{x}{1-m+x}\hat{x}_t\right)\right) \\
&+(1-\sigma_q)\left[\log\left(\frac{1}{m}\right)-\log\left(\frac{1}{1-m+x}\right)\right]\theta\hat{\theta}_t] \\
&-(1+\tau^c)\lambda_z\left[\frac{\tau^c}{1+\tau^c}\hat{\tau}_t^c+\hat{\lambda}_{z,t}\right]|\Omega_t\}=0
\end{aligned}
$$

With ACEL preferences, the middle term is replaced by zero. The reduced form wage equation is:

$$(19)\ E\left\{\eta_0\hat{w}_{t-1}+\eta_1\hat{w}_t+\eta_2\hat{w}_{t+1}+\eta_{\bar{3}}\hat{\pi}_{t-1}+\eta_3\hat{\pi}_t+\eta_4\hat{\pi}_{t+1}+\eta_5\hat{l}_t+\eta_6\left[\hat{\lambda}_{z,t}-\frac{\tau^l}{1-\tau^l}\hat{\tau}_t^l\right]+\eta_7\hat{\zeta}_t|\Omega_t\right\}=0$$

where

$$\eta=\begin{pmatrix} b_w\xi_w \\ -b_w\left(1+\beta\xi_w^2\right)+\sigma_L\lambda_w \\ \beta\xi_w b_w \\ b_w\xi_w \\ -\xi_w b_w\left(1+\beta\right) \\ b_w\beta\xi_w \\ -\sigma_L\left(1-\lambda_w\right) \\ 1-\lambda_w \\ -\left(1-\lambda_w\right)\end{pmatrix}=\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_{\bar{3}}\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\end{pmatrix}.$$

3.2.6. Aggregate Restrictions

The resource constraint is:

$$\begin{aligned}(20)\ 0= \quad & d_y\left[\frac{G'(\bar{\omega})}{G(\bar{\omega})}\bar{\omega}\hat{\bar{\omega}}_t+\frac{R^k}{1+R^k}\hat{R}_t^k+\hat{q}_{t-1}+\hat{\bar{k}}_t-\hat{\mu}_{z,t}-\hat{\pi}_t\right]+u_y\hat{u}_t+g_y\hat{g}_t+c_y\hat{c}_t+\bar{k}_y\frac{i}{\bar{k}}\hat{\imath}_t \\ & +\Theta(1-\gamma)v_y\hat{v}_t-\alpha\left(\hat{u}_t-\hat{\mu}_{z,t}+\hat{\bar{k}}_t+\hat{\nu}_t^k\right)-(1-\alpha)\left(\hat{l}_t+\hat{\nu}_t^l\right)-\hat{\epsilon}_t\end{aligned}$$

$$
\begin{aligned}
\alpha_1(20,6) &= d_y \frac{G'(\bar{\omega})}{G(\bar{\omega})}\bar{\omega} : \ \widehat{\bar{\omega}}_t \\
\alpha_1(20,7) &= \frac{R^k}{1+R^k} d_y : \ \hat{R}_t^k \\
\alpha_1(20,1) &= -d_y : \ \hat{\pi}_t \\
\alpha_1(20,5) &= u_y - \alpha : \ \hat{u}_t \\
\alpha_1(20,18) &= c_y : \ \hat{c}_t \\
\alpha_1(20,4) &= \bar{k}_y \frac{i}{k} : \ \hat{\imath}_t \\
\alpha_1(20,10) &= -\alpha - (1-\alpha) : \ \hat{\nu}_t^l \\
\alpha_1(20,20) &= -(1-\alpha) : \ \hat{l}_t \\
\alpha_2(20,21) &= d_y - \alpha : \ \widehat{\bar{k}}_t \\
\alpha_2(20,9) &= d_y : \ \hat{q}_{t-1} \\
\beta_1(20,46) &= -d_y + \alpha : \ \hat{\mu}_{z,t} \\
\beta_1(20,37) &= g_y : \ \hat{g}_t \\
\beta_1(20,52) &= -1 : \ \hat{\epsilon}_t
\end{aligned}
$$

where

$$\bar{k}_y = \frac{\bar{k}}{y+\phi+d},$$

and the object in square brackets corresponds to the resources used up in monitoring.

$$(21)\ \widehat{\bar{k}}_{t+1} - \frac{1-\delta}{\mu_z}\left(\widehat{\bar{k}}_t - \hat{\mu}_{z,t}\right) - \frac{i}{\bar{k}}\hat{\imath}_t = 0.$$

Monetary policy is represented by:

$$(22)\ \hat{m}_t^b + \frac{x}{1+x}\hat{x}_t - \hat{\pi}_{t+1} - \hat{\mu}_{z,t+1} - \hat{m}_{t+1}^b = 0$$

The parameters in the reduced form are:

$$
\begin{aligned}
\alpha_0(22,1) &= -1 : \ \hat{\pi}_{t+1}, \ \alpha_0(22,12) = -1 : \hat{m}_{t+1}^b, \\
\alpha_1(22,12) &= 1 : \hat{m}_t^b, \ \alpha_1(22,23) = \frac{x}{1+x} : \ \hat{x}_t, \\
\beta_0(22,46) &= -1 : \hat{\mu}_{z,t+1}.
\end{aligned}
$$

The timing of this equation could be changed to:

$$(22)'\ \hat{m}_{t-1}^b + \frac{x}{1+x}\hat{x}_{t-1} - \hat{\pi}_t - \hat{\mu}_{z,t} - \hat{m}_t^b = 0$$

3.2.7. Monetary Policy

Monetary policy has the following representation:

$$(23)\ \hat{x}_t = \sum_{i=1}^{p} x_{it},$$

where the x_{it}'s are functions of the underlying shocks.

3.2.8. Other Variables

The currency to deposit ratio is:

$$d_t^c = \frac{m_t}{1 - m_t + x_t + \psi_{l,t}\frac{w_t}{m_t^b}l_t + \psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\mu_{z,t} m_t^b}},$$

with

$$d^c = \frac{m}{1 - m + x + \psi_l \frac{w}{m^b} l + \psi_k \frac{r^k k}{\mu_z m^b}}$$

$$\begin{aligned}
\hat{d}_t^c &= \hat{m}_t - \left[1 - m + x + \widehat{\psi_l \frac{w}{m^b}} l + \psi_k \frac{r^k k}{\mu_z m^b}\right] \\
&= \hat{m}_t - \frac{d\left[1 - m_t + x_t + \psi_{l,t}\frac{w_t}{m_t^b}l_t + \psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\mu_{z,t} m_t^b}\right]}{1 - m + x + \psi_l \frac{w}{m^b} l + \psi_k \frac{r^k k}{\mu_z m^b}} \\
&= \hat{m}_t - \frac{d^c}{m} d\left[1 - m_t + x_t + \psi_{l,t}\frac{w_t}{m_t^b}l_t + \psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\mu_{z,t} m_t^b}\right]
\end{aligned}$$

so

$$\begin{aligned}
\hat{d}_t^c &= \hat{m}_t \\
&\quad + d^c \hat{m}_t - \frac{d^c x}{m}\hat{x}_t - \frac{d^c \psi_l}{m}\frac{w}{m^b} l \left[\hat{\psi}_{l,t} + \hat{w}_t - \hat{m}_t^b + \hat{l}_t\right] \\
&\quad - \frac{d^c \psi_k}{m}\frac{r^k k}{\mu_z m^b}\left[\hat{\psi}_{k,t} + \hat{r}_t^k + \hat{u}_t + \hat{\bar{k}}_t - \hat{\mu}_{z,t} - \hat{m}_t^b\right]
\end{aligned}$$

We are interested in obtaining a linearized representation for the external finance premium:

$$\begin{aligned}
P_t^e &= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) Q_{\bar{K}',t-1}\bar{K}_t}{Q_{\bar{K}',t-1}\bar{K}_t - \bar{N}_t} \\
&= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) P_{t-1}q_{t-1}z_{t-1}\bar{k}_t}{P_{t-1}q_{t-1}\bar{k}_t - z_{t-1}P_{t-1}n_t} \\
&= \frac{\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) \left(1+R_t^k\right) P_{t-1}q_{t-1}z_{t-1}\bar{k}_t}{P_{t-1}q_{t-1}\bar{k}_t - z_{t-1}P_{t-1}n_t} \\
&= \frac{\mu G(\bar{\omega}_t) \left(1+R_t^k\right) q_{t-1}\bar{k}_t}{q_{t-1}\bar{k}_t - n_t}.
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{P}_t^e &= G'(\bar{\omega}_t)\widehat{\bar{\omega}}_t + \frac{R^k \hat{R}_t^k}{1+R^k} + \hat{q}_{t-1} + \widehat{\bar{k}}_t - \left[\widehat{q_{t-1}\bar{k}_t - n_t}\right] \\
&= G'(\bar{\omega})\widehat{\bar{\omega}}_t + \frac{R^k}{1+R^k}\hat{R}_t^k + \hat{q}_{t-1} + \widehat{\bar{k}}_t - \frac{\bar{k}\left(\hat{q}_{t-1} + \widehat{\bar{k}}_t\right) - n\hat{n}_t}{\bar{k}-n}
\end{aligned}$$

We need a measure of output for the economy as a whole, and a measure of output for the two sectors in the economy. Here are some random notes to start thinking about this...Goods-producing sector resource costs are

$$\begin{aligned}
& \left(1+\psi_{l,t}R_t\right) W_t\nu_t l_t + \left(1+\psi_{k,t}R_t\right) P_t r_t^k \nu_t k_t \\
= \; & \underbrace{W_t\nu_t l_t + P_t r_t^k \nu_t k_t}_{\text{factor input costs}} + \underbrace{\psi_{l,t}R_t W_t \nu_t l_t + \psi_{k,t}R_t P_t r_t^k \nu_t k_t}_{\text{purchases of `intermediate inputs' from banking sector}}
\end{aligned}$$

Value-added of the goods producing sector is profits and factor incomes generated by that sector. In our case, profits are zero in steady state, but can be non-zero outside of steady state. So, in steady state at least, value-added in the goods producing sector is $W_t\nu_t l_t + P_t r_t^k \nu_t k_t$. $F - \phi z = sF$. The purchases of this gross output is divided between $c + i + g$+monitoring costs. Let $i = 1$ denote the goods producing industry and $i = 2$ is banking industry. a_{11} is the quantity of output of the goods producing industry going into goods. This is zero in our setup. a_{12} is the quantity of goods used in the banking industry. These are monitoring costs. a_{21} is the inputs going from banking to goods, which are the interest charges. a_{22} ....

3.3. Alternative Versions of the Model

We consider versions of the model which drop the entrepreneurial and banking sectors. To drop the entrepreneurial sector, simply drop equations 6 (a), 7 (a) and 8 (a), and add equation 24 (p). Letters in parentheses indicate whether the equation is full (a) or partial (p) information. The first three are the equations pertaining to the costly state verification contract. The last equation is the household's intertemporal Euler equation for accumulating capital. There are two variables to be dropped, 6 (a) and 8 (a). These are the cutoff productivity level implicit in the costly state verification contract and the law of motion for entrepreneur net worth. The household's intertemporal equation pertaining to time deposits remains as a way to define the risk free rate. Note that there is a sense in which more equations than variables have been dropped. Two 'a' variables are dropped, but, three a equations are dropped, with only a 'p' replacing one of them. To keep the number of 'a' equations and 'a' variables equal to each other, some variable has to be converted from 'a' to 'p'.

But, what variable? One option focusses on the labor market. In the model without banking or entrepreneurs, we know that aggregate employment is a 'p'. This is because aggregate demand, prices and wages (these enter the 'efficiency gaps' in the resource constraint), the physical stock of capital and the rate of capital utilization are all predetermined relative to a financial market shock. With everything else in the resource constraint predetermined, aggregate employment is predetermined too. But, this argument breaks down when there are banks. Because of this, the capital stock available to the goods sector is not predetermined. We could make this predetermined, of course, by requiring that firms choose the capital that they rent before the realization of the financial market shocks. But, this would change the nature of the firm problems and so would necessitate a change in the model. It's not clear this is the way we want to go. There is an alternative possibility, based on a change of variable.

Note from the scaled resource constraint, (2.58), that a financial market shock leaves

$$V_t^f = \nu_t^l (L_t)^{1-\alpha}$$

unaffected. In this notation, (2.58) is written:

$$d_t + a(u_t)\frac{1}{\mu_{z,t}}\bar{k}_t + g_t + c_t + i_t + \Theta(1-\gamma)v_t \leq \epsilon_t \left(u_t \frac{1}{\mu_{z,t}}\bar{k}_t\right)^\alpha V_t^f - \phi.$$

Linearizing this,

$$\hat{V}_t^f = \hat{\nu}_t^l + (1-\alpha)\hat{L}_t.$$

The proposed strategy would replace $\hat{L}_t$ (actually, $\hat{l}_t$) by $\hat{V}_t^f$ in z_t in (3.22). A simple way to do this proceeds as follows. Below, we argue that the linearized system leads to the following canonical form:

$$\mathcal{E}_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0,$$

where z_t is defined in (3.22), except that the two variables mentioned above are not included. Thus, z_t is a 21×1 column vector. Let $\tilde{z}_t$ be z_t, with $\hat{l}_t$ replaced by $\hat{V}_t^f$. The vectors, z_t and $\tilde{z}_t$, have a simple relationship. Let Q be a 21-dimensional square matrix which, with the exception of the 18^{th} row, is the 21-dimensional identity matrix. The 18^{th} row has $(1-\alpha)$ in the diagonal location and unity in the 8^{th}. Then,

$$Qz_t = \tilde{z}_t.$$

Note that,

$$\mathcal{E}_t[\alpha_0 Q^{-1} Q z_{t+1} + \alpha_1 Q^{-1} Q z_t + \alpha_2 Q^{-1} Q z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0,$$

or,

$$\begin{aligned} \mathcal{E}_t[\tilde{\alpha}_0 \tilde{z}_{t+1} + \tilde{\alpha}_1 \tilde{z}_t + \tilde{\alpha}_2 \tilde{z}_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] &= 0, \\ \tilde{\alpha}_i &= \alpha_i Q^{-1}, \; i = 0, 1, 2. \end{aligned}$$

The vector, $\tilde{z}_t$, is displayed below:

	$\tilde{z}_t$	information, $\tilde{z}$	information, equation (#)
1	$\hat{\pi}_t$	p	$p(1)$
2	$\hat{s}_t$	a	$a(2)$
3	$\hat{r}_t^k$	a	$a(3)$
4	$\hat{\imath}_t$	p	$p(4)$
5	$\hat{u}_t$	p	$p(5)$
6	$\hat{R}_t^k$	a	$a(9)$
7	$\hat{q}_t$	a	$a(10)$
8	$\hat{\nu}_t^l$	a	$a(11)$
9	$\hat{e}_{\nu,t}$	a	$a(12)$
10	$\hat{m}_t^b$	a	$a(13)$
11	$\hat{R}_t$	a	$a(14)$
12	$\hat{u}_{c,t}^z$	a	$a(15)$
13	$\hat{\lambda}_{z,t}$	a	$a(16)$
14	$\hat{m}_t$	a	$a(17)$
15	$\hat{R}_{a,t}$	a	$p(18)$
16	$\hat{c}_t$	p	$p(19)$
17	$\hat{w}_t$	p	$a(20)$
18	$\hat{V}_t^f$	p	$p(21)$
19	$\widehat{\bar{k}}_{t+1}$	p	$a(22)$
20	$\hat{R}_{t+1}^e$	a	$a(23)$
21	$\hat{x}_t$	a	$p(24)$

The right column reports the equations in the system, with corresponding equation numbers from the text, and an indication of what information set is associated with the the given equation. The left column gives the numerical location of a variable in $\tilde{z}_t$. Recall that the computational algorithm is set up so the i^{th} variable is an 'a' if, and only if, the i^{th} equation is an 'a'. Similarly for 'p'. At the moment, the equations are not lined up properly. However, if the 15^{th} and 17^{th} equations are interchanged, as well as the 19^{th} and 21^{st}, then the required alignment does occur.

When we drop the banks and entrepreneurs, there is still a 'financial sector'. Households start the period with the entire stock of high powered money, M^b. They set a part of this, M, aside. The rest, $M^b - M$, is deposited with a financial intermediary, where they earn R_a. The financial intermediary loans this out, plus the money injection, X, to firms for working capital, at a rate of interest, R. The intermediary has no costs and so the zero profit condition associated with competition implies $R = R_a$. Households still have access to risk free lending, with return, R_e. This market will have zero activity and simply defines the risk free rate. In steady state, it is just the usual intertemporal marginal rate of substitution, adjusted for inflation. If $\theta = 1$, so that households do not get utility from their deposits at the financial intermediary (as they would if these deposits produced transactions services) then, in steady state, $R_a = R = R_e$. If they did earn utility from deposits, then then $R_a < R_e$.

To also drop the banking sector, drop the 4 equations pertaining directly to the banks, 10 (a), 11 (a), 12 (a), 13 (a). One equation must be added, the clearing condition for the loan market, 25 (a). So, there is a net deletion of 3 'a' equations. We drop variables 10, 11 and 17. These are $\hat{\nu}_t^l$, $\hat{e}_{\nu,t}$ and $\hat{R}_{a,t}$. The last variable enters the household's M_t and M_{t+1}^b equations (equations 16 and 17). However, $\hat{R}_{a,t}$ there should simply be replaced by $\hat{R}_t$. All these variables are a's.

3.4. Computational Notes

We can write the 24 equations listed above in matrix form as follows:

$$\mathcal{E}_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0,$$

where z_t is defined above and $\mathcal{E}_t$ is the expectation operator which takes into account the information set associated with each equation. Also, s_t is constructed from the vector of shocks, Ψ_t, that impact on agents' environment, and it has the following representation:

$$s_t = P s_{t-1} + \tilde{\varepsilon}_t. \tag{3.24}$$

We now discuss the construction of the elements, s_t and P, of this time series representation.

There are $N = 20$ basic exogenous shocks, ς_t, in the model:

$$\hat{\lambda}_{f,t},\ \hat{\tau}_t,\ \hat{\psi}_{l,t},\ \hat{\psi}_{k,t},\ \hat{\xi}_t,\ \hat{x}_t^b,\ \hat{\tau}_t^T,\ \hat{\theta}_t,\ \hat{\tau}_t^D,\ \hat{\tau}_t^l,$$
$$\hat{\tau}_t^k,\ \hat{\zeta}_t,\ \hat{g}_t,\ \hat{\upsilon}_t,\ \hat{w}_t^e,\ \hat{\mu}_{z,t},\ \hat{\gamma}_t,\ \hat{\epsilon}_t,\ \hat{x}_{pt},\ \hat{\tau}_t^C$$

Here, λ_f is the steady state markup for intermediate good firms; τ is the reserve requirement for banks; ψ_l is the fraction of the wage bill that must be financed in advance; ψ_k is the fraction of the capital services bill that must be financed in advance; x_t is the growth rate of the monetary base; ξ_t is a shock influencing bank demand for reserves; x_t^b is a technology shock to the bank production function; τ_t^T is the tax rate on household earnings of interest on time deposits; θ_t is a shock to the relative preference for currency versus deposits; τ_t^D is the tax rate on household earnings of interest on deposits; τ_t^l is the tax rate on wage income; τ_t^k is the tax rate paid by entrepreneurs on their earnings of rent on capital services; ζ_t is a preference shock for household leisure; g_t is a shock to government consumption; $\hat{\upsilon}_t$

is a shock to the household demand for transactions services; $\hat{w}_t^e$ is a shock to the transfers received by entrepreneurs; $\hat{\mu}_{z,t}$ is a shock to the growth rate of technology; $\hat{\gamma}_t$ is a shock to the rate of survival of entrepreneurs; $\hat{\epsilon}_t$ is a stationary technology shock to intermediate good production; $\hat{x}_{pt}$ is the monetary policy shock; $\hat{\tau}_t^C$ is the tax on consumption.

In each case, we give the shock an ARMA(1,1) representation. In addition, we suppose that monetary policy corresponds to the innovation in a shock according to an ARMA(1,1), as in (2.53). Consider, for example, the first shock $\hat{\lambda}_{f,t}$. The following vector first order autoregression captures in its first row, the ARMA(1,1) representation of $\hat{\lambda}_{f,t}$ and in the third row the ARMA(1,1) representation of the response of monetary policy to the shock:

$$\begin{pmatrix} \hat{\lambda}_{f,t} \\ \epsilon_{f,t} \\ x_{f,t} \end{pmatrix} = \begin{bmatrix} \rho_f & \eta_f & 0 \\ 0 & 0 & 0 \\ 0 & \phi_f^1 & \phi_f^2 \end{bmatrix} \begin{pmatrix} \hat{\lambda}_{f,t-1} \\ \epsilon_{f,t-1} \\ x_{f,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{f,t} \\ \epsilon_{f,t} \\ \phi_f^0 \epsilon_{f,t} \end{pmatrix}.$$

There are 6 parameters associated with $\hat{\lambda}_{f,t}$: ρ_f, η_f, ϕ_f^2, ϕ_f^1, ϕ_f^0 and the standard deviation of $\epsilon_{f,t}$, σ_f. The parameters ϕ_f^0 and σ_f are only needed when the model is simulated, such as for computing impulse response functions or obtaining second moments. It is not required for computing the model solution. In this way, there are 6 parameters associated with each of the first 18 shocks, and the 20^{th}. Since logically there is no monetary policy response to a monetary policy shock, there are only three parameters for that shock. So, the total number of parameters associated with the exogenous shocks is $19 \times 6 + 3 = 117$.

We now discuss the construction of (3.24) in detail. Define the $3N \times 1$ vector Ψ_t as follows:

$$\Psi_t = \begin{pmatrix} \Psi_{1,t} \\ \vdots \\ \Psi_{N,t} \end{pmatrix}.$$

Here, $\Psi_{i,t}$ is 3×1 for $i = 1, ..., N$:

$$\Psi_{1,t} = \begin{pmatrix} \hat{\lambda}_{f,t} \\ \epsilon_{f,t} \\ x_{f,t} \end{pmatrix}, \ \Psi_{2,t} = \begin{pmatrix} \hat{\tau}_t \\ \epsilon_{\tau,t} \\ x_{\tau,t} \end{pmatrix}, \ \Psi_{3,t} = \begin{pmatrix} \hat{\psi}_{l,t} \\ \epsilon_{l,t} \\ x_{l,t} \end{pmatrix}, \ \Psi_{4,t} = \begin{pmatrix} \hat{\psi}_{k,t} \\ \epsilon_{k,t} \\ x_{k,t} \end{pmatrix}$$

$$\Psi_{5,t} = \begin{pmatrix} \hat{\xi}_t \\ \epsilon_{\xi,t} \\ x_{\xi,t} \end{pmatrix}, \ \Psi_{6,t} = \begin{pmatrix} \hat{x}_t^b \\ \epsilon_{b,t} \\ x_{b,t} \end{pmatrix}, \ \Psi_{7,t} = \begin{pmatrix} \hat{\tau}_t^T \\ \epsilon_{T,t} \\ x_{T,t} \end{pmatrix}, \ \Psi_{8,t} = \begin{pmatrix} \hat{\theta}_t \\ \epsilon_{\theta,t} \\ x_{\theta,t} \end{pmatrix},$$

$$\Psi_{9,t} = \begin{pmatrix} \hat{\tau}_t^D \\ \epsilon_{D,t} \\ x_{D,t} \end{pmatrix}, \ \Psi_{10,t} = \begin{pmatrix} \hat{\tau}_t^l \\ \epsilon_{\tau^l,t} \\ x_{\tau^l,t} \end{pmatrix}, \ \Psi_{11,t} = \begin{pmatrix} \hat{\tau}_t^k \\ \epsilon_{\tau^k,t} \\ x_{\tau^k,t} \end{pmatrix}, \ \Psi_{12,t} = \begin{pmatrix} \hat{\zeta}_t \\ \epsilon_{\zeta,t} \\ x_{\zeta,t} \end{pmatrix},$$

$$\Psi_{13,t} = \begin{pmatrix} \hat{g}_t \\ \epsilon_{g,t} \\ x_{g,t} \end{pmatrix}, \ \Psi_{14,t} = \begin{pmatrix} \hat{\upsilon}_t \\ \epsilon_{\upsilon,t} \\ x_{\upsilon,t} \end{pmatrix}, \ \Psi_{15,t} = \begin{pmatrix} \hat{w}_t^e \\ \epsilon_{w^e,t} \\ x_{w^e,t} \end{pmatrix}, \ \Psi_{16,t} = \begin{pmatrix} \hat{\mu}_{z,t} \\ \epsilon_{\mu_z,t} \\ x_{\mu_z,t} \end{pmatrix},$$

$$\Psi_{17,t} = \begin{pmatrix} \hat{\gamma}_t \\ \epsilon_{\gamma,t} \\ x_{\gamma,t} \end{pmatrix}, \ \Psi_{18,t} = \begin{pmatrix} \hat{\epsilon}_t \\ \epsilon_{\epsilon,t} \\ x_{\epsilon,t} \end{pmatrix}, \ \Psi_{19,t} = \begin{pmatrix} \hat{x}_{pt} \\ \epsilon_{p,t} \\ \hat{\tau}_{t-1}^k \end{pmatrix}, \ \Psi_{20,t} = \begin{pmatrix} \hat{\tau}_t^C \\ \epsilon_{\tau^C,t} \\ \hat{x}_{\tau^C,t} \end{pmatrix}$$

The non-financial market shocks are

$$\hat{\lambda}_{f,t}, \ \hat{\tau}_t, \ \hat{x}_t^b, \ \hat{\tau}_t^T, \ \hat{\tau}_t^D, \ \hat{\tau}_t^l, \ \hat{\tau}_t^k, \ \hat{\zeta}_t, \ \hat{g}_t, \ \hat{w}_t^e, \ \hat{\mu}_{z,t}, \ \hat{\epsilon}_t, \ \hat{\tau}_t^C$$

The financial market shocks are:

$$\hat{\psi}_{l,t}(7-9), \ \hat{\psi}_{k,t}(10-12), \ \hat{\xi}_t(13-15), \ \hat{\theta}_t(22-24), \ \hat{\upsilon}_t(40-42), \ \hat{\gamma}_t(49-51), \ \hat{x}_{pt}(55-56)$$

Numbers in parentheses correspond to the associated entries in Ψ_t.

The time series representation of Ψ_t is:

$$\Psi_t = \rho \Psi_{t-1} + \varepsilon_t^{\Psi},$$

where ρ is a $3N \times 3N$ matrix. With one exception, it is block diagonal in a way that is conformable with the partitioning of Ψ_t. Each block is 3×3. Thus, with one exception, ρ has the following structure:

$$\rho = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \rho_N \end{bmatrix},$$

with the partitioning being conformable with the partitioning of Ψ_t. The exception is the 31^{st} entry in the 57^{th} row of ρ, which is unity. For example,

$$\rho_1 = \begin{bmatrix} \rho_f & \eta_f & 0 \\ 0 & 0 & 0 \\ 0 & \phi_f^2 & \phi_f^1 \end{bmatrix},\ \rho_{19} = \begin{bmatrix} \rho_f & \eta_f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In general, ρ_i is 3×3 for $i = 1, ..., 18$, with zeros in the middle row and in the 1,3 and 3,1 elements. Similarly, we partition

$$\varepsilon_t^{\Psi} = \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{20t} \end{bmatrix},$$

where ε_{it} is 3×1 for $i = 1, ..., 20$, and the last element of $\varepsilon_{19,t}$ is zero. The first two entries of ε_{it} are equal and represent the innovation in the associated exogenous shock variable. The last entry is proportional to the second, where the factor of proportionality characterizes the contemporaneous response of monetary policy to the shock.

We now discuss the relation between s_t and Ψ_t. In the 'standard case' we assume that the information set in each equation is Ω_t^{μ}. In this case,

$$s_t = \theta_t,\ P = \rho,\ \tilde{\varepsilon}_t = \varepsilon_t^{\Psi}.$$

If any one of the information sets in any one of the equations contains less information than Ω_t^{μ}, then s_t is constructed slightly differently:

$$s_t = \begin{pmatrix} \Psi_t \\ \Psi_{t-1} \end{pmatrix},\ P = \begin{bmatrix} \rho & 0 \\ I & 0 \end{bmatrix},\ \tilde{\varepsilon}_t = \begin{pmatrix} \varepsilon_t^{\Psi} \\ 0 \end{pmatrix}. \tag{3.25}$$

The matrices, β_0 and β_1 provided to the computational algorithm are the ones that are suitable for the standard case. If the algorithm detects that some information sets are small, then it makes appropriate adjustments to the β's.

Monetary policy is a function of Ψ_t, according to equation (24) and (2.53):

$$\hat{x}_t = \left[\sum_{i=1,\ i\neq 19}^{20} (0,0,1)\Psi_{it} \right] + (1,0,0)\Psi_{19,t}.$$

A solution to the model is a set of matrices, A and B, in:

$$z_t = A z_{t-1} + B s_t,$$

where B is restricted to be consistent with our information set assumptions. In what follows, we first address the issue of computing the 'feedback matrix', A. We then turn to the computation of the 'feedforward matrix', B.

3.4.1. The Feedback Matrix, A

For now, we consider the problem of finding the 23×23 matrix A. For these purposes, we lose nothing by simply 'forgetting' exogenous shocks. So, we consider the case where $\Psi_t = 0$. The matrix A must satisfy the equilibrium conditions:

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0,$$

for $t = 0, 1, \ldots$. Here, the vector z_t is defined as the 23 by 1 vector displayed in (3.22). We require:

$$\left[\alpha_0 A^2 + \alpha_1 A + \alpha_2 I\right] z_{t-1} = 0,$$

for all z_{t-1}, so that

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = \underbrace{0}_{23\times 23}.$$

To find A, we apply the procedure developed in Blanchard and Kahn. First, set up the previous second order difference equation as a first order system. Let

$$Y_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}.$$

Then,

$$aY_{t+1} + bY_t = 0, \tag{3.26}$$

where

$$a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix},\ b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.$$

We seek a sequence, Y_0, Y_1, ..., which satisfies several conditions: (i) it must satisfy the above difference equation and the initial conditions of the system, which are contained in the bottom 23 elements of Y_0; (ii) it must be a 'minimal state sequence', namely, that there must exist a 23×46 matrix, D, such that $DY_t = 0$ for all t; (iii) it must satisfy 'convergence'

$$Y_t \to 0.$$

Finally, we would like (iv), that Y_0, Y_1, ... is the only sequence that satisfies (i)-(iii).

It is straightforward to determine if a sequence satisfying (i)-(iii) can be found and whether such a sequence is unique, i.e., whether it satisfies (iv). A particularly simple case occurs when a is invertible, which corresponds to the assumption that α_0 is invertible. In fact, α_0 in our example is singular so that a is not invertible. Still, even to explore the non-invertible case, it is instructive to study the invertible case first.

The Invertible a Case Define

$$\Pi = -a^{-1}b.$$

Then, the entire class of solutions to the difference equation (whether or not they satisfy (ii)-(iii)) can be written

$$Y_{t+1} = \Pi Y_t. \tag{3.27}$$

Each sequence, $Y_0, Y_1, ...,$ that satisfies (3.27) is differentiated by a different value for the first 23 elements of Y_0, namely, z_0 (recall, the initial conditions of the system are in z_{-1}, the last 23 elements of Y_0). Thus, the space of sequences that satisfy (i) is 23 dimensional. That's because z_0 is arbitrary, and $z_0 \in R^{23}$. We seek a sequence that satisfies not just (3.27) (i.e., (i)), but one that is a minimal state sequence, and which satisfies convergence.

We proceed now to identify sequences that satisfy (i)-(iii). Write the eigenvalue-eigenvector decomposition of Π :

$$\Pi = P\Lambda P^{-1},$$

where Λ is a diagonal matrix containing the eigenvalues and each column of P is the right eigenvector corresponding to the associated eigenvalue in Λ. Let

$$\tilde{P} = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_{46} \end{bmatrix}, \ \tilde{P} \equiv P^{-1}$$

where $\tilde{p}_i$ is the i^{th} left eigenvector of Π. Let

$$\tilde{Y}_t = \tilde{P} Y_t = \begin{bmatrix} \tilde{p}_1 Y_t \\ \vdots \\ \tilde{p}_{46} Y_t \end{bmatrix}.$$

Premultiplying (3.27) by $\tilde{P}$:

$$\tilde{Y}_t = \Lambda \tilde{Y}_{t-1} = \Lambda^t \tilde{Y}_0.$$

Recall that z_{-1}, the bottom 23 elements in Y_0, are free. So, it is possible to select Y_0 so that $\tilde{p}_i Y_0 = 0$ for up to 23 left eigenvectors. Note:

$$\tilde{p}_i Y_0 = 0 \rightarrow \tilde{p}_i Y_t = 0, \text{ for all } t.$$

So, to construct a sequence that satisfies (i) and (ii), simply construct a matrix D using any subset of 23 of the 46 left eigenvectors of Π. Thus, there are 46 choose 23 sequences that satisfy (i) and (ii).

We now consider convergence. Clearly, convergence requires that we 'suppress' the explosive eigenvalues of the system. This means that we select z_0 so that $\tilde{p}_i Y_0 = 0$ where λ_i is an eigenvalue with $|\lambda_i| > 1$. Suppose there are exactly 23 explosive eigenvalues. Then, there is exactly one sequence, Y_t, that satisfies (i)-(iii). This is because there is exactly one way to construct a matrix, D, that satisfies $DY_t = 0$ for all t, satisfies (3.27), and displays convergence. If there are fewer explosive eigenvalues there are many such solutions. If there are more than 23 explosive eigenvalues, then there is no sequence that satisfies (i)-(iii). Suppose the first scenario applies. Partition $D = [D^1 \vdots D^2]$, where D^1 and D^2 are each 23 by 23 matrices. In addition, D is constructed from the 23 left eigenvectors associated with the 23 explosive eigenvalues of Π. Then, $DY_t = 0$ implies:

$$z_t = Az_{t-1},\ A = -\left(D^1\right)^{-1} D^2.$$

The matrix A is the object that we seek.

The Non-Invertible a Case In our model, a is not invertible. The rank of the 23 by 23 matrix, α_0 is 8. It follows that the rank of a is 31=8+23, where 23 is the rank of the identify matrix in a. Still, a variant of the previous argument works, using the QZ decomposition, as implemented by Chris Sims. The first step is to find the orthonormal matrices Q and Z, and the upper triangular matrices H_0 and H_1 with the properties:

$$QaZ = H_0,\ QbZ = H_1.$$

The matrix H_0 is structured so that the zeros on its diagonal are located in the lower right part of H_0. Denote the upper 31×31 block of H_0 by G_0. This matrix has rank 31. The lower right 15 by 15 block of H_0 has rank zero and has zeros on the diagonal. Let the corresponding upper left 31×31 block in H_1 be denoted G_1. The diagonal terms in the lower right 15×15 block of H_1 are non-zero. Partition Z' as follows:

$$Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

where L_1 is 31×46 and L_2 is 15×46.

Inserting ZZ' $(= I)$ before Y_{t+1} and Y_t in (3.26), defining $\gamma_t \equiv Z'Y_t$, and pre-multiplying (3.26) by Q, (3.26) becomes:

$$H_0\gamma_{t+1} + H_1\gamma_t = 0,\ t = 0, 1, \ldots . \tag{3.28}$$

Partition γ_t as follows:

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}, \tag{3.29}$$

where γ_t^1 is 31×1 and γ_t^2 is 15×1. It is easy to verify that (3.28) implies $\gamma_t^2 = 0,\ t \geq 0$, i.e.,[14]

$$L_2 Y_t = 0,\ t = 0, 1, \ldots \ . \tag{3.30}$$

With (3.30) imposed, the last 15 equations in (3.28) are redundant, so (3.28) can be written

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = 0,\ t = 0, 1, \ldots \ . \tag{3.31}$$

The set of solutions to this system can be expressed as $\gamma_t^1 = (-G_0^{-1} G_1)^t \gamma_0^1$, $t \geq 0$, or,

$$P^{-1} \gamma_t^1 = \Lambda^t P^{-1} \gamma_0^1, \tag{3.32}$$

where $P \Lambda P^{-1} = -G_0^{-1} G_1$ is the eigenvector, eigenvalue decomposition of $-G_0^{-1} G_1$. The γ_t^1 that solve (3.32) converge to zero asymptotically if, and only if, $\tilde{p} \gamma_0^1 = 0$, where $\tilde{p}$ is composed of the rows of P^{-1} corresponding to diagonal terms in Λ that exceed 1 in absolute value. This condition is:

$$\tilde{p} L_1 Y_0 = 0. \tag{3.33}$$

Recall that the number of free elements in Y_0 is 23. Equation (3.30) for $t = 0$ represents 15 restrictions on Y_0, so that to pin Y_0 down uniquely, 8 more restrictions are needed. Thus, uniqueness requires that there be 8 explosive eigenvalues in Λ, i.e., that $\tilde{p} L_1$ contain 7 rows. Then, define

$$D = \begin{bmatrix} \tilde{p} L_1 \\ L_2 \end{bmatrix}. \tag{3.34}$$

The matrix A that we seek is then obtained by manipulating D in exactly the same way that was done before.

[14] To see this, let us temporarily adopt a simpler notation. Let the lower right 8×8 block of H_0 be denoted Γ and let the corresponding block of H_1 be denoted W. Write $\Gamma = [\Gamma_{ij}]$ and $W = [W_{ij}]$. The matrices, Γ and W, are upper triangular, with the former having zeros along its diagonal and the latter having non-zero terms along its diagonal. Also, write $x_t = \gamma_t^2$, with $x_t = [x_{1t}, \ldots, x_{8t}]'$. Then we have $\Gamma x_{t+1} + W x_t = 0$ for $t = 0, 1, 2, \ldots$. Note that the last row of Γ is composed of zeros, so that the last row of this system of equations is $W_{8,8} x_{8t} = 0$ for all t. Since $W_{8,8} \neq 0$, this implies $x_{8t} = 0$ for all t. Now consider the $8 - 1^{th}$ equation:

$$\Gamma_{7,8} x_{8,t+1} + W_{7,7} x_{7,t} + W_{7,8} x_{8,t} = 0,$$

for $t = 0, 1, 2, \ldots$. But, since $x_{\cdot,t} = 0$ for all t, this implies $W_{7,7} x_{7,t} = 0$ for all t. Since $W_{7,7} \neq 0$, this in turn implies $x_{7,t} = 0$ for $t = 0, 1, 2, \ldots$ .Proceeding in this way, we establish recursively that $x_{j,t} = 0$ for all t, for $j = 8, 7, \ldots, 1$.

3.4.2. The Feedforward Matrix, B

Our system of equations is, taking into account the matrix A already computed:

$$\mathcal{E}_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0.$$

We solution we seek has the form, $z_t = Az_{t-1} + Bs_t$. Note,

$$\begin{aligned} z_{t+1} &= Az_t + Bs_{t+1} \\ &= A\left[Az_{t-1} + Bs_t\right] + B\left[Ps_t + \tilde{\varepsilon}_{t+1}\right] \\ &= A^2 z_{t-1} + ABs_t + BPs_t + B\tilde{\varepsilon}_{t+1}. \end{aligned}$$

$$\mathcal{E}_t[\alpha_0\left(A\left[Az_{t-1} + Bs_t\right] + B\left[Ps_t + \tilde{\varepsilon}_{t+1}\right]\right) + \alpha_1\left(Az_{t-1} + Bs_t\right) + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

Substituting this, and the expression for z_t into the difference equation, and rearranging:

$$\mathcal{E}_t\{\alpha(A)z_{t-1} + Fs_t + \alpha_0 B\tilde{\varepsilon}_{t+1} + \beta_0\tilde{\varepsilon}_{t+1t}\} = 0,$$

where

$$F = \alpha_0 AB + \alpha_0 BP + \alpha_1 B + \beta_0 P + \beta_1 \tag{3.35}$$

Since $\mathcal{E}_t\tilde{\varepsilon}_{t+1} = 0$, a solution requires $\alpha(A) = 0$ and $\mathcal{E}_t F s_t = 0$, or,

$$\mathcal{E}_t F s_t = \tilde{F} s_t = 0, \tag{3.36}$$

for all s_t, so that

$$\tilde{F} = 0.$$

Here, $\tilde{F}$ is a 23 by 120 matrix constructed from F. If the i^{th} equation in our system is a full information equation, then the i^{th} row of $\tilde{F}$ is just the i^{th} row of F. When the i^{th} equation in our system is a partial information equation, then the entries of the i^{th} row of $\tilde{F}$ that correspond to elements that are not observed in period t are zero. We now discuss what the other entries of such a row of $\tilde{F}$ look like.

The following discussion has two parts. In the first part, we suppose we have candidate A and B in hand, and we wish to verify that these correspond to a solution. To do this, we evaluate $\alpha(A) = 0$ and $\tilde{F} = 0$. The former calculation is trivial. If $F = \tilde{F}$, then the latter is also trivial. However, in the partial information case, constructing $\tilde{F}$ from a candidate B is not trivial. In the second part of the discussion below, we describe a constructive approach to finding a B that sets $\tilde{F}$ to zero.

Checking a Given B Suppose that the exogenous shocks that are *not* observed in period t by a given equation are given by $D\Psi_t$. Each row of D is composed of all zeros except for a single entry which is unity, and which corresponds to the element of Ψ_t that is not observed. In particular, when an equation does not observe all current shocks (i.e., has a 'p' attached to it in (3.22)), it is specifically the 7 financial market shocks that are not observed. So, D is a 20×60 matrix. The fact that the left dimension of D is 20 reflects that each financial market shock, with the exception of the monetary policy shock, corresponds to three variables in Ψ_t (i.e., the variable itself, its innovation and the monetary policy response to the shock). So, there are 18 shocks coming from the non-monetary policy financial shocks. Then, there are another 2 coming from monetary policy.

The i^{th} row of D is the i^{th} row of the 60-dimensional identity matrix, where i is the location in Ψ_t of the shock that is not observed in period t to a partial information equation. We need to know what the date t conditional expectation these random variables is:

$$\begin{aligned} & E\left[D\Psi_t | \text{period } t \text{ non-financial market shocks and past information}\right] \\ = & DE\left[\Psi_t | \text{period } t \text{ non-financial market shocks and past information}\right] \\ = & DE\left[\Psi_t | \Psi_{t-1}\right]. \end{aligned}$$

The last equality reflects two assumptions: that the innovations in the variables are uncorrelated, and that Ψ_t is a first order Markov process. The property of the innovations implies that the current realization of the period t non-financial market shocks is of no use in computing the conditional expectation of the period t financial market shocks. The solution to the above conditional expectation is straightforward[15]. So, we conclude:

$$\begin{aligned} & E\left[D\Psi_t | \text{period } t \text{ non-financial market shocks}\right] \\ = & D\rho\Psi_{t-1}. \end{aligned}$$

From (3.22), the 'partial information' equations are $i = 1, 4, 5, 18, 19, 21$. In each case, the D matrix is the same, because the variables not observed in all partial information equations are the same. Write,

$$\begin{aligned} \mathcal{E}_t F_i s_t &= \mathcal{E}_t \left[F_i^1 \ F_i^2\right] \begin{pmatrix} \Psi_t \\ \Psi_{t-1} \end{pmatrix} \\ &= \mathcal{E}_t F_i^1 \Psi_t + F_i^2 \Psi_{t-1}, \end{aligned}$$

since $\mathcal{E}_t \Psi_{t-1} = \Psi_{t-1}$. Let $\tilde{D}$ be composed of the 40 rows of the 60 dimensional identity matrix

[15] Solveb is designed to handle the case where the correlations between the shocks are non-zero. However, I do not know if that part of the program has ever been properly tested.

which are not contained in D. That is, the matrix $\bar{D}$, where

$$\bar{D} = \begin{bmatrix} D \\ \tilde{D} \end{bmatrix},$$

is just the 60 dimensional identity matrix with the rows rearranged. Note that $\bar{D}'\bar{D} = I$. Taking this into account,

$$\begin{aligned} \mathcal{E}_t F_i s_t &= \mathcal{E}_t F_i^1 \Psi_t + F_i^2 \Psi_{t-1} \\ &= \mathcal{E}_t F_i^1 \bar{D}' \bar{D} \Psi_t + F_i^2 \Psi_{t-1} \\ &= \mathcal{E}_t \bar{F}_i^1 \bar{D} \Psi_t + F_i^2 \Psi_{t-1}, \end{aligned}$$

where $\bar{F}_i^1 = F_i^1 \bar{D}'$. That is, $\bar{F}_i^1$ is just F_i^1 with the columns reshuffled so that the first 20 columns correspond to the financial market shocks, and the others pertain to the non-financial market shocks. Similarly, $\bar{D}\Psi_t$ is Ψ_t rearranged, so the first 20 elements of $\bar{D}\Psi_t$ are the financial market parts of Ψ_t and the next 40 are the non-financial market shock parts of Ψ_t. Partition $\bar{F}_i^1$ appropriately, so :

$$\begin{aligned} \mathcal{E}_t F_i s_t &= \mathcal{E}_t \left[\bar{F}_{1i}^1 \vdots \bar{F}_{2i}^1 \right] \begin{bmatrix} D\Psi_t \\ \tilde{D}\Psi_t \end{bmatrix} + F_i^2 \Psi_{t-1} \\ &= \mathcal{E}_t \bar{F}_{1i}^1 D \Psi_t + \bar{F}_{2i}^1 \tilde{D} \Psi_t + F_i^2 \Psi_{t-1} \\ &= \bar{F}_{1i}^1 D \rho \Psi_{t-1} + \bar{F}_{2i}^1 \tilde{D} \Psi_t + F_i^2 \Psi_{t-1} \\ &= \bar{F}_{2i}^1 \tilde{D} \Psi_t + \left[F_i^2 + \bar{F}_{1i}^1 D \rho \right] \Psi_{t-1}, \\ &= \tilde{F}_i \begin{pmatrix} \Psi_t \\ \Psi_{t-1} \end{pmatrix}. \end{aligned}$$

where

$$F_i^1 \bar{D}' = \left[\bar{F}_{1i}^1 \vdots \bar{F}_{2i}^1 \right].$$

Note that the i^{th} row of $\tilde{F}_i$ has zeros in the entries corresponding to the elements of Ψ_t that are not observed in period t. Those entries of F_i (i.e., $\bar{F}_{1i}^1$) appear in the part of $\tilde{F}_i$ that multiplies Ψ_{t-1}.

To check the program that finds B, compute $\tilde{F}$ from A and B, and verify that $\tilde{F} = 0$. If it is zero, and A satisfies $\alpha(A) = 0$, then we know we have a solution. To evaluate $\tilde{F}$, first compute F. The latter 23×120 matrix is easy to compute using (3.35). The row dimension of F corresponds to the 23 equations in our system. For the full information equations, the corresponding row of $\tilde{F}$ is just the corresponding row of F. For the other equations, the corresponding row of $\tilde{F}$ has some zeros, and the non-zero elements of that row are functions of the elements of that row of F and of ρ. To construct such a row of $\tilde{F}$, first partition that row of F into $\left[F_{1i}^{1}\vdots F_{2i}^{1}\right]$. Then, construct D and $\tilde{D}$ and $\left[\bar{F}_{1i}^{1}\vdots\bar{F}_{2i}^{1}\right]$.

A Method for Finding B in the First Place We now develop a method for constructing B when we don't know it. Suppose we already have A in hand (it's a solution to $\alpha(A) = 0$). We seek a B which contains a pattern of zeros identical to the pattern in $\tilde{F}$, which implies that $\tilde{F} = 0$. We first address the 'standard case', when the information sets in each equation are full, and $\tilde{F} = F$, and there are no entries in B forced to be zero. We make heavy use of the following results:

$$\begin{array}{rcl} vec(A_1A_2A_3) & = & (A_3' \otimes A_1)vec(A_2) \\ vec(A+B) & = & vec(A) + vec(B), \end{array}$$

where $\otimes$ denotes the Kronecker product[16] and $vec(\cdot)$ denotes the vectorization operator.[17] Applying this to (3.35), we obtain:

$$F' = B'A'\alpha_0' + P'B'\alpha_0' + B'\alpha_1' + P'\beta_0' + \beta_1'$$

$$vec(F') = vec\left(I_{80}B'A'\alpha_0'\right) + vec(P'B'\alpha_0') + vec(I_{80}B'\alpha_1') + vec(P'\beta_0' + \beta_1') = 0,$$

where I_{80} is the 80-dimensional identify matrix. Then,

$$vec(F') = \left[(\alpha_0 A \otimes I_{80}) + (\alpha_0 \otimes P') + (\alpha_1 \otimes I_{80})\right] vec(B') + vec(P'\beta_0' + \beta_1')$$

That is,

$$d + q\delta = 0,$$

[16]The Kronecker product of two matrices, X and Y, is, if X is 3,2

$$\left[\begin{array}{cc} X(1,1)Y & X(1,2)Y \\ X(2,1)Y & X(2,3)Y \\ X(3,1)Y & X(3,2)Y \end{array}\right].$$

[17]If $X = [x_1\vdots x_2\vdots\cdots\vdots x_n]$, where x_i denotes the i^{th} column of X, then

$$vec(X) \equiv \left[\begin{array}{c} x_1 \\ \cdots \\ x_n \end{array}\right].$$

where

$$q = [(\alpha_0 A \otimes I) + (\alpha_0 \otimes P') + (\alpha_1 \otimes I)]\,,\ d = vec(P'\beta_0' + \beta_1'),\ \text{and } \delta = vec(B').$$

Evidently, we obtain δ in this case by solving:

$$\delta = -q^{-1}d.$$

It is useful to notice the dimension of these matrices. In our (benchmark) example, A and α_i are 23-dimensional. The matrix B is 23×120. So, the identity matrix in $vec(F')$ is 120×120. This means that the dimension of q is $23{\cdot}120 \times 23 \cdot 120$, or 2760×2760. This is an enormous matrix to invert. For example, on a 1.1Gig machine with plenty of RAM memory, it took 89 seconds to execute the instruction, $a\backslash b$, where a is a 2760×2760 matrix of random numbers and b is the conformable identity matrix.

One way to reduce the size of the matrices being computed would be to include as lags only the 20 elements of Ψ_t, i.e., the object $D\Psi_t$ above, which are not observed contemporaneously. This would reduce the s_t vector from 120-dimensional to 80-dimensional. This change in the definition of s_t would of course require a change in the definition of P in the structure of the s_t process, in (3.25). Then, the matrix being inverted above would be of dimension $23 \times 80 = 1840$ by 1840. This reduces the time requirement of computing $a\backslash b$ to 26 seconds.

Another way to reduce the size of the matrices might be to reduce the number of shocks. A relatively unobtrusive way to do this would be to impose that particular columns of B are zero, in the solution:

$$z_t = Az_t + Bs_t$$

To see how this impacts on the calculations, it is necessary to review how those calculations are done. To keep things simple, suppose there are only two equations. Then,

$$\begin{aligned}
\mathcal{E}_t F s_t &= \mathcal{E}_t \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{pmatrix} \Psi_t \\ \Psi_{t-1} \end{pmatrix} \\
&= \mathcal{E}_t \begin{bmatrix} F_{11}\Psi_t & F_{12}\Psi_{t-1} \\ F_{21}\Psi_t & F_{22}\Psi_{t-1} \end{bmatrix} \\
&= \mathcal{E}_t \begin{bmatrix} 0 & F_{12} + F_{11}\rho \\ F_{21} & F_{22} \end{bmatrix} \begin{pmatrix} \Psi_t \\ \Psi_{t-1} \end{pmatrix} \\
&= \tilde{F} s_t
\end{aligned}$$

Here, F_{ij} is a 1×120 column vector. Then,

$$\begin{aligned} vec(\tilde{F}') &= \begin{bmatrix} 0 \\ F_{12} + F_{11}\rho \\ F_{21} \\ F_{22} \end{bmatrix} = \tilde{R}vec(F') \\ &= \tilde{R}\,(q + d\delta) \end{aligned}$$

Let R denote a matrix that selects from q the non-zero elements. Thus, the dimension of R is $[1840 - (120 + 340)] = 1380$ by 1840. It is composed of the 1380 elements of the 1840− dimensional identity matrix which correspond to the entries of δ which are not set to zero.

Let the vector whose entries are sought be denoted $\bar{\delta}$. Then, $\bar{\delta} = R\delta$. We need to 'remove' the columns of q that correspond to the entries of δ that are identically zero, and which do not appear in $\bar{\delta}$. The corresponding rows of q need also be removed. We do this by defining, $\bar{q} = RqR'$. Also, let $\bar{d} = Rd$. Then, consider the following system:

$$\bar{d} + \bar{q}\bar{\delta} = 0.$$

Solve this for $\bar{\delta}$. The matrix B that we seek can be constructed from $\bar{\delta}$ computed in this way. As a check on the calculations, it is useful to apply the calculations described above.

To implement this strategy, we need a time series representation for the relevant s_t process, an efficient way to compute R, and an efficient way to map from $\bar{\delta}$ to B. We adopt the following structure:

$$s_t = \begin{pmatrix} \Psi_t \\ D\Psi_{t-1} \end{pmatrix},\ P = \begin{bmatrix} \rho & 0 \\ D & 0 \end{bmatrix},\ \tilde{\varepsilon}_t = \begin{pmatrix} \varepsilon_t^{\Psi} \\ 0 \end{pmatrix}, \tag{3.37}$$

where D was defined above. This structure makes heavy use of the recursive assumption, that lagged non-financial variables are not useful for predicting the current value of financial variables.

To construct D, let γ^p denote a vector containing the indeces of the equations for which there is full information. Thus,

$$\gamma^p = (1, 4, 5, 18, 19, 21).$$

Then, γ^f denotes the complementary vector, containing the indeces of the equations for which there is partial information. We also need to know the indeces of the elements of Ψ_t which are and are not, observed in period t by partial information equations. Let the indeces that are not observed be denoted by τ^p :

$$\tau^p = [[7:9][10:12][13:15][22:24][40:42][49:51][55:56]].$$

For each financial market shock, there are three indeces. Thus, the indeces corresponding to $\hat{\psi}_{l,t}$ are $7, 8,$ and 9, in Ψ_t. A more convenient way to construct τ^p may be to start from the basic vector of shocks, (3.25). The financial market shocks are elements 3, 4, 5, 8, 14, 17, 19 of this set. Then, τ^p can be constructed using the details of the mapping from the basic shocks to Ψ_t. To obtain D, just select the rows of the 60-dimensional identity matrix that correspond to the indeces in τ^p. It is also convenient to define the complementary vector, τ^f, which contains the indeces of the non-financial market shocks, in (3.25).

We now turn to R. This matrix is block diagonal, with $R_1, ..., R_{23}$, down the diagonal. The matrix, R_i, multiplies B_i' in δ. The matrix, R_i, is of one of two types, depending on whether i corresponds to a full or partial information equation. Either way, the rows of R_i are selections from the 80-dimensional identity matrix. For i corresponding to full information equations, R_i is the first 60 rows of this identity matrix. Such an R_i has the property that $R_i B_i'$ selects the elements of B_i' that correspond to Ψ_t, and ignores the ones that correspond to $D\Psi_{t-1}$. Now consider i's that correspond to partial information equations. Such an R_i has the property that $R_i B_i'$ ignores the elements of B_i' that correspond to current period financial market shocks. Since there are 40 current period non-financial market shocks and 20 lagged financial market shocks in s_t, this means that R_i is 60 by 80. The 60 rows of the 80 dimensional identity matrix in R_i are obtained as follows. The first 40 are the rows corresponding to the indeces in τ^f. The remaining 20 are the last 20 rows of the identity matrix.

The matrix, R, is $60 \cdot 23 \times 80 \cdot 23$, or 1380×1840. It is constructed as follows:

$$\underset{1380\times1840}{R} = \begin{bmatrix} \underset{60\times80}{R_1} & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{23} \end{bmatrix}$$

It turns out that the computational time needed to evaluate RqR' can be substantial. For example, to do it once for this example requires 26 seconds. Note, however, that there is substantial structure on this multiplication, and it makes sense to exploit this to get computational time down.

The new approach to computing B here necessitates a different checking procedure on B. The formulas, (3.35)-(3.36) continue to hold. There is one small difference. To see this, note first that:

$$\begin{aligned}
& E\left[D\Psi_t | \text{period } t \text{ non-financial market shocks}\right] \\
= \;& D\rho\Psi_{t-1} \\
= \;& D\rho\bar{D}'\bar{D}\Psi_{t-1} \\
= \;& D\rho\left[D' \vdots \tilde{D}'\right]\begin{pmatrix} D\Psi_{t-1} \\ \tilde{D}\Psi_{t-1} \end{pmatrix} \\
= \;& \left[D\rho D' \vdots D\rho\tilde{D}'\right]\begin{pmatrix} D\Psi_{t-1} \\ \tilde{D}\Psi_{t-1} \end{pmatrix}
\end{aligned}$$

Because of our assumption on the structure of ρ, $D\rho\tilde{D}' = 0_{20\times 40}$. So,

$$E_t\left[D\Psi_t\right] = \left(D\rho D'\right)D\Psi_{t-1}.$$

We use this in what follows. With the adjustment for the new structure of s_t,

$$\begin{aligned}
\mathcal{E}_t F_i s_t &= \mathcal{E}_t F_i^1 \Psi_t + F_i^2 D\Psi_{t-1} \\
&= \mathcal{E}_t \bar{F}_i^1 \bar{D}\Psi_t + F_i^2 D\Psi_{t-1} \\
&= \bar{F}_{1i}^1 D\rho\Psi_{t-1} + \bar{F}_{2i}^1 \tilde{D}\Psi_t + F_i^2 D\Psi_{t-1} \\
&= \bar{F}_{2i}^1 \tilde{D}\Psi_t + \left[F_i^2 D\bar{D}' + \bar{F}_{1i}^1 D\rho\bar{D}'\right]\bar{D}\Psi_{t-1}.
\end{aligned}$$

This expression can be simplified using a couple of results:

$$\begin{aligned}
D\bar{D}' &= D\left[D' \; \tilde{D}'\right] \\
&= \left[DD' \; D\tilde{D}'\right] \\
&= \left[\; I_{20} \vdots \underset{20\times 40}{0}\right],
\end{aligned}$$

and,

$$\begin{aligned}
D\rho\bar{D}' &= D\rho\left[D' \; \tilde{D}'\right] \\
&= \left[D\rho D' \; D\rho\tilde{D}'\right] \\
&= \left[D\rho D' \vdots \underset{20\times 40}{0}\right].
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}_t F_i s_t &= \bar{F}_{2i}^1 \tilde{D} \Psi_t + \left[F_i^2 D \bar{D}' + \bar{F}_{1i}^1 D \rho \bar{D}' \right] \bar{D} \Psi_{t-1} \\
&= \bar{F}_{2i}^1 \tilde{D} \Psi_t + \left\{ F_i^2 [\ I_{20} \vdots \underset{20\times 40}{0}] + \bar{F}_{1i}^1 [D \rho D' \vdots \underset{20\times 40}{0}] \right\} \begin{pmatrix} D\Psi_{t-1} \\ \tilde{D}\Psi_{t-1} \end{pmatrix} \\
&= \bar{F}_{2i}^1 \tilde{D} \Psi_t + \left\{ F_i^2 [\ I_{20} \vdots \underset{20\times 40}{0}] + \bar{F}_{1i}^1 [D \rho D' \vdots \underset{20\times 40}{0}] \right\} \begin{pmatrix} D\Psi_{t-1} \\ \tilde{D}\Psi_{t-1} \end{pmatrix} \\
&= \bar{F}_{2i}^1 \tilde{D} \Psi_t + \left[F_i^2 + \bar{F}_{1i}^1 D \rho D' \right] D \Psi_{t-1} \\
&= \left(\bar{F}_{2i}^1 \tilde{D} \vdots \left[F_i^2 + \bar{F}_{1i}^1 D \rho D' \right] \right) s_t .
\end{aligned}$$

As before, for i corresponding to equations with full information, F_i is just the i^{th} row of F in (3.35).

3.5. Simulations

In many cases, the actual variables produced by simulating $z_t = A z_{t-1} + B s_t$ are not the ones we're directly interested in. For example, we may want the percent deviation of consumption from its unshocked path. Instead, we get a simulation of $\hat{c}_t$. To 'unwind' this:

$$\hat{c}_t = \frac{c_t - c}{c},$$

where c denotes steady state consumption and c_t denotes consumption, scaled by z_t. So,

$$\hat{c}_t = \frac{\frac{\tilde{C}_t}{\tilde{z}_t} - \frac{C_t}{z_t}}{\frac{C_t}{z_t}},$$

where a tilde signifies the value of the variable along the shocked path, and the absence of a tilde means the variable along an unshocked steady state growth path. Supposing the system starts up at

$$\hat{c}_t = \frac{\frac{\tilde{C}_t}{\tilde{z}_t} - \frac{C_t}{z_t}}{\frac{C_t}{z_t}} = \frac{z_t}{C_t} \frac{\tilde{C}_t}{\tilde{z}_t} - 1,$$

or,

$$\frac{\tilde{C}_t}{C_t} = \frac{\tilde{z}_t}{z_t} \left(1 + \hat{c}_t \right).$$

But,

$$\frac{z_t}{z_{t-1}} = \mu_{zt}$$

so,

$$z_t = \mu_{zt} z_{t-1},$$

or,

$$\begin{aligned} z_0 &= \mu_{z,0} z_{-1} \\ z_1 &= \mu_{z,1} z_0 = \mu_{z,1} \mu_{z,0} z_{-1} \\ z_2 &= \mu_{z,2} z_1 = \mu_{z,2} \mu_{z,1} \mu_{z,0} z_{-1} \\ &\ldots \\ z_t &= \mu_{z,t} z_{t-1} = \mu_{z,t} \cdots \mu_{z,0} z_{-1}. \end{aligned}$$

Along the perturbed path:

$$\tilde{z}_t = \tilde{\mu}_{z,t} \cdots \tilde{\mu}_{z,0} z_{-1},$$

so,

$$\frac{\tilde{z}_t}{z_t} = \frac{\tilde{\mu}_{z,t}}{\mu_z} \cdots \frac{\tilde{\mu}_{z,0}}{\mu_z}$$

But,

$$\hat{\mu}_{z,t} = \frac{\tilde{\mu}_{z,t} - \mu_z}{\mu_z},$$

so,

$$\frac{\tilde{\mu}_{z,t}}{\mu_z} = \hat{\mu}_{z,t} + 1,$$

and

$$\frac{\tilde{z}_t}{z_t} = (\hat{\mu}_{z,t} + 1) \cdots (\hat{\mu}_{z,0} + 1)$$

Finally,

$$\frac{\tilde{C}_t - C_t}{C_t} = \left[(\hat{\mu}_{z,t} + 1) \cdots (\hat{\mu}_{z,0} + 1) \right] (1 + \hat{c}_t) - 1.$$

Note that when there is no shock to technology, so that $\hat{\mu}_{z,t} = 0$, then

$$\frac{\tilde{C}_t - C_t}{C_t} = \hat{c}_t,$$

and no adjustment is required.

Total output, Y_t, is:

$$Y_t = G_t + C_t + I_t,$$

or, after scaling:

$$\frac{Y_t}{z_t} = \frac{G_t}{z_t} + \frac{C_t}{z_t} + \frac{I_t}{z_t},$$

so that

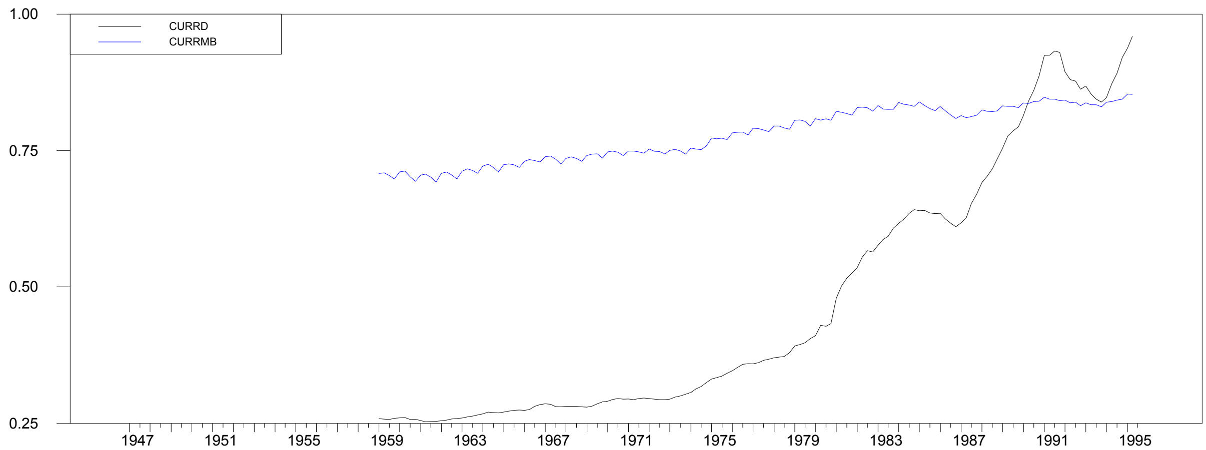
$$y\hat{y}_t = g\hat{g}_t + c\hat{c}_t + i\hat{\imath}_t$$

$$\hat{y}_t = \frac{g}{y}\hat{g}_t + \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{\imath}_t.$$

We'd also like the ratio of currency to deposits.

4. Econometric Methodology

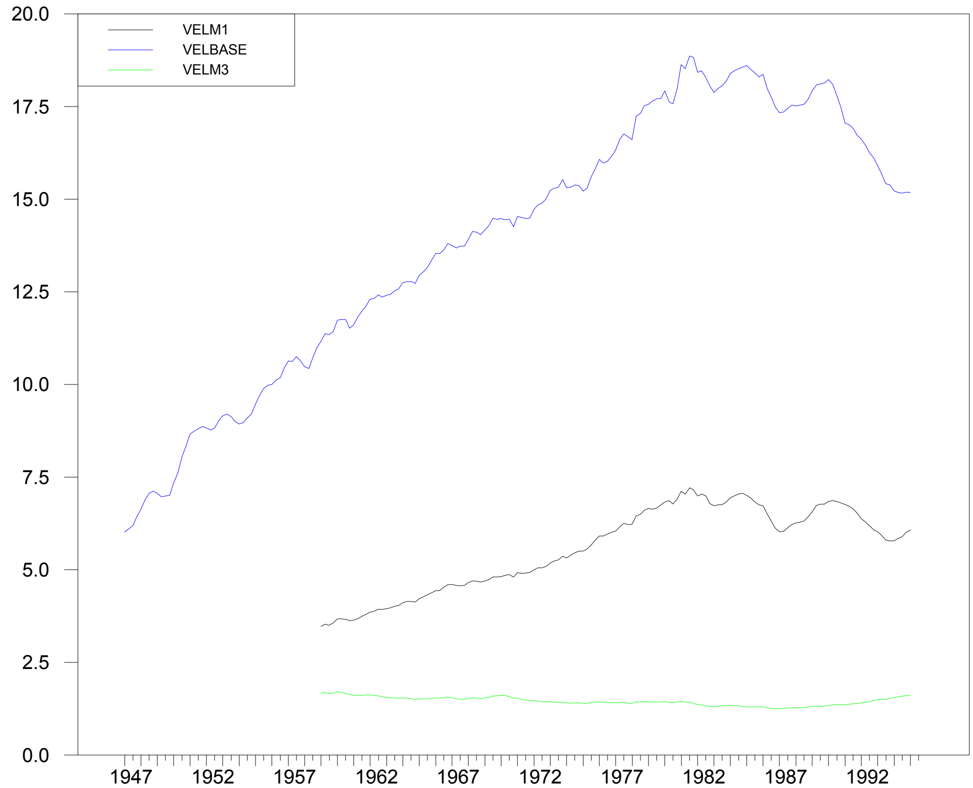
We plan to explore various econometric methodologies. The basic strategy is to begin with an array of limited information estimation and testing procedures. These might focus on a subset of the model's second moment implications, or on a subset of the model's implications for responses to various economic shocks. The distinguishing feature of limited information methods is that they do not impose all of the restrictions of the model, and permit a relatively informal assessment of the model's strengths and weaknesses. We expect that at this stage, various changes to the model structure will be needed to improve its empirical implications. At later stages, we expect to be able to move to full information methods such as maximum likelihood.



The previous graph displays data on the currency to deposit ratio (the steeply trending line) and the currency to monetary base ratio (the other line). [18]

[18]Data are taken from Citibase, and curency is measured by FMSCU; the monetary base is measured by FZFBA; deposits are measured by FMSD.

Following are the velocity of the monetary base, $M1$ and $M3$.



The following two pictures depict a set of parameter values for the model and the associated steady state. The capital output ratio is a little low here. But, the equity to debt ratio seems 'ball-park'. Also, note how the velocity of the different monetary aggregates corresponds roughly to the numbers reported in the previous figure. The velocity of the base is 14, the velocity of M1 is a little over 4. The velocity of $M3$ is 0.9.

&&&&&&&&&&&&&&&

Following is a set of steady state parameter values. Eventually, they should be put in a nice table...

year = 2002, month = 9, day = 14, hour = 12, minute = 41, second = 37.02, 14-Sep-2002

Variant of the model solved

entr = 1, model with entrepreneurs

pref = 1, so work with benchmark preferences

Fomegabar = 0.0300, muzˆ4 = 1.0150

y = c + i + g = 2.6383, we/y = 0.0569
we = 0.1500, ((1+R)^4-1)*100 = 10.4007
sigma = 0.3992
Gamma = 0.4340, psiL = 0.4627
n = 12.1030, n/k = 0.5637
n/(k-n) = 1.2922, ksi = 0.9549
Total Assets of Banks equals reserves plus working capital plus entrepreneur loans = 11.5535
Loans to Entrepreneurs and Time Deposits (as fraction of bank assets) = 0.7987
fraction of gnp devoted to monitoring = mu*G*(1+Rk)*k/(muz*pi))/y1
fraction = 0.00559713
Gross Interest rate paid by non-bankrupt entrepreneurs^4 = 1.11812812
Gross Average Interest rate paid by all entrepreneurs^4 = 1.10006389
Annualized velocity of M3 = 0.8702
mu = 0.0600, lambdaf = 1.2000
((1+x)^4-1)*100 = 6.1364, lambdaw = 1.05
alpha = 0.3600, psik = 0.5000
psil = 1.0000, delta = 0.0200
tauk = 0.2800, taul = 0.2500
gamma = 0.9700, l = 1.1254
tauT = 0.0000, tauD = 0.0000
sigmaL = 1.0000, zeta = 1.0000
etag = 0.1800, theta = 0.8000
v = 0.0050, beta^(-4)= 1.0300
sigmaq = 0.6600, b = 0.6300
m = 0.8000, vl = 0.9800
vk = 0.9800, tau = 0.0250
k/y = 8.1377, c/y = 0.6276
i/y = 0.1924, g/y = 0.1800
k/l = 19.0778, inflation^4 = 1.0457
elasticity of demand for currency (should be negative) = -0.0156
Annualized velocity of base = 4*y/mb 14.7182
Annualized velocity of M1 = 3.6399
household currency / household demand deposits = 3.7209
household deposits / firm demand deposits = 0.0710
household currency / total demand deposits = 0.2466
((1+Ra)^4-1)*100 = 0.63018
((1+Re)^4-1)*100 = 9.32045

((1+Rk)ˆ4-1)*100 = 12.27501
rk = 0.0450 mb = 0.7170
xb = 35.1704, lamz = 0.4854
Banking Sector Balance Sheet in SS (Fraction of Total Assets)
Total Reserves = 0.0133
Required Reserves = 0.0050
Excess Reserves = 0.0083
Working Capital Loans = 0.1880
Working Capital Loans (Cap. Rental) = 0.0417
Working Capital Loans (Labor costs) = 0.1463
Demand Deposit Liabilities = 0.2013
Household Demand Deposits = 0.0133
Firm Demand Deposits = 0.1880
****

A. Appendix 1: Alternative Specification of Entrepreneur's Rate of Return

We now consider what happens to the CSV contract when we specify the entrepreneur's rate of return to have the following form:

$$1 + R_{t+1}^{k,j} = (1 + \tilde{R}_{t+1}^{k})\omega + \tau_t^k \delta,$$

where $(1 + \tilde{R}_{t+1}^{k})$ is defined in section 2.4.

As before, total borrowing by a type-N entrepreneur is:

$$B_{t+1}^{N} = Q_{\bar{K}',t}\bar{K}_{t+1}^{N} - N_{t+1}.$$

The parameters of the $N_{t+1}-$type standard debt contract, B_{t+1}^{N} Z_{t+1}^{N}, imply a cutoff value of ω, $\bar{\omega}_{t+1}^{N}$, as follows:

$$\left[(1 + \tilde{R}_{t+1}^{k})\bar{\omega}_{t+1}^{N} + \tau_t^k \delta\right] Q_{\bar{K}',t}\bar{K}_{t+1}^{N} = Z_{t+1}^{N} B_{t+1}^{N}.$$

Note that it is possible for $\bar{\omega}_{t+1}^{N}$ in this equation to be negative. The following condition guarantees $\bar{\omega}_{t+1}^{N} > 0$:

$$\tau_t^k \delta < Z_{t+1}^{N} \frac{B_{t+1}^{N}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}}.$$

The object on the left of the equality is roughly 1/2.[19] The object on the left should be an order of magnitude smaller. We proceed under the assumption, $\bar{\omega}_{t+1}^{N} > 0$.

For $\omega < \bar{\omega}_{t+1}^{N}$, the entrepreneur pays all its revenues to the bank:

$$\left[(1 + \tilde{R}_{t+1}^{k})\omega + \tau_t^k \delta\right] Q_{\bar{K}',t}\bar{K}_{t+1}^{N},$$

which is less than $Z_{t+1}^{N} B_{t+1}^{N}$. In this case, the bank must monitor the enterpreneur, at cost

$$\mu \left[(1 + \tilde{R}_{t+1}^{k})\omega + \tau_t^k \delta\right] Q_{\bar{K}',t}\bar{K}_{t+1}^{N}.$$

Zero profits for banks implies:

$$\left[1 - F\left(\bar{\omega}_{t+1}^{N}\right)\right] Z_{t+1}^{N} B_{t+1}^{N} + (1-\mu)\int_0^{\bar{\omega}_{t+1}^{N}} \left[(1 + \tilde{R}_{t+1}^{k})\omega + \tau_t^k \delta\right] Q_{\bar{K}',t}\bar{K}_{t+1}^{N} dF(\omega) = \left(1 + R_{t+1}^{e}\right) B_{t+1}^{N},$$

[19]This corresponds to a debt to equity ratio of unity, which is in line with estimates reported for the US. See Benninga and Protopapadakis (1990) and literature cited there.

or,

$$\left[1-F\left(\bar{\omega}_{t+1}^{N}\right)\right]\left[\bar{\omega}_{t+1}^{N}+\frac{\tau_t^k\delta}{1+\tilde{R}_{t+1}^{k}}\right]+(1-\mu)\int_0^{\bar{\omega}_{t+1}^{N}}\left[\omega+\frac{\tau_t^k\delta}{1+\tilde{R}_{t+1}^{k}}\right]dF\left(\omega\right)=\frac{1+R_{t+1}^{e}}{1+\tilde{R}_{t+1}^{k}}\frac{B_{t+1}^{N}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}},$$

or,

$$\begin{aligned}&\frac{\left[1-\mu F\left(\bar{\omega}_{t+1}^{N}\right)\right]\tau_t^k\delta}{1+\tilde{R}_{t+1}^{k}}+\left[1-F\left(\bar{\omega}_{t+1}^{N}\right)\right]\bar{\omega}_{t+1}^{N}+(1-\mu)\int_0^{\bar{\omega}_{t+1}^{N}}\omega dF\left(\omega\right)\\=\;&\frac{1+R_{t+1}^{e}}{1+\tilde{R}_{t+1}^{k}}\frac{B_{t+1}^{N}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}}.\end{aligned}$$

which can be written:

$$\frac{\left[1-\mu F\left(\bar{\omega}_{t+1}^{N}\right)\right]\tau_t^k\delta}{1+\tilde{R}_{t+1}^{k}}+\Gamma(\bar{\omega}_{t+1}^{N})-\mu G(\bar{\omega}_{t+1}^{N})=\frac{1+R_{t+1}^{e}}{1+\tilde{R}_{t+1}^{k}}\frac{B_{t+1}^{N}}{Q_{\bar{K}',t}\bar{K}_{t+1}^{N}},\tag{A.1}$$

where $\Gamma(\bar{\omega}_{t+1}^{N})$ and $\mu G(\bar{\omega}_{t+1}^{N})$ are defined as follows:

$$\begin{aligned}G(\bar{\omega}_{t+1}^{N})&=\int_0^{\bar{\omega}_{t+1}^{N}}\omega dF(\omega).\\\Gamma(\bar{\omega}_{t+1}^{N})&=\bar{\omega}_{t+1}^{N}\left[1-F(\bar{\omega}_{t+1}^{N})\right]+G(\bar{\omega}_{t+1}^{N})\end{aligned}$$

It is useful to work out the derivative of Γ :

$$\begin{aligned}\Gamma'(\bar{\omega}_{t+1}^{N})&=1-F(\bar{\omega}_{t+1}^{N})-\bar{\omega}_{t+1}^{N}F'(\bar{\omega}_{t+1}^{N})+G'(\bar{\omega}_{t+1}^{N})\\&=1-F(\bar{\omega}_{t+1}^{N})>0,\end{aligned}$$

where we have used

$$G'(\bar{\omega}_{t+1}^{N})=\bar{\omega}_{t+1}^{N}F'(\bar{\omega}_{t+1}^{N}).$$

Then

$$\begin{aligned}\Gamma'(\bar{\omega}_{t+1}^{N})-\mu G'(\bar{\omega}_{t+1}^{N})&=1-F(\bar{\omega}_{t+1}^{N})-\mu\bar{\omega}_{t+1}^{N}F'(\bar{\omega}_{t+1}^{N})\\&=\left[1-F(\bar{\omega}_{t+1}^{N})\right]\left[1-\mu\frac{\bar{\omega}_{t+1}^{N}F'(\bar{\omega}_{t+1}^{N})}{1-F(\bar{\omega}_{t+1}^{N})}\right]\\&=\left[1-F(\bar{\omega}_{t+1}^{N})\right]\left[1-\mu\bar{\omega}_{t+1}^{N}h(\bar{\omega}_{t+1}^{N})\right],\end{aligned}$$

where

$$h(\bar{\omega}_{t+1}^N) = \frac{F'(\bar{\omega}_{t+1}^N)}{1 - F(\bar{\omega}_{t+1}^N)}.$$

We follow BGG in adopting the assumption (page 10, equation 3.10) that h is increasing in $\bar{\omega}_{t+1}^N$. They show (see appendix A in BGG) that h increasing implies there is a unique $\bar{\omega}^*$ such that $\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N)$ is increasing for $\bar{\omega}_{t+1}^N < \bar{\omega}^*$, has zero slope for $\bar{\omega}_{t+1}^N = \bar{\omega}^*$ and is decreasing for $\bar{\omega}_{t+1}^N > \bar{\omega}^*$.

For sufficiently small $\tau_t^k \delta$, the left side of (A.1) will share this property of $\Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N)$. We infer that the zero profit condition of banks resembles the usual Laffer curve setup in some respects. The left side of (A.1) is an inverted U shape. The right side is a horizontal line. If there is one value of $\bar{\omega}_{t+1}^N$ that solves (A.1), then generically there is another one too.

The non-negativity constraint on bank profits is, from (A.1):

$$\frac{1 + \tilde{R}_{t+1}^k}{1 + R_{t+1}^e} \left\{ \frac{\left[1 - \mu F\left(\bar{\omega}_{t+1}^N\right)\right] \tau_t^k \delta}{1 + \tilde{R}_{t+1}^k} + \Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N) \right\} \geq \frac{Q_{\bar{K}',t} \bar{K}_{t+1}^N - N_{t+1}}{Q_{\bar{K}',t} \bar{K}_{t+1}^N} = 1 - \frac{N_{t+1}}{Q_{\bar{K}',t} \bar{K}_{t+1}^N},$$

or,

$$\frac{Q_{\bar{K}',t} \bar{K}_{t+1}^N}{N_{t+1}} \frac{1 + \tilde{R}_{t+1}^k}{1 + R_{t+1}^e} \left\{ \frac{\left[1 - \mu F\left(\bar{\omega}_{t+1}^N\right)\right] \tau_t^k \delta}{1 + \tilde{R}_{t+1}^k} + \Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N) \right\} \geq \frac{Q_{\bar{K}',t} \bar{K}_{t+1}^N}{N_{t+1}} - 1$$

Let

$$\tilde{u}_{t+1} \equiv \frac{1 + R_{t+1}^k}{E\left(1 + R_{t+1}^k | \Omega_t^\mu\right)},\ s_{t+1} \equiv \frac{E\left(1 + R_{t+1}^k | \Omega_t^\mu\right)}{1 + R_{t+1}^e},\ k_{t+1}^N = \frac{Q_{\bar{K}',t} \bar{K}_{t+1}^N}{N_{t+1}},$$

so that

$$k_{t+1}^N \tilde{u}_{t+1} s_{t+1} \left\{ \frac{\left[1 - \mu F\left(\bar{\omega}_{t+1}^N\right)\right] \tau_t^k \delta}{\tilde{u}_{t+1} E\left(1 + R_{t+1}^k | \Omega_t^\mu\right)} + \Gamma(\bar{\omega}_{t+1}^N) - \mu G(\bar{\omega}_{t+1}^N) \right\} \geq k_{t+1}^N - 1 \tag{A.2}$$

Competition implies that the CSV loan contract is the best possible one, from the point of view of the entrepreneur, conditional on (A.2). That is, it maximizes the entrepreneur's 'utility' subject to the zero profit constraint just stated. The entrepreneur's expected revenues over the period in which the standard debt contract applies are:[20]

[20] We treat this as the entrepreneur's utility function, even though the entrepreneur will be around in the future (either he will be around as a condemmed person eating his last meal in the next period, or he will be around with at least one more period after that). Still, we drop all reference to the future in our expression of his utility function. A possible rationale for this is that future utility is a linear function of future net worth. We hope to show this in a future draft.

$$
\begin{aligned}
& E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\left((1+\tilde{R}_{t+1}^k)\omega+\tau_t^k\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^N-Z_{t+1}^N B_{t+1}^N\right]dF(\omega)|\Omega_t,X_t\right\} \\
= \;& E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\left((1+\tilde{R}_{t+1}^k)\omega+\tau_t^k\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^N-\left((1+\tilde{R}_{t+1}^k)\bar{\omega}_{t+1}^N+\tau_t^k\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^N\right]dF(\omega)|\Omega_t,X_t\right\} \\
= \;& E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\omega-\bar{\omega}_{t+1}^N\right]dF(\omega)\left(1+\tilde{R}_{t+1}^k\right)|\Omega_t,X_t\right\}Q_{\bar{K}',t}\bar{K}_{t+1}^N.
\end{aligned}
$$

Note that[21]

$$
1=\int_0^{\infty}\omega dF(\omega)=\int_{\bar{\omega}_{t+1}^N}^{\infty}\omega dF(\omega)+G(\bar{\omega}_{t+1}^N),
$$

so that the objective can be written:

$$
E\left\{\left[1-\Gamma(\bar{\omega}_{t+1}^N)\right]\left(1+\tilde{R}_{t+1}^k\right)|\Omega_t^{\mu}\right\}Q_{\bar{K}',t}\bar{K}_{t+1}^N,
$$

or, after dividing by $(1+R_{t+1}^e)N_{t+1}$ (which is a constant with respect to date $t+1$ aggregate uncertainty), and rewriting:

$$
E\left\{\left[1-\Gamma(\bar{\omega}_{t+1}^N)\right]\tilde{u}_{t+1}|\Omega_t^{\mu}\right\}s_{t+1}k_{t+1}^N,\;\;\tilde{u}_{t+1}=\frac{1+\tilde{R}_{t+1}^k}{E\left(1+\tilde{R}_{t+1}^k|\Omega_t^{\mu}\right)},\;\;s_{t+1}=\frac{E\left(1+\tilde{R}_{t+1}^k|\Omega_t^{\mu}\right)}{1+R_{t+1}^e}, \tag{A.3}
$$

where Ω_t^{μ} denotes all period t shocks. From this expression and the fact, $\Gamma'>0$, it is evident that the objective is decreasing in $\bar{\omega}_{t+1}^N$ for given k_{t+1}^N.

The problem solved by the CSV is to choose k_{t+1}^N and $\bar{\omega}_{t+1}^N$ to maximize (A.3) subject to (A.2). Note that N_{t+1} does not appear in the problem, so that, as before, we can delete N.

[21] Under the alternative treatment of depreciation,

$$
\begin{aligned}
& E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\left((1+\tilde{R}_{t+1}^k)\omega+\tau_t^k\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^N-\left(\left[1+\tilde{R}_{t+1}^k\right]\bar{\omega}_{t+1}^N+\tau_t^k\delta\right)Q_{\bar{K}',t}\bar{K}_{t+1}^N\right]dF(\omega)|\Omega_t,X_t\right\} \\
= \;& E\left\{\int_{\bar{\omega}_{t+1}^N}^{\infty}\left[\omega-\bar{\omega}_{t+1}^N\right]dF(\omega)\left(1+\tilde{R}_{t+1}^k\right)|\Omega_t,X_t\right\}Q_{\bar{K}',t}\bar{K}_{t+1}^N.
\end{aligned}
$$

In addition, for notational convenience we delete the time subscripts too. Thus, the problem is to solve the following Lagrangian problem:

$$\max_{k,\bar{\omega}} E\left\{[1-\Gamma(\bar{\omega})]\,\tilde{u}sk+\lambda\left\{k\tilde{u}s\left[\frac{[1-\mu F(\bar{\omega})]\,\tau^k\delta}{\tilde{u}E\left(1+\tilde{R}^k_{t+1}|\Omega^\mu_t\right)}+\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right]+1-k\right\}\right\}$$

B. Appendix 2: Indeterminacy In Money-in-the-Utility Function

We consider indeterminacy in simple monetary models. One is a cash in advance model, and the other has money in the utility function like the one in our benchmark model.

B.1. Cash in Advance

Consider a simple monetary economy in which households have the following preferences:

$$\sum_{t=0}^{\infty}\beta^t u(c_t,l_t),$$

where

$$u(c,l)=\frac{c^{1-\sigma}}{1-\sigma}-\psi_0\frac{l^{1+\psi}}{1+\psi}.$$

Households must finance consumption purchases with cash obtained by setting aside part of their claims to the monetary base, M_t^b, and obtained with current wages:

$$P_tc_t\leq W_tl_t+M_t,$$

where M_t is cash. Their cash evolution equation is:

$$M^b_{t+1}=(1+R_t)(M^b_t-M_t+X_t)+M_t+W_tl_t-P_tC_t,$$

where X_t denotes a transfer from the monetary authority. Households' first order conditions include the usual static one:

$$\frac{-u_{l,t}}{u_{c,t}}=\frac{W_t}{P_t},$$

as well as the dynamic one:

$$\frac{u_{c,t}}{P_t}=\beta\left(1+R_t\right)\frac{u_{c,t+1}}{P_{t+1}}$$

Firms are required to finance the labor input using loans. The technology is linear and one-for-one in labor, so that the resource constraint implies

$$c_t = l_t.$$

Firms' first order condition for labor implies:

$$\frac{(1+R_t)W_t}{P_t} = 1.$$

Loan market clearing implies:

$$W_t l_t = M_t^b - M_t + X_t$$

Note that when the cash in advance constraint is binding (as it will whenever $R_t > 0$),

$$P_t c_t = M_t^b + X_t = M_{t+1}^b, \tag{B.1}$$

where the latter is the law of motion of the aggregate monetary base.

We now characterize the equilibrium with a first order difference equation in l_t. substitute the two static euler equations into the household intertemporal equation:

$$\frac{-u_{l,t}}{P_t} = \beta \frac{u_{c,t+1}}{P_{t+1}},$$

or, making use of (B.1)

$$\frac{-u_{l,t}c_t}{M_{t+1}^b} = \beta \frac{u_{c,t+1}c_{t+1}}{M_{t+1}^b + X_{t+1}}.$$

Multiply by M_{t+1}^b

$$-u_{l,t}c_t = \beta \frac{u_{c,t+1}c_{t+1}}{1+x_{t+1}}, 1+x_{t+1} \equiv \frac{M_{t+1}^b + X_{t+1}}{M_{t+1}^b}.$$

Making use of our functional forms and the resource constraint, we find:

$$\psi_0 l_t^{1+\psi} = \beta \frac{l_{t+1}^{1-\sigma}}{1+x_{t+1}}, 1+x_{t+1} \equiv \frac{M_{t+1}^b + X_{t+1}}{M_{t+1}^b},$$

or, after solving in terms of l_{t+1}, with a constant money growth rate:

$$l_{t+1} = \left[\psi_0 \frac{1+x}{\beta}\right]^{\frac{1}{1-\sigma}} (l_t)^{\frac{1+\psi}{1-\sigma}}.$$

In this example, there is a unique steady state, and that steady state is indeterminate if, and only if,

$$\sigma \geq \psi + 2.$$

In this case,

$$\left|\frac{1+\psi}{1-\sigma}\right| < 1,$$

so the map cuts the 45^0 from above at the steady state.

The equilibrium rate of interest is given by:

$$1 + R_t = \frac{1}{\psi_0} l_t^{-(\psi+\sigma)}.$$

The intuition for the multiplicity of equilibria in the neighborhood of steady state is as follows. Consider the steady state equilibrium itself. That obviously involves constant everything. Now, suppose low current employment were an equilibrium. In this economy, this can only happen if the nominal rate of interest were high. But, if the nominal rate of interest is high, then (as long as the real rate is high too), people will choose a high consumption growth rate. That is, employment in the next period has to be relatively high. For that to be true requires that next period's interest rate is low. If that corresponds to a low real rate, then, consumption growth starting in the next period has to be low.

Notice that in this model, indeterminacy is less likely the greater is ψ. This corresponds to a low labor supply elasticity. In this case, fluctuations in demand for labor arising from fluctuations in the nominal interest rate cannot induce substantial fluctations in employment.

B.2. Money in the Utility Function

We now consider an alternative version of the model, in which the reason M_t is set aside is that it appears in the utility function. We drop the cash in advance constraint on the household. The new preferences have the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t, \frac{M_t}{P_t}),$$

where

$$u(c, l, \frac{M}{P}) = \frac{c^{1-\sigma}}{1-\sigma} - \psi_0 \frac{l^{1+\psi}}{1+\psi} - \upsilon \frac{\left(\frac{Pc}{M}\right)^{1-\sigma_q}}{1-\sigma_q}.$$

The cash evolution equation is the same as before. The Lagrangian representation of the household problem is:

$$\sum_{t=0}^{\infty}\beta^t\{u(c_t,l_t,\frac{M_t}{P_t})$$
$$+\lambda_t\left[(1+R_t)(M_t^b-M_t+X_t)+M_t+W_tl_t-P_tc_t-M_{t+1}^b\right]\}$$

The first order condition for c_t is:

$$c_t^{-\sigma}-\upsilon c_t^{-\sigma_q}\left(\frac{P_t}{M_t}\right)^{1-\sigma_q}-\lambda_tP_t=0.$$

The first order condition for l_t is:

$$-\psi_0l_t^{\psi}+\lambda_tW_t=0$$

The first order condition for M_{t+1}^b is:

$$-\lambda_t+\beta\lambda_{t+1}(1+R_{t+1})=0$$

Note that it is next period's interest rate that enters the intertemporal Euler equation. This reflects that consumption is a credit good in this economy. That is, purchasing a good in period t does not entail setting aside cash in period t and therefore losing period t interest. The first order condition for M_t is:

$$\upsilon\left(P_tc_t\right)^{1-\sigma_q}M_t^{\sigma_q-2}-\lambda_tR_t=0.$$

Substituting out for the multiplier, these reduce to:

$$c_t^{-\sigma}-\upsilon c_t^{-\sigma_q}\left(\frac{P_t}{M_t}\right)^{1-\sigma_q}-\psi_0l_t^{\psi}\frac{P_t}{W_t}=0. \tag{B.2}$$

$$\upsilon\left(P_tc_t\right)^{1-\sigma_q}M_t^{\sigma_q-2}-\psi_0l_t^{\psi}\frac{R_t}{W_t}=0. \tag{B.3}$$

$$\frac{l_t^{\psi}}{W_t}=\beta\frac{l_{t+1}^{\psi}}{W_{t+1}}(1+R_{t+1}) \tag{B.4}$$

The money demand equation here is obtained by substituting out for λ_t from the consumption first order condition into the M_t first order condition:

$$\frac{\upsilon\left(c_t\right)^{1-\sigma_q}\left(m_t^b m_t\right)^{\sigma_q-2}}{c_t^{-\sigma}-\upsilon c_t^{-\sigma_q}\left(m_t^b m_t\right)^{\sigma_q-1}}=R_t$$

The interest elasticity of money demand is

$$\eta=-\frac{d\log\left(m_t^b m_t\right)}{d\log R_t}=\frac{\widehat{\left(m_t^b m_t\right)}}{\hat{R}_t},$$

holding fixed c_t. Expanding the money demand equation:

$$\upsilon\left(c_t\right)^{\widehat{1-\sigma_q\left(m_t^b m_t\right)}^{\sigma_q-2}}-\left[c_t^{-\sigma}-\upsilon \widehat{c_t^{-\sigma_q}\left(m_t^b m_t\right)}^{\sigma_q-1}\right]=\hat{R}_t.$$

or,

$$\left[\left(\sigma_q-2\right)+\frac{\upsilon c^{-\sigma_q}\left(m^b m\right)^{\sigma_q-1}\left(\sigma_q-1\right)}{c^{-\sigma}-\upsilon c^{-\sigma_q}\left(m^b m\right)^{\sigma_q-1}}\right]\widehat{\left(m_t^b m_t\right)}=\hat{R}_t,$$

or, using (B.2),

$$\left[\left(\sigma_q-2\right)+\frac{\upsilon c^{-\sigma_q}\left(m^b m\right)^{\sigma_q-1}\left(\sigma_q-1\right)}{\psi_0 l^{\psi}/w}\right]\widehat{\left(m_t^b m_t\right)}=\hat{R}_t,$$

so that the elasticity is:

$$\begin{aligned}\eta &= \frac{1}{2-\sigma_q}\frac{1}{1+\frac{\sigma_q-1}{\sigma_q-2}\frac{\upsilon c^{-\sigma_q}\left(m^b m\right)^{\sigma_q-1}}{\psi_0 l^{\psi}/w}}=\frac{1}{2-\sigma_q}\frac{1}{1+\frac{\sigma_q-1}{\sigma_q-2}\frac{\upsilon l^{-(\sigma_q+\psi)}\left(m^b m\right)^{\sigma_q-1}}{(1+R)\psi_0}} \\ &= \frac{1}{2-\sigma_q+\left(1-\sigma_q\right)\frac{\upsilon l^{-(\sigma_q+\psi)}\left(m^b m\right)^{\sigma_q-1}}{(1+R)\psi_0}}\end{aligned}$$

Note that this is guaranteed to be positive only if $\sigma_q \leq 1$. If $\sigma_q \geq 2$, then the elasticity is negative.

The firms and technology are just like in the previous example. That is, the resource constraint is:

$$c_t=l_t$$

and the firms' efficiency condition is:

$$\frac{(1+R_t)W_t}{P_t} = 1.$$

In addition, loan market clearing requires:

$$M_t^b - M_t + X_t = W_t l_t,$$

Let's convert all these equations into real terms. Let

$$m_t = \frac{M_t}{M_t^b},\ m_t^b = \frac{M_t^b}{P_t},\ 1 + x_t = \frac{M_{t+1}^b}{M_t^b},\ w_t = \frac{W_t}{P_t}.$$

Rewriting (B.2):

$$c_t^{-\sigma} - \upsilon c_t^{-\sigma_q}\left(\frac{1}{m_t^b m_t}\right)^{1-\sigma_q} - \psi_0 l_t^{\psi}\frac{1}{w_t} = 0. \tag{B.5}$$

Rewriting (B.3):

$$\upsilon\left(\frac{c_t}{m_t m_t^b}\right)^{1-\sigma_q}\frac{1}{m_t m_t^b} - \psi_0 l_t^{\psi}\frac{R_t}{w_t} = 0. \tag{B.6}$$

Rewriting (B.4):

$$l_t^{1+\psi}\frac{l_{t+1}W_{t+1}}{l_t W_t} = \beta l_{t+1}^{1+\psi}(1+R_{t+1}),$$

or, after substituting from the loan market clearing condition:

$$\frac{l_t^{1+\psi}(1+x_t)}{1-m_t+x_t} = \frac{\beta l_{t+1}^{1+\psi}(1+R_{t+1})}{1-m_{t+1}+x_{t+1}}. \tag{B.7}$$

The firm first order condition is:

$$(1+R_t)w_t = 1. \tag{B.8}$$

Also, the loan market clearing condition is:

$$m_t^b(1-m_t+x_t) = w_t l_t \tag{B.9}$$

The equations available to us are the resource constraint, $c_t = l_t$, and the 5: (B.5)-(B.9). The variables to be determined are 5: l_t, w_t, R_t, m_t^b, m_t. The strategy is to use (B.5), (B.6), (B.7), (B.8) to define a mapping from l_t to w_t, R_t, m_t^b and m_t. With R_{t+1} and m_{t+1} functions (hopefully, they're not correspondences!) of l_{t+1} and m_t a function of l_t, (B.7) becomes a first order difference equation in l_t and l_{t+1}.

Substitute out for w_t from (B.8) into (B.5), (B.6), (B.7), and replace c_t by l_t to obtain:

$$\begin{aligned} l_t^{-\sigma} - \upsilon l_t^{-\sigma_q}\left(\frac{1}{m_t^b m_t}\right)^{1-\sigma_q} - \psi_0 l_t^{\psi}(1+R_t) &= 0 \\ \upsilon\left(\frac{l_t}{m_t m_t^b}\right)^{1-\sigma_q}\frac{1}{m_t m_t^b} - \psi_0 l_t^{\psi}(1+R_t)R_t &= 0 \\ (1+R_t) &= \frac{l_t}{m_t^b\left(1-m_t+x_t\right)} \end{aligned} \qquad \text{(B.10)}$$

For given l_t, this represents three equations in m_t^b, m_t and R_t. We can substitute out for R_t using the last equation:

$$\begin{aligned} l_t^{-\sigma} - \upsilon l_t^{-\sigma_q}\left(m_t^b m_t\right)^{\sigma_q-1} &= \frac{\psi_0 l_t^{1+\psi}}{m_t^b\left(1-m_t+x_t\right)} \\ \upsilon\left(\frac{m_t}{l_t}\right)^{\sigma_q-1}\frac{\left(m_t^b\right)^{\sigma_q}}{m_t} &= \psi_0 l_t^{\psi}\frac{l_t}{(1-m_t+x_t)}\left[\frac{l_t}{(1-m_t+x_t)} - m_t^b\right] \end{aligned}$$

The second equation implies a unique m_t^b for each fixed m_t. To see this, note that the right side is strictly decreasing in m^b, while the left side is strictly increasing, for $\sigma_q > 1$. Also, note that the left side is strictly smaller than the right at $m^b = 0$, while at $m^b = l/(1-m+x) > 0$ the left side is bigger than the right. There must be a unique intersection, for each possible l_t, m_t. The first equation then can be thought of as a single non-linear equation in m_t. Suppose there is a unique solution. Existence and uniqueness is easy to check numerically for given values of the model parameters, since m_t is constrained to the compact set, $[0,1]$.

Suppose $x_t = x$ for all t. We now have a mapping, $m^b = m^b(l)$ and $m = m(l)$. Equation (B.10) can be used to define a mapping, $R = R(l)$. Taken together, these results imply that (B.7) can be written:

$$B(l) = A(l').$$

This implicitly defines a map from l to l'. The way to evaluate it numerically is to select an arbitrary value of l, say $\bar{l}$, for which B exists. Then, find l' such that $A(l') = B(\bar{l})$.

It is useful to have the steady state for l. From (B.7), we have

$$\frac{1+x}{\beta} = 1+R$$

The steady state value of l is and l^* such that

$$1 + R(l^*) = \frac{1+x}{\beta}.$$

We found numerically that existence of a steady state is not assured for all parameter values. We constructed an example with

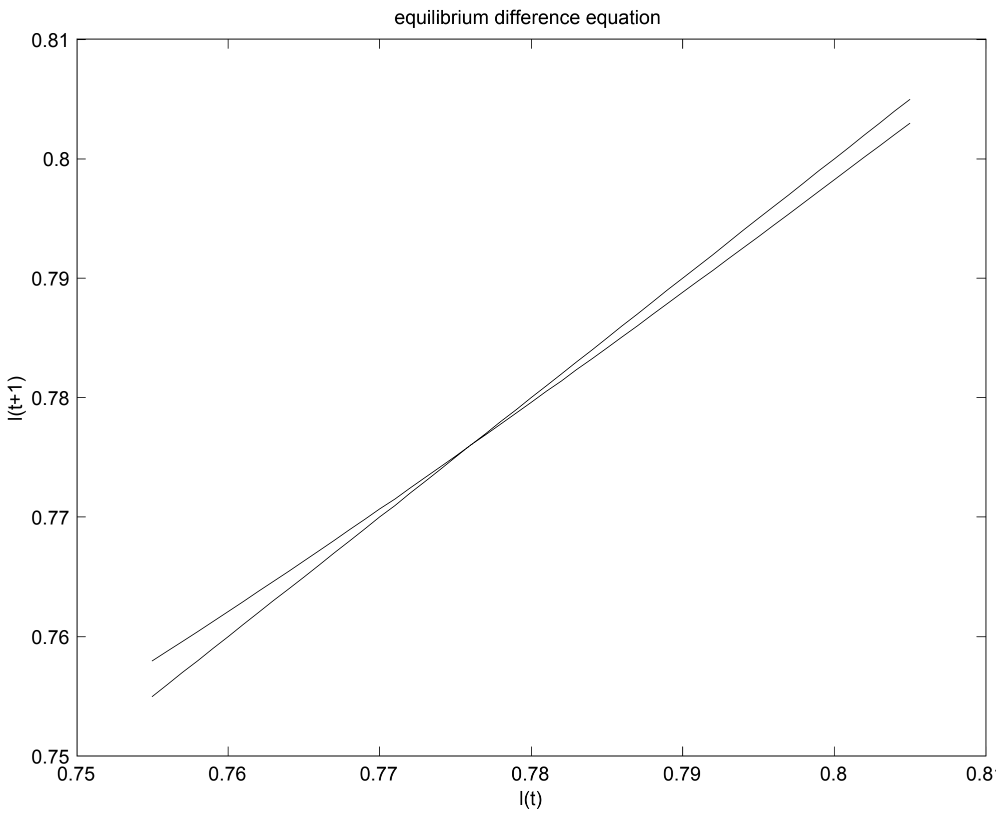
$$\sigma_q = 2,\ \sigma = 1,\ \psi_0 = 1,\ \psi = 1,\ x = 0.015,\ \upsilon = 0.05,\ \beta = \frac{1+x}{1+R^*}.$$

The steady states are:

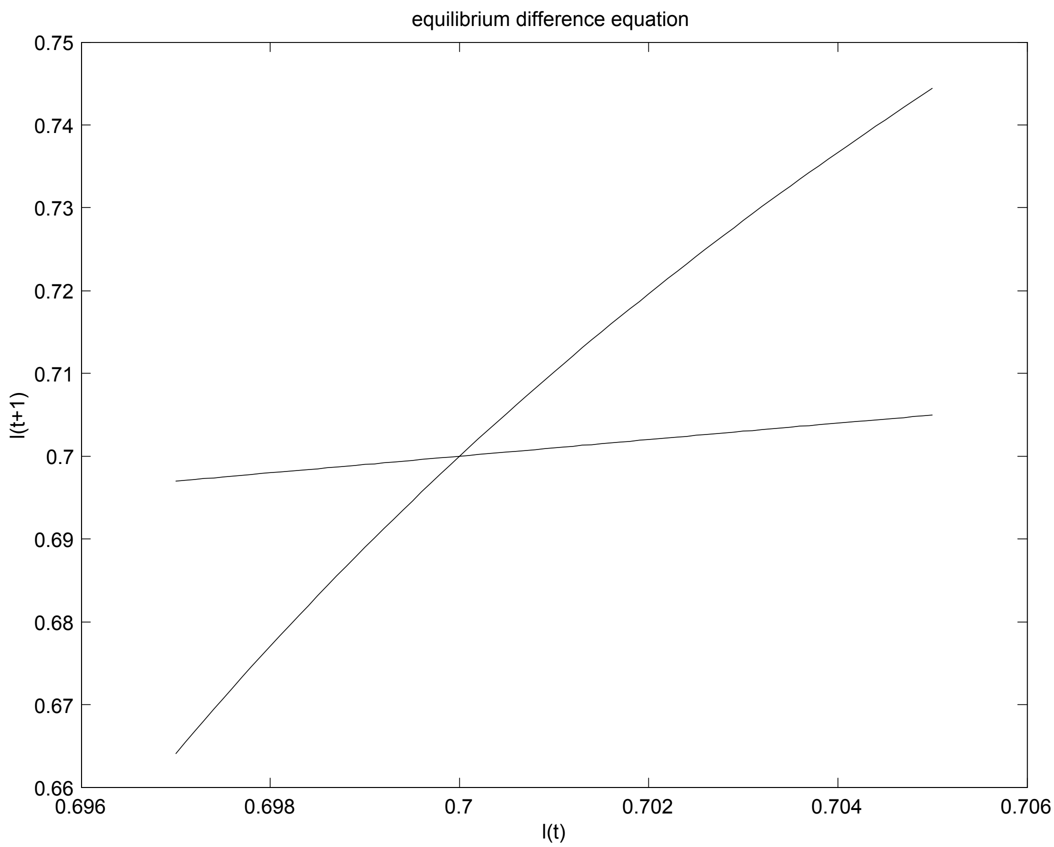
$$\begin{aligned} R^* &= 0.07708942392812 \\ l^* &= 0.7760 \\ m^{b,*} &= 6.08,\ m^* = 0.897. \end{aligned}$$

The money demand elasticity, η, is -1.85 for this example. This is obviously undesirable, since the money demand curve slopes up in this case.

The following figure displays a graph of the equilibrium difference equation, against the 45^0 degree line. Note how the equilibrium difference equation cuts the 45^0 from above. This equilibrium is indeterminate.



We now consider a second example, one in which $\sigma_q = 1$. Also, we set $l^* = 0.70$ and found a steady state which rationalizes this by computing $R(l^*) = 0.93877551020408$, and then finding the value of β such that $\beta = (1+x)/(1+R(l^*))$. In this case, $\beta = 0.52352631578947$. These are obviously a very high interest rate, and low value for β. Still, the exercise is interesting, because the money demand elasticity is now unity, i.e., $\eta = 1$. The equilibrium difference equation for this example is displayed in the following figure:



The curved line is the equilibrium difference equation. Note how now it cuts the 45^0 line from below, indicating that the steady state is determinate.

We can also evaluate determinacy of the steady state by differentiating the equilibrium difference equation in the neighborhood of steady state. Thus, totaly differentiating the equations that characterize the equilibrium:

$$-\sigma l^{-\sigma}\hat{l}_t + \upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q}\left[\sigma_q\hat{l}_t + (1-\sigma_q)\left(\hat{m}_t^b + \hat{m}_t\right)\right] = \psi_0 l^{\psi}(1+R)\left[\psi\hat{l}_t + (\widehat{1+R_t})\right]$$

$$\begin{aligned}(1-\sigma_q)\left[\hat{l}_t - \hat{m}_t - \hat{m}_t^b\right] - \hat{m}_t - \hat{m}_t^b &= \psi\hat{l}_t + (\widehat{1+R_t}) + \hat{R}_t \\ (\widehat{1+R_t}) &= \hat{l}_t - \hat{m}_t^b - (1-\widehat{m_t + x_t})\end{aligned}$$

Substituting:

$$-\sigma l^{-\sigma}\hat{l}_t + \upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q}\left[\sigma_q \hat{l}_t + (1-\sigma_q)\left(\hat{m}_t^b + \hat{m}_t\right)\right] = \psi_0 l^{\psi}(1+R)\left[\psi\hat{l}_t + \hat{l}_t - \hat{m}_t^b - (1 - \widehat{m_t + x_t})\right]$$

$$(1-\sigma_q)\left[\hat{l}_t - \hat{m}_t - \hat{m}_t^b\right] - \hat{m}_t - \hat{m}_t^b = \psi\hat{l}_t + \hat{l}_t - \hat{m}_t^b - (1 - \widehat{m_t + x_t}) + \hat{R}_t$$

But,

$$(1 - \widehat{m_t + x_t}) = \frac{-m\hat{m}_t}{1-m+x}$$

so that,

$$-\sigma l^{-\sigma}\hat{l}_t + \upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q}\left[\sigma_q \hat{l}_t + (1-\sigma_q)\left(\hat{m}_t^b + \hat{m}_t\right)\right] = \psi_0 l^{\psi}(1+R)\left[\psi\hat{l}_t + \hat{l}_t - \hat{m}_t^b + \frac{m\hat{m}_t}{1-m+x}\right] \tag{B.11}$$

and

$$(1-\sigma_q)\left[\hat{l}_t - \hat{m}_t - \hat{m}_t^b\right] - \hat{m}_t - \hat{m}_t^b = \psi\hat{l}_t + \hat{l}_t - \hat{m}_t^b + \frac{m\hat{m}_t}{1-m+x} + \frac{1+R}{R}\left[\hat{l}_t - \hat{m}_t^b + \frac{m\hat{m}_t}{1-m+x}\right] \tag{B.12}$$

In the last expression, we have used

$$(\widehat{1 + R_t}) = \frac{R\hat{R}_t}{1+R},$$

so that

$$\hat{R}_t = \frac{1+R}{R}\left[\hat{l}_t - \hat{m}_t^b + \frac{m\hat{m}_t}{1-m+x}\right]$$

Collecting terms in (B.11):

$$\hat{l}_t\left[-\sigma l^{-\sigma} + \upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q}\sigma_q - \psi_0 l^{\psi}(1+R)(1+\psi)\right] + \hat{m}_t^b\left[(1-\sigma_q)\upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q} + \psi_0 l^{\psi}(1+\right.$$

$$+\hat{m}_t\left[(1-\sigma_q)\upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q} - \psi_0 l^{\psi}(1+R)\frac{m}{1-m+\right.$$

Collecting terms in (B.12):

$$\begin{aligned}
& \hat{l}_t\left[(1-\sigma_q) - (1+\psi) - \frac{1+R}{R}\right] + \hat{m}_t\left[-(1-\sigma_q) - 1 - \frac{m}{1-m+x} - \frac{1+R}{R}\frac{m}{1-m+x}\right] \\
& +\hat{m}_t^b\left[-(1-\sigma_q) - 1 + 1 + \frac{1+R}{R}\right] \\
= \;& 0
\end{aligned}$$

Write these two equations as:

$$\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \begin{pmatrix} \hat{m}_t^b \\ \hat{m}_t \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \hat{l}_t,$$

where

$$\begin{aligned}
a_0 &= (1-\sigma_q)\,\upsilon l^{-\sigma_q} \left(\frac{1}{m^b m}\right)^{1-\sigma_q} + \psi_0 l^{\psi}(1+R) \\
a_1 &= (1-\sigma_q)\,\upsilon l^{-\sigma_q} \left(\frac{1}{m^b m}\right)^{1-\sigma_q} - \psi_0 l^{\psi}(1+R)\frac{m}{1-m+x} \\
a_2 &= -\left[-\sigma l^{-\sigma} + \upsilon l^{-\sigma_q}\left(\frac{1}{m^b m}\right)^{1-\sigma_q}\sigma_q - \psi_0 l^{\psi}(1+R)\,(1+\psi)\right] \\
b_0 &= -(1-\sigma_q) - 1 + 1 + \frac{1+R}{R} \\
b_1 &= -(1-\sigma_q) - 1 - \frac{m}{1-m+x} - \frac{1+R}{R}\frac{m}{1-m+x} \\
b_2 &= -\left[(1-\sigma_q) - (1+\psi) - \frac{1+R}{R}\right]
\end{aligned}$$

so that

$$\begin{pmatrix} \hat{m}_t^b \\ \hat{m}_t \end{pmatrix} = \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}^{-1} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \hat{l}_t = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \hat{l}_t$$

Expanding (B.7):

$$(1+\psi)\,\hat{l}_t + \frac{m\hat{m}_t}{1-m+x} = (1+\psi)\,\hat{l}_{t+1} + \hat{l}_{t+1} - \hat{m}_{t+1}^b + \frac{2m\hat{m}_{t+1}}{1-m+x},$$

or,

$$\left[(1+\psi) + \frac{mb_3}{1-m+x}\right]\hat{l}_t = \left[(1+\psi) + 1 - a_3 + \frac{2mb_3}{1-m+x}\right]\hat{l}_{t+1},$$

or,

$$\begin{aligned}
\hat{l}_{t+1} &= \frac{(1+\psi) + \frac{mb_3}{1-m+x}}{(1+\psi) + 1 - a_3 + \frac{2mb_3}{1-m+x}}\hat{l}_t \\
&= f\hat{l}_t,
\end{aligned}$$

where

$$f = \frac{(1+\psi) + \frac{mb_3}{1-m+x}}{(1+\psi) + 1 - a_3 + \frac{2mb_3}{1-m+x}}.$$

Determinacy requires $|f| > 1$. The equilibrium is (locally) indeterminate if $|f| < 1$.

C. Appendix 3: Calvo Pricing When Steady State Inflation is Non-Zero and Non-Optimizers Can't Change Their Price

This section works out the first order condition for Calvo price-setters when non-optimizers can't change their price. Let the price set by a firm that reoptimizes in period t be denoted $\tilde{P}_t$. Its price in period $t+1$ in case it does not reoptimize then is $\tilde{P}_t$. In period $t+j$ it is $\tilde{P}_t$, $j = 1, 2, ...$, and so on. The firm's objective is:

$$E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{t+j}[\left(\tilde{P}_t\right)^{1-\frac{\lambda_f}{\lambda_f-1}} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} - MC_{t+j}\left(Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} \left(\tilde{P}_t\right)^{-\frac{\lambda_f}{\lambda_f-1}} + \phi z_{t+j}\right)].$$

Differentiating this expression with respect to $\tilde{P}_t$:

$$\begin{aligned} E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{t+j}[\left(1 - \frac{\lambda_f}{\lambda_f - 1}\right) \left(\tilde{P}_t\right)^{-\frac{\lambda_f}{\lambda_f-1}} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} & \\ -\left(-\frac{\lambda_f}{\lambda_f - 1}\right) MC_{t+j} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} \left(\tilde{P}_t\right)^{-\frac{\lambda_f}{\lambda_f-1}-1}] &= 0 \end{aligned}$$

Multiply by $-\left(\tilde{P}_t\right)^{\frac{\lambda_f}{\lambda_f-1}+1} (\lambda_f - 1)$

$$E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{t+j} \left[\tilde{P}_t Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} - \lambda_f s_{t+j} Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}+1}\right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \left(\lambda_{t+j} P_{t+j}\right) Y_{t+j} P_{t+j}^{\frac{\lambda_f}{\lambda_f-1}} \left[\tilde{P}_t P_{t+j}^{-1} - \lambda_f s_{t+j}\right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{z,t+j} y_{t+j} \left(P_t \pi_{t+1} ... \pi_{t+j}\right)^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1} ... \pi_{t+j}} - \lambda_f s_{t+j}\right] = 0,$$

where $\lambda_{z,t+j} = \lambda_{t+j} P_{t+j} z_{t+j}$, and $y_{t+j} = Y_{t+j}/z_{t+j}$ and it is understood that when $j = 0$, $\pi_{t+1} ... \pi_{t+j} = 1$. Finally,

$$E_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{z,t+j} y_{t+j} \left(\pi_{t+1} ... \pi_{t+j}\right)^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1} ... \pi_{t+j}} - \lambda_f s_{t+j}\right] = 0. \tag{C.1}$$

Now, we proceed to linearize this expression about the steady state of the variables. We adopt the following convention:

$$\hat{x}_t = \frac{dx_t}{x},$$

where x is the steady state value of x_t.

First, we compute the value of $\tilde{p}$. Replacing the variables by their steady state values, we obtain:

$$\sum_{j=0}^{\infty} (\beta\xi_p)^j \, \bar{\pi}^{j\frac{\lambda_f}{\lambda_f - 1}} \left[\frac{\tilde{p}_t}{\bar{\pi}^j} - \lambda_f s \right] = 0,$$

or

$$\tilde{p}_t \sum_{j=0}^{\infty} (\beta\xi_p)^j \, \bar{\pi}^{j\left(\frac{\lambda_f}{\lambda_f - 1} - 1\right)} = \lambda_f s \sum_{j=0}^{\infty} (\beta\xi_p)^j \, \bar{\pi}^{j\frac{\lambda_f}{\lambda_f - 1}},$$

or,

$$\tilde{p} = \lambda_f s \frac{\frac{1}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}}{\frac{1}{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}} = \lambda_f s \frac{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}.$$

Note that when $\bar{\pi} = 1$, $\tilde{p} = \lambda_f s$. Also, when $\bar{\pi} > 1$, then $\tilde{p} > \lambda_f s$. This is to be expected. When $\bar{\pi} > 1$, then in steady state the firm that optimizes in a static sense, by setting $\tilde{p} = \lambda_f s$ systematically undershoots its desired price in the future in the event that it cannot reoptimize then. So, it 'splits the difference' by setting its price a little too high in the period when it has the opportunity to reoptimize. Note that for $\tilde{p}$ to be a finite number, we must have

$$\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}} < 1$$

or,

$$\bar{\pi} < \left(\frac{1}{\beta\xi_p}\right)^{\frac{\lambda_f-1}{\lambda_f}}.$$

Let's see how tight this constraint is. We minimize the right hand side by setting β and ξ_p to their biggest reasonable values.

Table: upper bounds on quarterly inflation

λ_f	$\xi_p = 0.75$	$\xi_p = 0.50$
1.02	1.0059	1.0139
1.50	1.1043	1.2641
1.30	1.0711	1.1762
1.10	1.0274	1.066

We view these as reasonably high upper bounds on inflation, particularly since $\xi_p = 0.50$ seems like an empirically reasonable estimate.

Now we proceed to linearize the first order condition of the firm. For convenience, we repeat its first order condition here:

$$\sum_{j=0}^{\infty} (\beta\xi_p)^j \lambda_{z,t+j} y_{t+j} (\pi_{t+1}...\pi_{t+j})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}...\pi_{t+j}} - \lambda_f s_{t+j}\right] = 0.$$

It is useful to write this expression out:

$$\begin{aligned}
&\lambda_{z,t} y_t [\tilde{p}_t - \lambda_f s_t] \\
&+ (\beta\xi_p) \lambda_{z,t+1} y_{t+1} (\pi_{t+1})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}} - \lambda_f s_{t+1}\right] \\
&+ (\beta\xi_p)^2 \lambda_{z,t+2} y_{t+2} (\pi_{t+1}\pi_{t+2})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}\pi_{t+2}} - \lambda_f s_{t+2}\right] \\
&+ (\beta\xi_p)^3 \lambda_{z,t+3} y_{t+3} (\pi_{t+1}\pi_{t+2}\pi_{t+3})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}\pi_{t+2}\pi_{t+3}} - \lambda_f s_{t+3}\right] \\
&+....
\end{aligned}$$

Differentiating with respect to s_{t+j} :

$$-(\beta\xi_p)^j \lambda_z y \left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}} \lambda_f s \hat{s}_{t+j}$$

Differentiating with respect to π_{t+1} :

$$\begin{aligned}
&\bar{\pi}\hat{\pi}_{t+1}\{(\beta\xi_p) \lambda_{z,t+1} y_{t+1} \frac{\lambda_f}{\lambda_f-1} (\pi_{t+1})^{\frac{\lambda_f}{\lambda_f-1}-1} \left[\frac{\tilde{p}_t}{\pi_{t+1}} - \lambda_f s_{t+1}\right] - (\beta\xi_p) \lambda_{z,t+1} y_{t+1} \pi_{t+1}^{\frac{\lambda_f}{\lambda_f-1}} \frac{\tilde{p}_t}{\pi_{t+1}^2} \\
&+ (\beta\xi_p)^2 \lambda_{z,t+2} y_{t+2} \frac{\lambda_f}{\lambda_f-1} \pi_{t+1}^{\frac{\lambda_f}{\lambda_f-1}-1} (\pi_{t+2})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}\pi_{t+2}} - \lambda_f s_{t+2}\right] \\
&- (\beta\xi_p)^2 \lambda_{z,t+2} y_{t+2} (\pi_{t+1}\pi_{t+2})^{\frac{\lambda_f}{\lambda_f-1}} \frac{\tilde{p}_t}{\pi_{t+1}^2\pi_{t+2}} \\
&+ (\beta\xi_p)^3 \lambda_{z,t+3} y_{t+3} \frac{\lambda_f}{\lambda_f-1} \pi_{t+1}^{\frac{\lambda_f}{\lambda_f-1}-1} (\pi_{t+2}\pi_{t+3})^{\frac{\lambda_f}{\lambda_f-1}} \left[\frac{\tilde{p}_t}{\pi_{t+1}\pi_{t+2}\pi_{t+3}} - \lambda_f s_{t+3}\right] \\
&- (\beta\xi_p)^3 \lambda_{z,t+3} y_{t+3} (\pi_{t+1}\pi_{t+2}\pi_{t+3})^{\frac{\lambda_f}{\lambda_f-1}} \frac{\tilde{p}_t}{\pi_{t+1}^2\pi_{t+2}\pi_{t+3}} \\
&+....\}
\end{aligned}$$

Evaluate the expression in braces in steady state, and multiply through by $\bar{\pi}$:

$$\begin{aligned}
&\hat{\pi}_{t+1}\{(\beta\xi_p)\,\lambda_z y \frac{\lambda_f}{\lambda_f-1}(\bar{\pi})^{\frac{\lambda_f}{\lambda_f-1}}\left[\frac{\tilde{p}}{\pi}-\lambda_f s\right]-(\beta\xi_p)\,\lambda_z y \pi^{\frac{\lambda_f}{\lambda_f-1}}\frac{\tilde{p}}{\pi} \\
&+(\beta\xi_p)^2\,\lambda_z y \frac{\lambda_f}{\lambda_f-1}\left(\pi^2\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\frac{\tilde{p}}{\pi^2}-\lambda_f s\right]-(\beta\xi_p)^2\,\lambda_z y\left(\pi^2\right)^{\frac{\lambda_f}{\lambda_f-1}}\frac{\tilde{p}}{\pi^2} \\
&+(\beta\xi_p)^3\,\lambda_z y \frac{\lambda_f}{\lambda_f-1}\left(\pi^3\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\frac{\tilde{p}}{\pi^3}-\lambda_f s_t\right]-(\beta\xi_p)^3\,\lambda_z y\left(\pi^3\right)^{\frac{\lambda_f}{\lambda_f-1}}\frac{\tilde{p}}{\pi^3}+....\}
\end{aligned}$$

or,

$$\begin{aligned}
&\hat{\pi}_{t+1}\sum_{j=1}^{\infty}\left\{(\beta\xi_p)^j\,\lambda_z y\frac{\lambda_f}{\lambda_f-1}\left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\frac{\tilde{p}}{\pi^j}-\lambda_f s\right]-(\beta\xi_p)^j\,\lambda_z y\left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}}\frac{\tilde{p}}{\pi^j}\right\} \\
=\;&\hat{\pi}_{t+1}\lambda_z y\left\{\left[\frac{\lambda_f}{\lambda_f-1}-1\right]\tilde{p}\sum_{j=1}^{\infty}(\beta\xi_p)^j\left(\pi^j\right)^{\left(\frac{\lambda_f}{\lambda_f-1}-1\right)}-\frac{\lambda_f}{\lambda_f-1}\lambda_f s\sum_{j=1}^{\infty}(\beta\xi_p)^j\left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}}\right\} \\
=\;&\hat{\pi}_{t+1}\lambda_z y\left\{\left[\frac{\lambda_f}{\lambda_f-1}-1\right]\tilde{p}\frac{\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}-\frac{\lambda_f}{\lambda_f-1}\lambda_f s\frac{\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}\right\} \\
=\;&\hat{\pi}_{t+1}\frac{\lambda_z y}{\lambda_f-1}\left\{\tilde{p}\frac{\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}-\lambda_f\lambda_f s\frac{\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}\right\} \\
=\;&\hat{\pi}_{t+1}\frac{\lambda_z y}{\lambda_f-1}\left\{\lambda_f s\frac{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\frac{\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}-\lambda_f\lambda_f s\frac{\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}\right\} \\
=\;&\hat{\pi}_{t+1}\frac{\lambda_z y\lambda_f s}{\lambda_f-1}\left\{\frac{\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}-\lambda_f\frac{\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}{1-\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}}\right\} \\
=\;&\hat{\pi}_{t+1}\frac{\lambda_z y\lambda_f s}{\lambda_f-1}\frac{\beta\xi_p}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\left\{\pi^{\frac{1}{\lambda_f-1}}-\lambda_f\pi^{\frac{\lambda_f}{\lambda_f-1}}\right\}
\end{aligned}$$

We conjecture that the derivative corresponding to π_{t+i} is:

$$\hat{\pi}_{t+i}\frac{\lambda_z y\lambda_f s}{\lambda_f-1}\frac{1}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\left\{\left(\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}\right)^i-\lambda_f\left(\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}\right)^i\right\}$$

Differentiation with respect to $\lambda_{z,t+j}$:

$$\left(\beta\xi_p\right)^j \lambda_z y \left(\bar{\pi}^{j\frac{\lambda_f}{\lambda_f-1}}\right) \left[\frac{\tilde{p}}{\bar{\pi}^j} - \lambda_f s\right] \hat{\lambda}_{z,t+j}$$

Differentiation with respect to y_{t+j} :

$$\left(\beta\xi_p\right)^j \lambda_z y \left(\bar{\pi}^{j\frac{\lambda_f}{\lambda_f-1}}\right) \left[\frac{\tilde{p}}{\bar{\pi}^j} - \lambda_f s\right] \hat{y}_{t+j}$$

Differentiation with respect to $\tilde{p}_t$:

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \lambda_z y \left(\bar{\pi}^j\right)^{\frac{1}{\lambda_f-1}} \tilde{p}\widehat{\tilde{p}}_t &= \frac{\lambda_z y \tilde{p}}{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}\widehat{\tilde{p}}_t \\
&= \frac{\lambda_z y}{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}\lambda_f s \frac{1-\beta\xi_p\bar{\pi}^{\frac{1}{\lambda_f-1}}}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\widehat{\tilde{p}}_t \\
&= \frac{\lambda_z y \lambda_f s}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\widehat{\tilde{p}}_t
\end{aligned}$$

The linearization of the firm's Euler equation is:

$$\begin{aligned}
\frac{\lambda_z y \lambda_f s}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\widehat{\tilde{p}}_t + \sum_{j=1}^{\infty}\hat{\pi}_{t+j}\frac{\lambda_z y \lambda_f s}{\lambda_f-1}\frac{1}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\left\{\left(\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}\right)^j - \lambda_f\left(\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}\right)^j\right\} \\
+\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \lambda_z y \left(\bar{\pi}^{j\frac{\lambda_f}{\lambda_f-1}}\right)\left[\frac{\tilde{p}}{\bar{\pi}^j} - \lambda_f s\right]\left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j}\right) \\
-\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \lambda_z y \left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}} \lambda_f s \hat{s}_{t+j} = 0
\end{aligned}$$

Solving this for $\widehat{\tilde{p}}_t$:

$$\begin{aligned}
\widehat{\tilde{p}}_t &= \frac{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}{\lambda_z y \lambda_f s}\{\sum_{j=1}^{\infty}\hat{\pi}_{t+j}\frac{\lambda_z y \lambda_f s}{\lambda_f-1}\frac{1}{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}\left[\lambda_f\left(\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}\right)^j - \left(\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}\right)^j\right] \\
&\quad + \sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \lambda_z y \left(\bar{\pi}^{j\frac{\lambda_f}{\lambda_f-1}}\right)\left[\lambda_f s - \frac{\tilde{p}}{\bar{\pi}^j}\right]\left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j}\right) + \sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \lambda_z y \left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}} \lambda_f s \hat{s}_{t+j}\}
\end{aligned}$$

or,

$$\begin{aligned}\widehat{\tilde{p}}_t &= \frac{1}{\lambda_f - 1}\sum_{j=1}^{\infty}\left[\lambda_f\left(\beta\xi_p\pi^{\frac{\lambda_f}{\lambda_f-1}}\right)^j - \left(\beta\xi_p\pi^{\frac{1}{\lambda_f-1}}\right)^j\right]\hat{\pi}_{t+j} \\ &+\frac{1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}}{\lambda_f s}\sum_{j=0}^{\infty}(\beta\xi_p)^j\left(\bar{\pi}^{j\frac{\lambda_f}{\lambda_f-1}}\right)\left[\lambda_f s - \frac{\tilde{p}}{\bar{\pi}^j}\right]\left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j}\right) \\ &+\left(1-\beta\xi_p\bar{\pi}^{\frac{\lambda_f}{\lambda_f-1}}\right)\lambda_z y\sum_{j=0}^{\infty}(\beta\xi_p)^j\left(\pi^j\right)^{\frac{\lambda_f}{\lambda_f-1}}\hat{s}_{t+j}\end{aligned}$$

The aggregate price index is:

$$P_t = \left[(1-\xi_p)\left(\tilde{P}_t\right)^{\frac{1}{1-\lambda_f}} + \xi_p\left(P_{t-1}\right)^{\frac{1}{1-\lambda_f}}\right]^{1-\lambda_f}.$$

Dividing by P_t :

$$1 = \left[(1-\xi_p)\left(\tilde{p}_t\right)^{\frac{1}{1-\lambda_f}} + \xi_p\left(\frac{1}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}\right]^{1-\lambda_f}. \tag{C.2}$$

Linearizing:

$$0 = \frac{1}{1-\lambda_f}(1-\xi_p)\left(\tilde{p}_t\right)^{\frac{1}{1-\lambda_f}-1}\tilde{p}\widehat{\tilde{p}}_t - \xi_p\frac{1}{1-\lambda_f}\left(\frac{1}{\pi_t}\right)^{\frac{1}{1-\lambda_f}-1}\frac{1}{\pi_t^2}\pi\hat{\pi}_t$$

Evaluating the derivatives in steady state,

$$0 = (1-\xi_p)\left(\tilde{p}\right)^{\frac{1}{1-\lambda_f}}\widehat{\tilde{p}}_t - \xi_p\left(\frac{1}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}\hat{\pi}_t$$

or,

$$\left(1-\xi_p\left(\frac{1}{\pi}\right)^{\frac{1}{1-\lambda_f}}\right)\widehat{\tilde{p}}_t = \xi_p\left(\frac{1}{\pi}\right)^{\frac{1}{1-\lambda_f}}\hat{\pi}_t,$$

or,

$$\widehat{\tilde{p}}_t = \frac{\xi_p\pi^{\frac{1}{\lambda_f-1}}}{1-\xi_p\pi^{\frac{1}{\lambda_f-1}}}\hat{\pi}_t$$

Combining the previous expression with the linearized first order condition for $\widehat{\tilde{p}}_t$, we obtain:

$$\begin{aligned}
\frac{\xi_p \pi^{\frac{1}{\lambda_f - 1}}}{1 - \xi_p \pi^{\frac{1}{\lambda_f - 1}}} \hat{\pi}_t &= \frac{1}{\lambda_f - 1} \sum_{j=1}^{\infty} \left[\lambda_f \left(\beta \xi_p \pi^{\frac{\lambda_f}{\lambda_f - 1}} \right)^j - \left(\beta \xi_p \pi^{\frac{1}{\lambda_f - 1}} \right)^j \right] \hat{\pi}_{t+j} \\
&+ \frac{1 - \beta \xi_p \bar{\pi}^{\frac{\lambda_f}{\lambda_f - 1}}}{\lambda_f s} \sum_{j=0}^{\infty} (\beta \xi_p)^j \left(\bar{\pi}^{j \frac{\lambda_f}{\lambda_f - 1}} \right) \left[\lambda_f s - \frac{\tilde{p}}{\bar{\pi}^j} \right] \left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j} \right) \\
&+ \left(1 - \beta \xi_p \bar{\pi}^{\frac{\lambda_f}{\lambda_f - 1}} \right) \lambda_z y \sum_{j=0}^{\infty} (\beta \xi_p)^j \left(\pi^j \right)^{\frac{\lambda_f}{\lambda_f - 1}} \hat{s}_{t+j}
\end{aligned}$$

or,

$$\begin{aligned}
\hat{\pi}_t &= \frac{1}{\lambda_f - 1} \frac{1 - \xi_p \pi^{\frac{1}{\lambda_f - 1}}}{\xi_p \pi^{\frac{1}{\lambda_f - 1}}} \sum_{j=1}^{\infty} \left[\lambda_f \left(\beta \xi_p \pi^{\frac{\lambda_f}{\lambda_f - 1}} \right)^j - \left(\beta \xi_p \pi^{\frac{1}{\lambda_f - 1}} \right)^j \right] \hat{\pi}_{t+j} \\
&+ \frac{1 - \beta \xi_p \bar{\pi}^{\frac{\lambda_f}{\lambda_f - 1}}}{\lambda_f s} \frac{1 - \xi_p \pi^{\frac{1}{\lambda_f - 1}}}{\xi_p \pi^{\frac{1}{\lambda_f - 1}}} \sum_{j=0}^{\infty} \left[\left(\beta \xi_p \pi^{\frac{\lambda_f}{\lambda_f - 1}} \right)^j \lambda_f s - \left(\beta \xi_p \pi^{\frac{1}{\lambda_f - 1}} \right)^j \tilde{p} \right] \left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j} \right) \\
&+ \left(1 - \beta \xi_p \bar{\pi}^{\frac{\lambda_f}{\lambda_f - 1}} \right) \frac{1 - \xi_p \pi^{\frac{1}{\lambda_f - 1}}}{\xi_p \pi^{\frac{1}{\lambda_f - 1}}} \lambda_z y \sum_{j=0}^{\infty} (\beta \xi_p)^j \left(\pi^j \right)^{\frac{\lambda_f}{\lambda_f - 1}} \hat{s}_{t+j}
\end{aligned}$$

or, setting

$$\begin{aligned}
\chi &= \beta \xi_p \pi^{\frac{\lambda_f}{\lambda_f - 1}}, \gamma = \beta \xi_p \pi^{\frac{1}{\lambda_f - 1}}, \ \chi = \gamma / \pi \\
a &= \frac{1 - \xi_p \pi^{\frac{1}{\lambda_f - 1}}}{\xi_p \pi^{\frac{1}{\lambda_f - 1}}}, \ b = 1 - \beta \xi_p \bar{\pi}^{\frac{\lambda_f}{\lambda_f - 1}}
\end{aligned}$$

$$\begin{aligned}
\hat{\pi}_t &= \frac{1}{\lambda_f - 1} a \sum_{j=1}^{\infty} \left[\lambda_f \chi^j - \gamma^j \right] \hat{\pi}_{t+j} + ba \lambda_z y \sum_{j=0}^{\infty} \chi^j \hat{s}_{t+j} \\
&+ ba \sum_{j=0}^{\infty} \left[\chi^j - \gamma^j \frac{\tilde{p}}{\lambda_f s} \right] \left(\hat{\lambda}_{z,t+j} + \hat{y}_{t+j} \right).
\end{aligned}$$

Writing this in lag-operator notation:

$$\left[1-\frac{1}{\lambda_f-1}a\left(\frac{\lambda_f\chi L^{-1}}{1-\chi L^{-1}}-\frac{\gamma L^{-1}}{1-\gamma L^{-1}}\right)\right]\hat{\pi}_t \quad =$$
$$+\frac{ba\lambda_z y}{1-\chi L^{-1}}\hat{s}_t+ba\left[\frac{1}{1-\chi L^{-1}}-\frac{1}{1-\gamma L^{-1}}\frac{\tilde{p}}{\lambda_f s}\right]\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)$$

Multiply both sides by $\left(1-\chi L^{-1}\right)\left(1-\gamma L^{-1}\right):$

$$\left[\left(1-\chi L^{-1}\right)\left(1-\gamma L^{-1}\right)-\frac{1}{\lambda_f-1}a\left(\frac{\left(1-\gamma L^{-1}\right)\lambda_f\chi L^{-1}}{1}-\frac{\left(1-\chi L^{-1}\right)\gamma L^{-1}}{1}\right)\right]\hat{\pi}_t \quad =$$
$$+\frac{\left(1-\gamma L^{-1}\right)ba\lambda_z y}{1}\hat{s}_t+ba\left[\frac{\left(1-\gamma L^{-1}\right)}{1}-\frac{\left(1-\chi L^{-1}\right)}{1}\frac{\tilde{p}}{\lambda_f s}\right]\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)$$

$$\left[1-(\chi+\gamma)L^{-1}+\chi\gamma L^{-2}-\frac{1}{\lambda_f-1}a\left(\frac{\left(1-\gamma L^{-1}\right)\lambda_f\chi L^{-1}}{1}-\frac{\left(1-\chi L^{-1}\right)\gamma L^{-1}}{1}\right)\right]\hat{\pi}_t \quad =$$
$$+\frac{\left(1-\gamma L^{-1}\right)ba\lambda_z y}{1}\hat{s}_t+ba\left[\frac{\left(1-\gamma L^{-1}\right)}{1}-\frac{\left(1-\chi L^{-1}\right)}{1}\frac{\tilde{p}}{\lambda_f s}\right]\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)$$

or,

$$\hat{\pi}_t=\left[(\chi+\gamma)-\chi\gamma L^{-1}+\frac{1}{\lambda_f-1}a\left(\frac{\left(1-\gamma L^{-1}\right)\lambda_f\chi}{1}-\frac{\left(1-\chi L^{-1}\right)\gamma}{1}\right)\right]\hat{\pi}_{t+1}$$
$$+\frac{\left(1-\gamma L^{-1}\right)ba\lambda_z y}{1}\hat{s}_t+ba\left[\frac{\left(1-\gamma L^{-1}\right)}{1}-\frac{\left(1-\chi L^{-1}\right)}{1}\frac{\tilde{p}}{\lambda_f s}\right]\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)$$

$$\hat{\pi}_t=\left[(\chi+\gamma)+\frac{1}{\lambda_f-1}a\left(\lambda_f\chi-\gamma\right)\right]\hat{\pi}_{t+1}-\chi\gamma\left[1+a\right]\hat{\pi}_{t+2}$$
$$+ba\lambda_z y\left(\hat{s}_t-\gamma\hat{s}_{t+1}\right)+ba\left[\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)-\gamma\left(\hat{\lambda}_{z,t+1}+\hat{y}_{t+1}\right)\right]-\frac{\tilde{p}}{\lambda_f s}ba\left[\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)-\chi\left(\hat{\lambda}_{z,t+1}+\hat{y}_{t+1}\right)\right]$$

The final, reduced form representation of inflation is:

$$\hat{\pi}_t=\gamma\left[(\frac{1}{\pi}+1)+\frac{\frac{\lambda_f}{\pi}-1}{\lambda_f-1}a\right]\hat{\pi}_{t+1}-\frac{\gamma\gamma}{\pi}\left[1+a\right]\hat{\pi}_{t+2}$$
$$+ba\lambda_z y\left(\hat{s}_t-\gamma\hat{s}_{t+1}\right)+ba\left[\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)-\gamma\left(\hat{\lambda}_{z,t+1}+\hat{y}_{t+1}\right)\right]-\frac{\tilde{p}}{\lambda_f s}ba\left[\left(\hat{\lambda}_{z,t}+\hat{y}_t\right)-\frac{\gamma}{\pi}\left(\hat{\lambda}_{z,t+1}+\hat{y}_{t+1}\right)\right]$$

D. Appendix 4: An Overlapping Generations Example

Here we work on a very simple dynamic economy to try and get a handle on the nature of the optimal contract when there is aggregate uncertainty.

D.1. Households

Let the utility of the generation born in period t be given by:

$$u(c_t^y, l_t, c_t^o),$$

where c_t^y and c_t^o denote consumption when young and old, respectively, and l_t denotes hours worked. People only work when young. The budget constraint when young is

$$c_t^y + s_t = w_t l_t,$$

where s_t denotes saving. The budget constraint when old is:

$$c_t^o = (1 + R_t) s_t,$$

where R_t denotes the rate of return, from t to $t+1$ on savings in period t.

D.2. Capital Producers

Capital producers buy the outstanding stock of capital in period t, $(1-\delta)K_t$, and investment goods, I_t, and produce new capital, K_{t+1}, using the following capital accumulation equation:

$$K_{t+1} = (1 - \delta)K_t + I_t.$$

They make zero profits. The price of capital is unity in each date. This reflects the linearity of the capital accumulation technology and that the marginal rate of technical substitution between investment goods and consumption goods is unity.

D.3. Entrepreneurs

Entrepreneurs at the end of period t purchase the outstanding stock of capital, K_{t+1}, using net worth, N_{t+1}, and loans. When an entreneur purchases K_{t+1}, ωK_{t+1} is available for use, $\omega \geq 0$. Here, ω is iid across entrepreneurs and over time, with distribution function, $F(\omega)$, and mean unity. The entrepreneurs rent their capital in a homogeneous capital market in period $t+1$, where they earn $r_{t+1}\omega K_{t+1}$. At the end of period $t+1$, they sell the undepreciated component of their capital, $(1-\delta)\omega K_{t+1}$, to the capital producers, and pay back their loans.

The rate of return on capital enjoyed by the entrepreneurs of type ω is $(1+R_t^k)\omega$ where:

$$1 + R_t^k = r_t + 1 - \delta,$$

where we have imposed that the price of capital is always unity. Later, we will see that in any equilibrium, $r_t = \theta_t$, so that

$$1 + R_t^k = \theta_t + 1 - \delta$$

D.4. Lending Contract with Entrepreneur

There are banks which take the savings of households and lend them to entrepreneurs. We adopt the usual CSV setup. So, zero profits for the representative bank in period t is:

$$[1 - F(\bar{\omega}_t)]\,\bar{\omega}_t + (1-\mu)\int_0^{\bar{\omega}_t} \omega dF(\omega) = \frac{(1+R_t)s_t}{\theta_t + 1 - \delta}$$

The bank takes R_t and θ_t as given and chooses s_t and $\bar{\omega}_t$.

References

[1] Basu, Susanto, and John Fernald, 1994, Constant returns and small markups in US manufacturing, Working paper, International Finance Discussion Paper No. 483, Board of Governors of the Federal Reserve System.

[2] Benninga, Simon, and Aris Protopapadakis, 1990, 'Leverage, time preference, and the 'equity premium puzzle,' *Journal of Monetary Economics*, vol. 25, pp. 49–58.

[3] Bernanke, Ben, Mark Gertler and Simon Gilchrist, 1999, 'The Financial Accelerator in a Quantitative Business Cycle Framework,' in Taylor and Woodford, editors, *Handbook of Macroeconomics*, North-Holland.

[4] Burnside, Craig and Martin Eichenbaum, 1996, 'Factor-Hoarding and the Propagation of Business Cycle Shocks', *American Economic Review*, 86(5), December, pages 1154-74.

[5] Carlstrom, Chuck, and Timothy Fuerst, 1997, 'Agency Costs, Net Worth and Business Fluctuations: A Computable General Equilibrium Analysis,' *American Economic Review*, December, 87(5), 893-910.

[6] Carlstrom, Chuck, and Timothy Fuerst, 2000, 'Monetary Shocks, Agency Costs and Business Cycles,' Federal Reserve Bank of Cleveland Working Paper 00-11.

[7] Calvo, Guillermo, 1983, 'Staggered Prices and in a Utility-Maximizing Framework, *Journal of Monetary Economics*, 12(3): 383-98.

[8] Casares, Miguel and Bennett T. McCallum, 2000, 'An Optimizing IS-LM Framework with Endogenous Investment,' National Bureau of Economic Research Working Paper #7908.

[9] Chari, V.V., Christiano, Lawrence J. and Martin Eichenbaum, 1995, 'Inside Money, Outside Money and Short Term Interest Rates', *Journal of Money, Credit and Banking*, 27(4), Part 2 Nov., pages 1354-86.

[10] Christiano, Lawrence J., 2002, 'Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients,' forthcoming, *Computational Economics.*

[11] Christiano, Lawrence J., Eichenbaum, Martin and Charles Evans, 1997, 'Sticky Price and Limited Participation Models: A Comparison', *European Economic Review*,Vol. 41, No. 6, June, pp. 1173-1200.

[12] Christiano, Eichenbaum and Evans, 2001, 'Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy,' National Bureau of Economic Research Working paper.

[13] Erceg, Christopher, J., Henderson, Dale, W. and Andrew T. Levin, 2000, 'Optimal Monetary Policy with Staggered Wage and Price Contracts', *Journal of Monetary Economics*, 46(2), October, pages 281-313.

[14] Fisher, Jonas, 1996, 'Credit Market Imperfections and the Heterogeneous Response of Firms to Monetary Shocks,' Federal Reserve Bank of Chicago Working Paper wp-96-23.

[15] Fuhrer, Jeffrey, 2000, 'Habit Formation in Consumption and Its Implications for Monetary-Policy Models', *American Economic Review*, 90(3), June, pages 367-90.

[16] Gale, Douglas, and Martin Hellwig, 1985, 'Incentive-Compatible Debt Contracts I: The One-Period Problem,' *Review of Economic Studies* 52 (4), pp. 647-64.

[17] Lucas, Robert E., Jr., 1993, 'On the Welfare Costs of Inflation', unpublished manuscript.

[18] Townsend, Robert, 1979, 'Optimal Contracts and Competitive Markets with Costly State Verification,' *Journal of Economic Theory* 21 (2), pp. 265-93.

[19] Williamson, Stephen, 1987, 'Financial Intermediation, Business Failures and Real Business Cycles,' *Journal of Political Economy* 95 (6), pp. 1196-1216.

[20] Woodford, Michael, 1994, 'Monetary Policy and Price Level Determinacy in a Cash-In-Advance Economy,' *Economic Theory,* 4, 345-389.

[21] Woodford, Michael, 1996, 'Control of the Public Debt: A Requirement for Price Stability,' NBER Working Paper 5684.