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416, Fall 2005
Homework 1, due Monday, October 3.

1. Consider a model in which preferences are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \gamma_t \log(1 - l_t) \right],$$

where $\gamma_t \geq 0$ is a random variable with the following representation:

$$\begin{aligned} \gamma_t &= \exp(x_t), \\ x_t &= (1 - \rho_x) x + \rho_x x_t + \varepsilon_{x,t}, \end{aligned}$$

where $x = Ex_t$ and $\varepsilon_{x,t}$ is mean zero, iid with $E\varepsilon_{x,t}^2 = \sigma_x^2$. The resource constraint is:

$$c_t + k_{t+1} \leq k_t^\alpha (z_t l_t)^{1-\alpha} = y_t,$$

where the technology shock, z_t, has the following representation:

$$\begin{aligned} \eta_t &= \log(z_t) - \log(z_{t-1}) \\ \eta_t &= (1 - \rho_\eta) \eta + \rho_\eta \eta_{t-1} + \varepsilon_{\eta,t}, \end{aligned}$$

where $\eta = E\eta_t$ and $\varepsilon_{\eta,t}$ is mean zero, iid with $E\varepsilon_{\eta,t}^2 = \sigma_\eta^2$. Set

$$\beta = 1.03^{-0.25}, \ \rho_\eta = 0, \ \sigma_\eta^2 = 0.01^2, \ \alpha = 1/3, \ \eta = 1 + 0.015/4$$

$\rho_\eta = 0,$

(a) Show that the efficient allocations satisfy:

$$\begin{aligned} k_{t+1} &= \alpha\beta k_t^\alpha (z_t l_t)^{1-\alpha} \\ c_t &= (1 - \alpha\beta) k_t^\alpha (z_t l_t)^{1-\alpha} \\ l_t &= \frac{1 - \alpha}{1 - \alpha + \gamma_t (1 - \beta\alpha)} \end{aligned}$$

Note that l_t does not respond to the technology shock, $\varepsilon_{\eta,t}$.

(b) Drawing $\varepsilon_{\eta,t}$ from a random number generator (randn in MATLAB will do), generate 10,000 artificial observations on output. Compute the variance of the growth rate of output. Repeatedly do the same thing, generating data using both $\varepsilon_{\eta,t}$ and $\varepsilon_{x,t}$. Each time, try a different value of σ_x^2 until the variance of the growth rate of output when there are only technology shocks is 70 percent of the variance of the growth rate output with both shocks. Use this value of σ_x^2 for the remainder of this question.

(c) Generate 1000 artificial data sets, each of length 150 observations, on

$$Y_t = \left[\Delta \log\left(y_t/l_t\right), \log\left(l_t\right)\right].$$

In each artificial data set, fit a four-lag VAR, and estimate the first four responses of $\log(l_t)$ to a technology shock using only the long-run restriction for identification.

(d) Graph the mean estimated responses across the 1,000 data sets. In addition, for each lag sort (use the sort command in MATLAB) the responses from smallest to largest and store the 25^{th} and 975^{th} responses in the ordering.

How well does the VAR do at recovering the response of hours worked to a technology shock? (Note that the true response is zero.)

(e) What happens to your conclusion in (d) if you drive ρ_x very close to unity (always adjusting σ_x^2 so that the variance of x_t remains constant).

2. Consider the time series representation for x_t in question 1. Note that the variance of x_t is:

$$E\left(x_t - x\right)^2 = \frac{\sigma_x^2}{1-\rho_x^2},$$

and the τ^{th} lag covariance is

$$C\left(\tau\right) = E\left(x_t - x\right)\left(x_{t-\tau} - x\right) = \rho_x^{|\tau|} E\left(x_t - x\right)^2, \ \tau = 0, \pm 1, \pm 2, \ldots .$$

3. Note that the spectral density of x_t is:

$$S_x\left(e^{-i\omega}\right) = \frac{\sigma_x^2}{\left(1-\rho_x e^{-i\omega}\right)\left(1-\rho_x e^{i\omega}\right)},$$

and recall result #3 in the handout:

$$C\left(\tau\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x\left(e^{-i\omega}\right)e^{i\omega\tau}d\omega.$$

Approximate this integral by the sum of the area of N rectangles, based on the Riemann sum interpretation of the integral. That is, if $\omega_j = 2\pi j/N$, with $j = -(N/2)+1,...,N/2$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_x\left(e^{-i\omega}\right)e^{i\omega\tau}d\omega \approx \frac{1}{2\pi}\sum_{j=-(N/2)+1}^{N/2} S_x\left(e^{-i\omega_j}\right)e^{i\omega_j\tau}\left(\omega_j-\omega_{j-1}\right)$$

Evaluate $C\left(0\right),\ C\left(2\right),\ C\left(4\right)$ using the exact, time domain formula and the Riemann approximation for various values of N. What is the smallest values of N for which the Riemann approximation is good?

4. Suppose the data vector in a VAR is:

$$Y_t = \left[\Delta\log p_t^I, \Delta\log\left(y_t/l_t\right), \log\left(l_t\right)\right].$$

where p_t^I is the price of investment goods. Suppose there are two shocks to technology that have a long-run impact on labor productivity: a neutral technology shock like the one analyzed above, and an investment-specific shock. No other shocks have an impact on labor productivity in the long-run. The investment specific shock differs from the neutral shock in that it is the only shock that has a permanent impact on the level of $\log p_t^I$ in the long run. Describe a suitable adaptation to the identification strategy discussed in lecture, that will allow us to identify both technology shocks.