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416, Fall 2005
Homework 5, due Thursday, November 10

1. Consider the model of human capital analyzed in class last year. There are two sectors. One produces a homogeneous output good, which is transformed one-for-one into consumption and investment using a Cobb-Douglas production function:

$$c_t + k_{t+1} - (1-\delta)\,k_t = k_t^{\alpha} n_t^{1-\alpha}.$$

Another sector produces human capital according to the following accumulation equation:

$$h_{t+1} = h_t + \lambda\,(h_t - n_t)\,,$$

where $\lambda > 0$, $c_t \geq 0$, $k_{t+1} \geq (1-\delta)\,k_t$, $0 \leq n_t \leq h_t$, and h_0, k_0 are given. Preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

$\gamma > 0$. To ensure boundedness, we require $\beta\,(1+\lambda)^{1-\gamma} < 1$. In class, the problem was reformulated in recursive form:

$$v\,(x) = \max_{(x',y)\in\Gamma(x)} u(x,x',y) + \beta y^{1-\gamma} v\,(x')\,,$$

where $x = k/h$, $y = h'/h$ and

$$u(x,x',y) = x^{\alpha}\left(\frac{1}{\lambda}\right)^{1-\alpha}[(1+\lambda)-y]^{1-\alpha} + (1-\delta)\,x - x'y$$

$$\Gamma\,(x) = \{x', y : 1 \leq y \leq 1+\lambda,$$

$$\left(x^{\alpha}\left(\frac{1}{\lambda}\right)^{1-\alpha}[(1+\lambda)-y]^{1-\alpha} + (1-\delta)\,x\right)/y \geq x' \geq (1-\delta)\,x/y\}$$

It was shown that there are policy rules of the form, $x' = f\,(x)\,,\ y = g\,(x)\,,$ where $x = k/h$ and $y = h'/h$.

- Set $\alpha = 1/3$, $\delta = 0.10$, $\beta = 0.97$, $\lambda = 0.04$, $\gamma = 1.1$. Compute steady state values of x, y : $\bar{x}$, $\bar{y}$.

- There are four equilibrium conditions that can be used to determine the equilibrium functions, f, g, v' and v. These are the fixed point of the dynamic programming problem, the two first order conditions and the envelope condition. The envelope condition can be eliminated by using it to substitute out for v' in one of the Euler conditions. Write out the resulting three equilibrium conditions in the three unknown functions, f, g, and v.
- The functional equations described in the previous question can be 'perturbed' to obtain the Taylor series expansion of the policy rules about steady state:

$$
\begin{aligned}
x' &= \bar{x} + f'(\bar{x})(x - \bar{x}) \\
y &= \bar{y} + g'(\bar{x})(x - \bar{x}).
\end{aligned}
$$

  Compute $f'(\bar{x})$ and $g'(\bar{x})$ by differentiating the three functional equations with respect to x and evaluating the result at $x = \bar{x}$.
- Consider two economies. Both have the same initial stock of human capital, $h_0 = 1$. One economy has an amount of physical capital, $k_0 = \bar{x}h_0$. The other economy has an amount of physical capital that is 10 percent below the level of the first economy.
  - Use the approximate policy rules to compute sequences, $h_1, ..., h_{10}$, $k_1, k_2, ..., k_{10}$ for the two economies.
  - Do the two economies eventually converge to the same physical to human capital ratio?
  - Compute the one-period rate of return on physical and human capital for the two economies for dates 1, 2, ..., 10 (hint: the period t rate of return on an asset is the ratio of two quantities. In the numerator, there is the period $t+1$ payoff of the asset, which is defined as the maximum that consumption can be increased in period $t+1$ with a one-unit increase in the asset, while leaving consumption opportunities in periods $t+2$ and on unchanged. In the denominator, there is the period t consumption price of the asset.) Use the time pattern in these rates of return to provide an economic analysis of the differences in the accumulation paths for the two economies.

2. This question will get you started with symbolic algebra in MATLAB. The relevant tutorials in MATLAB are reasonably good. At the MATLAB prompt, type 'doc'. That will bring up the help menu. Select the 'search' tab, and type symbolic in the 'search for' window. Work through the 'symbolic objects' file, in 'getting started' (you have to scroll down a bit to get to the symbolic math toolbox. After working through several of these files, and you start to feel comfortable with symbolic math in MATLAB, select the 'contents' tab and go to 'using the symbolic math toolbox'. Read through the section on calculus, in particular, the section on Taylor series and the extended calculus example. Now, you're in a position to do this homework question, using symbolic algebra in MATLAB.

   Consider the following function:

$$f(x) = \frac{1}{5 + 4\cos(x)}, \; x \in (-2\pi, 2\pi).$$

   (a) Enter this function as a symbolic object, and then compute a high order Taylor series expansion around $x = 0$. Graph the Taylor series expansion of the function, as well as the function itself. How high does the Taylor series expansion have to go in order to start capturing the curvature in the two 'humps' in this function.

   (b) Now, fit an n^{th} order polynomial to this function, by choosing $n + 1$ points in the domain of the function and selecting the $n + 1$ parameters of the Taylor series expansion so that the polynomial coincides with $f(x)$ at the points. How does the polynomial approximation compare with the approximation based on the Taylor series expansion?