Notes on Ramsey-Optimal Monetary Policy

Lawrence J. Christiano, Roberto Motto and Massimo Rostagno

January 9, 2007

Contents

These notes describe a set of monetary models which have been coded into Dynare, and which can be solved for the optimal monetary policy using code recently written for use in Levin, Lopez-Salido, (2004) and Levin, Onatski, Williams and Williams (2005) (LL-SLOWW). The first section below describes the logic of the algorithm. The sequence of models goes from the least complicated to the most complicated, a version of the monetary model in Christiano, Eichenbaum and Evans (2004). In each case, the code is provided in a zip file available on the website where this document is posted.

1. Ramsey-Optimal Policy

Let x_t denote a set of N endogenous variables in a dynamic economic model. Let the private sector equilibrium conditions be represented by the following $N-1$ conditions:

$$\sum_{s^{t+1}|s^t} \frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)} f\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right) = 0, \tag{1.1}$$

for all t and all s^t. Here, s^t denotes a history:

$$s^t = \left(s_0, s_1, ..., s_t\right),$$

and s_t denotes the time t realization of uncertainty, which can take on n possible values:

$$\begin{aligned} s_t &\in \left\{s\left(1\right), ..., s\left(n\right)\right\} \\ \mu\left(s^t\right) &= \text{prob}[s^t], \end{aligned}$$

so that $\mu\left(s^{t+1}\right)/\mu\left(s^t\right)$ is the probability of history s^{t+1}, conditional on s^t. This economy is not ‘closed’ because there are fewer equations than unknowns. One way to close it would be to add an equation which characterizes policy, perhaps a Taylor rule. Instead, we consider the Ramsey optimal equilibrium.

Suppose preferences over $x\left(s^t\right)$ are follows:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) U\left(x\left(s^t\right), s_t\right). \tag{1.2}$$

The Ramsey problem is to maximize preference by choice of $x\left(s^t\right)$ for each s^t, subject to (1.1). We express the Ramsey problem in Lagrangian form as follows:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu\left(s^t\right) \left\{ U\left(x\left(s^t\right), s_t\right) + \underbrace{\lambda\left(s^t\right)}_{1\times N-1} \sum_{s^{t+1}|s^t} \frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)} \underbrace{f\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right)}_{N-1\times 1} \right\},$$

where $\lambda\left(s^t\right)$ is the row vector of multipliers on the equilibrium conditions. Consider a particular history, $s^t = \left(s^{t-1}, s_t\right)$, with $t > 0$. The first order necessary condition for optimality of $x\left(s^t\right)$ is

$$\begin{aligned} &\underbrace{U_1\left(x\left(s^t\right), s_t\right)}_{1\times N} + \underbrace{\lambda\left(s^t\right)}_{1\times N-1} \sum_{s^{t+1}|s^t} \frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)} \underbrace{f_1\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right)}_{N-1\times N} \\ &\qquad + \beta^{-1} \underbrace{\lambda\left(s^{t-1}\right)}_{1\times N-1} \underbrace{f_2\left(x\left(s^{t-1}\right), x\left(s^t\right), s_{t-1}, s_t\right)}_{N-1\times N} = \underbrace{0}_{1\times N} \end{aligned} \tag{1.3}$$

after dividing by $\mu(s^t)\beta^t$. In more conventional notation,

$$U_1(x_t, s_t) + \lambda_t E_t f_1(x_t, x_{t+1}, s_t, s_{t+1}) + \beta^{-1}\lambda_{t-1} f_2(x_{t-1}, x_t, s_{t-1}, s_t) = 0.$$

The first order necessary condition for optimality at $t = 0$ is (1.3) with $\lambda_{-1} \equiv 0$.

The equations that characterize the Ramsey equilibrium are the $N-1$ equations, (1.1), and the N equations (1.3). The unknowns are the N elements of x and the $N-1$ multipliers, λ. We will solve these equations by first or second order perturbation using Dynare.

To apply the perturbation method, we require the nonstochastic steady state value of x. We compute this in two steps. First, fix one of the elements of x, say the inflation rate, π. We then solve for the remaining $N-1$ elements of x by imposing the $N-1$ equations, (1.1). In the next step we compute the $N-1$ vector of multipliers using the steady state version of (1.3):

$$U_1 + \lambda\left[f_1 + \beta^{-1} f_2\right] = 0,$$

where a function without an explicit argument is understood to mean it is evaluated in steady state. Write

$$\begin{aligned} Y &= U_1' \\ X &= \left[f_1 + \beta^{-1} f_2\right]' \\ \beta &= \lambda', \end{aligned}$$

so that Y is an $N \times 1$ column vector, X is an $N \times (N-1)$ matrix and β is an $(N-1) \times 1$ column vector. Compute β and u as

$$\begin{aligned} \beta &= (X'X)^{-1} X'Y \\ u &= Y - X\beta. \end{aligned}$$

Note that this regression will not in general fit perfectly, because there are $N-1$ 'explanatory variables' and N elements of Y to 'explain'. We vary the value of π until $\max|u_i| = 0$. This completes the discussion of the calculation of the steady state.

Equations (1.1) and (1.3) form a system of dynamic equations in the endogenous variables, $x(s^t)$ and $\lambda(s^t)$. Dynare can approximate the solution to these equations using first or second order perturbations about the nonstochastic steady state. To do this, one provides a Dynare-formated code with the equilibrium conditions, (1.1), and with the utility function and discount rate in (1.2). The code written by (LLSLOWW) takes this as input, computes the equations in (1.3) symbolically and sets up (1.1) and (1.3) as a new set of Dynare-formated code. Dynare can be applied to the result.

2. Rotemberg-Sticky Prices

We describe the agents in the model, and then summarize the equilibrium conditions. The example is sufficiently simple that (1.3) can be computed by hand, and the law of motion of the multipliers can be established anlytically.[1]

[1] We are grateful to Ippei Fujiwara for suggesting this example to us.

2.1. Household

Household i maximizes discounted utility, where the period utility function is:

$$\log\left(C_{i,t}\right) - \frac{\chi}{2}h_{i,t}^2.$$

The budget constraint is:

$$\frac{B_{i,t}}{P_t} = \left(1 + R_{t-1}\right)\frac{B_{i,t-1}}{P_t} - C_{i,t} + \frac{W_t}{P_t}h_{i,t} + \Pi_{i,t},$$

where $\Pi_{i,t}$ denotes lump-sum profits and taxes. The first order necessary conditions for household optimality are:

$$\chi h_t C_t = \frac{W_t}{P_t}, \tag{2.1}$$

and

$$\frac{1}{1+R_t} = \beta E_t \frac{P_t C_t}{P_{t+1}C_{t+1}}, \tag{2.2}$$

for $t = 0, 1, 2, ...$.

2.2. Firms

Firm j maximizes profit:

$$\left(1+\tau\right)\frac{P_{j,t}}{P_t}C_{j,t} - MC_t \times C_{j,t} - \frac{\phi}{2}\left(\frac{P_{j,t}}{P_{j,t-1}} - 1\right)^2 C_t. \tag{2.3}$$

The first term is firm revenues, including a tax subsidy, τ, received from the government (this is financed by a lump-sum tax on the household). The term after the first minus sign corresponds to the labor costs incurred in producing $C_{j,t}$. We assume that to produce 1 unit of $C_{j,t}$, $\exp\left(-Z_t\right)$ units of labor are required, where Z_t denotes a shock to technology. Thus

$$MC_t = \frac{W_t}{P_t \exp\left(Z_t\right)} = \frac{\chi h_t C_t}{\exp\left(Z_t\right)},$$

after substituting out for the real wage using (2.1). The term after the second minus sign in (2.3) is the quantity of of the final good lost when the firm chooses to adjust its prices. The quantity of goods lost is positively related to the aggregate level of output, C_t. Firm j faces the following demand curve:

$$C_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\theta} C_t, \ \theta \geq 1.$$

The Lagrangian representation of the firm's problem is:

$$\max_{\{P_{j,t+n}\}_{n=0}^{\infty}} E_t \sum_{n=0}^{\infty} \beta^n \frac{C_t}{C_{t+n}} [\left(1+\tau\right)\left(\frac{P_{j,t+n}}{P_{t+n}}\right)^{1-\theta} C_{t+n}$$
$$-MC_{t+n} \times \left(\frac{P_{j,t+n}}{P_{t+n}}\right)^{-\theta} C_{t+n} - \frac{\phi}{2}\left(\frac{P_{j,t+n}}{P_{j,t-1+n}} - 1\right)^2 C_{t+n}],$$

where $\beta^n C_t/C_{t+n}$ represents the state-contingent value, to households, of profits. This is taken as exogenous, by the firm. The first order necessary condition associated with the optimal choice of the price level is:

$$
\begin{aligned}
(1-\theta)(1+\tau)\left(\frac{P_{j,t}}{P_t}\right)^{-\theta}\frac{C_t}{P_t}+\theta\times MC_t\times\left(\frac{P_{j,t}}{P_t}\right)^{-\theta-1}\frac{C_t}{P_t}-\phi\left(\frac{P_{j,t}}{P_{j,t-1}}-1\right)\frac{C_t}{P_{j,t-1}}& \\
+\beta E_t\frac{C_t}{C_{t+1}}\phi\left(\frac{P_{j,t+1}}{P_{j,t}}-1\right)\frac{P_{j,t+1}}{P_{j,t}^2}C_{t+1} &= 0.
\end{aligned}
$$

In a symmetric equilibrium, $P_{j,t}=P_t$, for all j, so that the efficiency condition associated with firms is:

$$\left[\tau-\frac{1}{(\theta-1)}\right](1-\theta)+\theta\left(MC_t-1\right)-\phi\left(\pi_t-1\right)\pi_t+\beta E_t\phi\left(\pi_{t+1}-1\right)\pi_{t+1}=0.$$

2.3. Equilibrium Conditions

The equilibrium conditions of the model are the household's intertemporal Euler equation,

$$\frac{1}{1+R_t}=\beta E_t\frac{C_t}{\pi_{t+1}C_{t+1}}, \tag{2.4}$$

the firm's efficiency condition:

$$\left[\tau-\frac{1}{(\theta-1)}\right](1-\theta)+\theta\left(\frac{\chi h_t C_t}{\exp\left(Z_t\right)}-1\right)=\phi\left(\pi_t-1\right)\pi_t-\beta E_t\phi\left(\pi_{t+1}-1\right)\pi_{t+1}, \tag{2.5}$$

and the resource constraint:

$$C_t\left[1+\frac{\phi}{2}\left(\pi_t-1\right)^2\right]=\exp\left(Z_t\right)h_t. \tag{2.6}$$

According to the latter, final goods are partly consumed by households, and partly they are used up in adjustment costs, if prices are being adjusted, i.e., if $\pi_t\neq 1$. The law of motion for the exogenous shock is:

$$
\begin{aligned}
Z_t &= \rho Z_{t-1}+u_t, \\
u_t &\sim N\left(0,\sigma_u\right).
\end{aligned} \tag{2.7}
$$

In the case where monetary policy is exogenous, we specify the following Taylor rule:

$$R_t=\frac{\pi^*}{\beta}-1+\alpha(\pi_t-\pi^*), \tag{2.8}$$

where π^* is the target inflation rate. According to this, when inflation is above target, $\pi_t>\pi^*$, they raise the nominal rate of interest above what the rate of interest would be expected to be at the target inflation rate and in steady state, $\pi^*/\beta-1$.

2.4. Ramsey Policy

We have three private sector equilibrium conditions and four endogenous variables, C_t, π_t, h_t and R_t. Absent a monetary policy rule, we do not have enough relations to determine all three variables. The Ramsey optimum is the value $\{C_t, \pi_t, h_t, R_t\}$ that maximizes household utility, subject to the three equilibrium conditions. It is perhaps obvious what the optimum is. Note that if we set

$$\tau = \frac{1}{\theta - 1},\ \pi_t = 1,\ \chi h_t^2 = 1,\ C_t = \exp\left(Z_t\right) h_t,$$

then (2.5) and (2.6) are satisfied. If we let the intertemporal Euler equation define the nominal rate of interest, R_t, then (2.4) is satisfied too. This is the Ramsey equilibrium because this setting of h_t and C_t solves the planning problem: maximize discounted utility subject to the version of the technology in which there are no losses to price adjustment. With these preferences and technology, you cannot do better than to set $\chi_t h_t^2 = 1$. Note that $\pi_t = 1$, $\chi h_t^2 = 1$ is the Ramsey equilibrium only if $\tau = 1/(\theta - 1)$. If, for example, $\tau = 0$, then (2.5) indicates there must be some deviation from $\pi_t = 1$, $\chi h_t^2 = 1$, given firm profit maximization.

The Lagrangian representation of the Ramsey problem is:

$$\max_{\{R_t, h_t, \pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_n^t \{\log\left(C_t\right) - \frac{\chi}{2} h_t^2 + \lambda_{1,t}\left[\frac{1}{1+R_t} - \beta E_t \frac{C_t}{\pi_{t+1} C_{t+1}}\right]$$
$$+\lambda_{2,t}\left[\tau\left(1-\theta\right) + 1 + \theta\left(\frac{\chi h_t C_t}{\exp\left(Z_t\right)} - 1\right) - \phi\left(\pi_t - 1\right)\pi_t + \beta\phi E_t\left(\pi_{t+1} - 1\right)\pi_{t+1}\right]$$
$$+\lambda_{3,t}\left(\exp\left(Z_t\right) h_t - C_t\left[1 + \frac{\phi}{2}\left(\pi_t - 1\right)^2\right]\right)\},$$

where β_n is the planner's discount rate, which may in principle be different from β. In principle, we should also add as an additional constraint, $R_t \geq 1$. However, ignore that in the hope that it is non-binding.

The first order necessary condition associated with R_t is:

$$\lambda_{1,t} \frac{1}{\left(1+R_t\right)^2} = 0,\ t = 0, 1, 2, ... \tag{2.9}$$

From this it is evident that

$$\lambda_{1,t} = 0,$$

for all t. We simplify the derivatives by imposing this from here on. The first order necessary condition associated with h_t is, for $t = 1, 2, ...,$

$$-\chi_t h_t + \lambda_{2,t}\theta \frac{\chi C_t}{\exp\left(Z_t\right)} + \lambda_{3,t}\exp\left(Z_t\right) \quad = \quad 0 \tag{2.10}$$

The first order condition associated with π_t is:

$$\lambda_{2,t}\left(1 - 2\pi_t\right)\phi + \beta_n^{-1}\lambda_{2,t-1}\beta\phi\left(2\pi_t - 1\right) - \lambda_{3,t} C_t \phi\left(\pi_t - 1\right) = 0. \tag{2.11}$$

with the understanding, $\lambda_{2,-1} \equiv 0$.

Finally, the first order condition with respect to C_t is

$$\frac{1}{C_t} + \lambda_{2,t}\theta \frac{\chi h_t}{\exp(Z_t)} - \lambda_{3,t}\left[1 + \frac{\phi}{2}(\pi_t - 1)^2\right] = 0 \tag{2.12}$$

The equations that characterize the Ramsey equilibrium are the 7, (2.4)-(**??**), (2.10)-(2.12). There are 7 unknowns, $\lambda_{2,t}$, $\lambda_{3,t}$ and $x_t = (C_t, \pi_t, h_t, R_t, Z_t)$, for $t = 0, 1, 2, ...$. The required initial conditions are $\lambda_{2,-1} = 0$ and Z_{-1}. We solve these equations by linearizing around steady state. This requires first computing the steady state of the Ramsey equilibrium, and then linearizing those equations about the steady state.

We compute the steady state of the Ramsey problem by first fixing an arbitrary value for π. Then, we solve for the remaining 4 elements of x using the four equations, (2.4)-(**??**). Solving (2.4) for steady state R :

$$1 + R = \frac{\pi}{\beta}.$$

Solving (2.5) and (2.6) for h:

$$h = \left\{\frac{1}{\theta\chi}\left[1 + \frac{\phi}{2}(\pi - 1)^2\right]\left[\phi(1-\beta)(\pi - 1)\pi + (\theta - 1)(1+\tau)\right]\right\}^{\frac{1}{2}}. \tag{2.13}$$

Given h, C can be recovered from (2.6). Also (2.7) can be solved for Z. Thus, we have solved for x as a function of π :

$$x(\pi).$$

We now use two of the three equations, (2.10)-(2.12), to solve for the two multipliers, λ_2, $\lambda_{3.}$. We adjust π until values for the two multipliers can be found which set these three equations to zero in steady state. In case the model we're working with is actually the one with exogenous monetary policy, (2.8), then the steady state inflation rate is just the target rate, π^*.

2.5. Numerical Examples

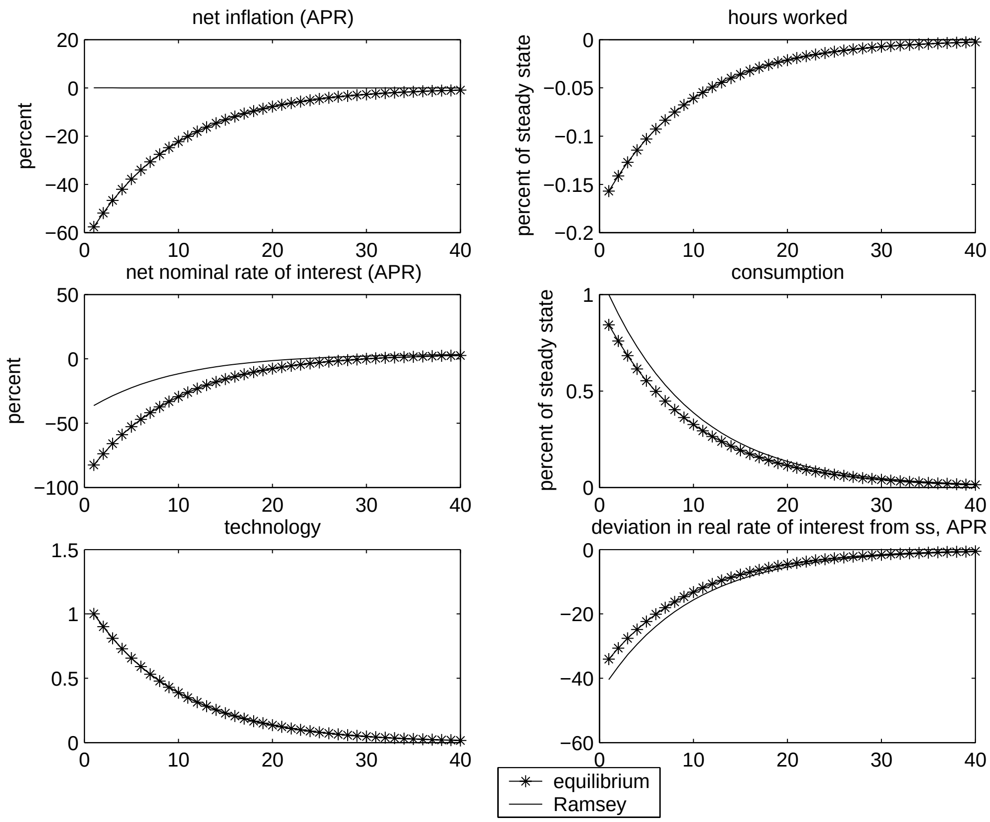
We computed some examples, to illustrate the calculations. We set

$$\beta = 0.99,\ \theta = 5,\ \phi = 100,\ \rho = 0.9,\ \alpha = 1.5,\ \beta_n = 0.99,\ \tau = \frac{1}{\theta - 1},\ \chi = 1,\ \pi^* = 1.$$

As explained above, in this example the Ramsey optimal policy is

$$\pi_t = 1,\ \chi h_t^2 = 1,\ C_t = \exp(Z_t)\, h_t,$$

and the steady state values of h_t and C_t are both unity in both the Ramsey and exogenous monetary policy equilibrium. The following figure displays the path of the actual variables in the Ramsey and the exogenous monetary policy (‘equilibrium’) equilibria



The shock is a one percent jump in technology, which decays over time. Hours worked and consumption correspond to the percent deviation from steady state. In the Ramsey equilibrium, the percent deviation in hours is zero and the percent deviation in consumption corresponds exactly to what technology. The technology shock creates an expectation that current consumption is high and later consumption is lower. Other things the same, this creates an intertemporal smoothing motive, which makes people want to consume less in the current period and save for the future. The Ramsey equilibrium responds to this by reducing the interest rate by precisely the amount that is required to induce people to follow the Ramsey-optimal consumption path. In the example, the reduction is quite large, nearly 40 percent! That is, the steady state interest rate is 0.01, and the reduction in the first period is -0.10. So, the actual quarterly nominal rate of interest in the first period is -0.09. When multiplied by 400 to convert to annualized percent terms, this is the -36 percent that we see for Ramsey in the figure. (This example draws obvious attention to the fact that we ignore the non-negativity constraint on the nominal rate of interest.) The monetary policy rule evidently cuts the interest rate more than what is called for in the Ramsey equilibrium. However, expected inflation falls by a lot too. Consumption is determined by the real rate of interest. The figure indicates that in the exogenous monetary policy equilibrium, the real interest rate is cut by less than in the Ramsey. As a result, the consumption smoothing motive is not undercut by enough in the exogenous monetary policy equilibrium. In particular, they cut their consumption relative to the Ramsey optimum. This leads to a fall in demand for goods, which leads to a fall in employment and marginal costs. The fall in marginal costs induces firms to cut prices and so inflation falls relative to the Ramsey optimum.

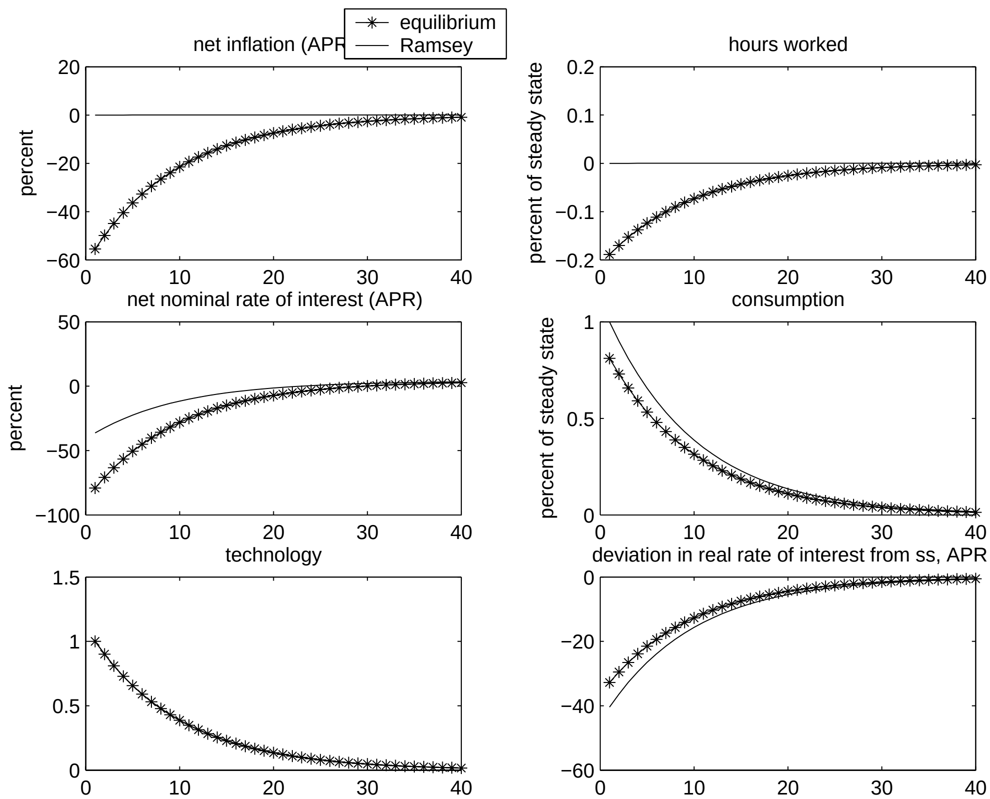
Next, we set $\tau = 0$ and redid the calculations. The steady state inflation rate in both equilibria is unity. Consumption and hours worked are both lower, now, at 0.8944. To see why, consider the firm efficiency condition for prices:

$$1 + \theta \left(h^2 - 1 \right) = \phi \left(\pi - 1 \right) \pi - \beta \phi \left(\pi - 1 \right) \pi.$$

The term on the right of the equality is zero. As a result,

$$h = \left[1 - \frac{1}{\theta} \right]^{\frac{1}{2}}, \; C_t = \exp \left(Z_t \right) h,$$

in the Ramsey equilibrium. So, in terms of percent deviations from steady state, the Ramsey equilibrium is (numerically) the same with $\tau = 0$ and $\tau = 1/\left(\theta - 1\right)$. In the exogenous monetary policy equilibrium, the drop in hours and consumption is a little bigger than what it was with the tax subsidy.



Interestingly, not only is $\lambda_{1,t} = 0$ for all t, but in the above numerical experiments, we found $\lambda_{2,t} = 0$ also. By contrast, $\lambda_{3,t}$ is non-zero. Interestingly, however, the lagged values of $\lambda_{3,t}$ do not appear in the state of the system, and so therefore the optimal plan is not time-inconsistent. This is true if τ is set to ensure an efficient steady state, or not.

3. Model with Calvo-Sticky Prices and No other Frictions

3.1. Firms

We adopt the usual assumption that a representative final good producer manufactures final output using the following linear homogenous technology:

$$Y_t = \left[\int_0^1 Y_{jt}{}^{\frac{1}{\lambda_t}} dj \right]^{\lambda_f}, \ 1 \leq \lambda_f < \infty, \tag{3.1}$$

Intermediate good j is produced by a price-setting monopolist according to the following technology:

$$Y_{jt} = \begin{cases} \epsilon_t K_{jt}^{\alpha} \left(z_t l_{jt} \right)^{1-\alpha} - \Phi z_t & \text{if } \epsilon_t K_{jt}^{\alpha} \left(z_t l_{jt} \right)^{1-\alpha} > \Phi z_t \\ 0, & \text{otherwise} \end{cases}, \ 0 < \alpha < 1, \tag{3.2}$$

where Φz_t is a fixed cost and K_{jt} and l_{jt} denote the services of capital and homogeneous labor. Capital and labor services are hired in competitive markets at nominal prices, $P_t r_t^k$, and W_t, respectively. The object, z_t, in (3.2), is assumed to evolve deterministically:

$$z_t = z_{t-1} \mu_z. \tag{3.3}$$

For now, we assume that $z_t = 1$, constant, with $\mu_z = 1$. In the last section below, we consider the possibility, $\mu_z > 1$.

In (3.2), the shock to technology, ϵ_t, has the time series representation in (**??**). We adopt a variant of Calvo sticky prices. In each period, t, a fraction of intermediate-goods firms, $1 - \xi_p$, can reoptimize their price. If the i^{th} firm in period t cannot reoptimize, then it sets price according to:

$$P_{it} = \tilde{\pi}_t P_{i,t-1},$$

where

$$\tilde{\pi}_t = \pi_{t-1}^{\iota} \bar{\pi}^{1-\iota}. \tag{3.4}$$

Here, π_t denotes the gross rate of inflation, $\pi_t = P_t / P_{t-1}$, and $\bar{\pi}$ denotes steady state inflation. If the i^{th} firm is permitted to optimize its price at time t, it chooses $P_{i,t} = \tilde{P}_t$ to optimize discounted profits:

$$E_t \sum_{j=0}^{\infty} \left(\beta \xi_p \right)^j \lambda_{t+j} \left[P_{i,t+j} Y_{i,t+j} - P_{t+j} s_{t+j} \left(Y_{i,t+j} + \Phi z_{t+j} \right) \right]. \tag{3.5}$$

Here, λ_{t+j} is the multiplier on firm profits in the household's budget constraint. Also, $P_{i,t+j}$, $j > 0$ denotes the price of a firm that sets $P_{i,t} = \tilde{P}_t$ and does not reoptimize between $t+1, ..., t+j$. The equilibrium conditions associated with firms appear in the next subsection.

3.2. Households

The household maximizes utility

$$E_t^j \sum_{l=0}^{\infty} \beta^{l-t} \left\{ u(c_{t+l}) - \psi_L \frac{h_t^{1+\sigma_L}}{1+\sigma_L} - \upsilon \frac{\left(\frac{P_{t+l} c_{t+l}}{M_{t+l}^d} \right)^{1-\sigma_q}}{1-\sigma_q} \right\} \tag{3.6}$$

subject to the constraint

$$P_t\left(c_t+i_t\right)+M_{t+1}^d-M_t^d+T_{t+1}\leq W_{t,j}l_{t,j}+P_tr_t^kk_t+\left(1+R_t^e\right)T_t, \tag{3.7}$$

where M_t^d denotes the household's beginning-of-period stock of money and T_t denotes nominal bonds issued in period $t-1$, which earn interest, R_t^e, in period t. This nominal interest rate is known at $t-1$. In the interest of simplifying, we suppose that υ in (3.6) is positive, but so small that the distortions to consumption, labor and capital first order conditions introduced by money can be ignored. The household's problem is to maximize (3.6) subject to the standard capital accumulation technology linking investment, i, to capital.

3.3. Monetary Authority

The monetary authority controls the supply of money, M_t^s. When policy is exogenous, it does so to implement a following Taylor rule. The target interest rate is:

$$R_t^*=\frac{\bar{\pi}}{\beta}-1+\alpha_\pi\left[E_t\left(\pi_{t+1}\right)-\bar{\pi}\right]+\alpha_y\log\left(\frac{Y_t}{Y_t^+}\right),$$

where Y_t^+ is aggregate output on a nonstochastic steady state growth path. The monetary authority manipulates the money supply to ensure that the equilibrium nominal rate of interest, R_t, satisfies:

$$R_t=\rho_iR_{t-1}+\left(1-\rho_i\right)R_t^*. \tag{3.8}$$

3.4. Equilibrium Conditions of the Model

Real marginal cost, s_t, can be represented as the ratio of the real cost of capital to its marginal product and the real cost of labor to its marginal product:

$$s_t = \frac{r_t^k}{\alpha\epsilon_t\left(\frac{h_t}{k_{t-1}}\right)^{1-\alpha}} \tag{3.9}$$

$$s_t = \frac{\tilde{w}_t}{\left(1-\alpha\right)\epsilon_t\left(\frac{h_t}{k_{t-1}}\right)^{-\alpha}}, \tag{3.10}$$

where r_t^k and $\tilde{w}_t$ denote the real rental rate on capital and the real wage rate, respectively. The household's first order condition for labor:

$$\psi_Lh_t^{\sigma_L}c_t=\tilde{w}_t \tag{3.11}$$

The household's intertemporal first order condition for the nominal return on capital:

$$-\frac{1}{c_t}+\frac{\beta}{\pi_{t+1}}\frac{1}{c_{t+1}}\left[1+R_{t+1}^k\right]=0 \tag{3.12}$$

The capital accumulation equation:

$$\bar{k}_t-(1-\delta)\bar{k}_{t-1}=i_t \tag{3.13}$$

The definition of the nominal rate of return on capital:

$$R_t^k = \left(r_t^k + (1-\delta)\right)\pi_t - 1 \tag{3.14}$$

The resource constraint, including possible distortions due to price dispersion:

$$c_t + i_t \leq (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left\{\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right\} \tag{3.15}$$

Here, ϵ_t is a technology shock, with the following law of motion:

$$\log \epsilon_t = \rho_\varepsilon \log \epsilon_{t-1} + \varepsilon_t, \tag{3.16}$$

where ε_t is iid with variance σ_ε^2.

Substitute out for $\tilde{w}_t$ from (3.11) into (3.10):

$$s_t = \frac{1}{1-\alpha} \frac{\left(\frac{h_t}{k_{t-1}}\right)^{\alpha}}{\epsilon_t} \psi_L h_t^{\sigma_L} c_t \tag{3.17}$$

Substitute out for R_{t+1}^k in (3.12) using (3.9) and (3.14), to obtain:

$$-\frac{1}{c_t} + \frac{\beta}{c_{t+1}} \left[\alpha \epsilon_{t+1} \left(\frac{h_{t+1}}{k_t}\right)^{1-\alpha} s_{t+1} + (1-\delta)\right] = 0 \tag{3.18}$$

Eliminate i_t in the resource constraint using (3.13):

$$c_t + \bar{k}_t - (1-\delta)\bar{k}_{t-1} = (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left\{\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right\} \tag{3.19}$$

We now turn to the equations pertaining to sticky prices. The equilibrium condition pertaining to p_t^* is:

$$p_t^* - \left[(1-\xi_p)\left(\frac{1-\xi_p\left(\frac{\pi_{t-1}^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right)^{\lambda_f} + \xi_p\left(\frac{\pi_{t-1}^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi_t} p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]^{\frac{1-\lambda_f}{\lambda_f}} = 0 \tag{3.20}$$

Note that when there are no sticky prices, so that $\xi_p = 0$, then $p_t^* = 1$ and (3.19) reduces to a more standard-looking resource constraint.

We also have the following equations:

$$\begin{aligned}
K_{p,t} - F_{p,t}\left[\frac{1-\xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{(1-\xi_p)}\right]^{1-\lambda_f} &= 0 \\
E_t\left\{\lambda_{z,t}Y_{z,t} + \left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}} \beta\xi_p F_{p,t+1} - F_{p,t}\right\} &= 0 \\
E_t\left\{\lambda_f \lambda_{z,t} Y_{z,t} s_t + \beta\xi_p\left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{-\frac{\lambda_f}{\lambda_f-1}} K_{p,t+1} - K_{p,t}\right\} &= 0 \\
\tilde{\pi}_t = \pi_{t-1}^{\iota_2}\bar{\pi}^{1-\iota_2} \quad (\text{we set } \iota_1 = 0)
\end{aligned}$$

Note that when there are no sticky prices, then $K_{p,t} = F_{p,t}$ and

$$s_t = \frac{1}{\lambda_f},$$

so that the markup is a constant (being real marginal cost, s_t is the reciprocal of the markup). Sticky prices in effect make the markup fluctuate.

Substitute out K_p and $\tilde{\pi}_t$, replace s_t using (3.17) above and replace $Y_{z,t}$, λ_{zt} using

$$\begin{aligned} Y_{z,t} &= (p_t^*)^{\frac{\lambda_f}{\lambda_f-1}} \left[\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right] \\ \lambda_{zt} &= \frac{1}{c_t}, \end{aligned}$$

so that we end up with the following two equations:

$$E_t \left\{ \frac{1}{c_t} (p_t^*)^{\frac{\lambda_f}{\lambda_f-1}} \left[\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right] + \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}} \beta \xi_p F_{p,t+1} - F_{p,t} \right\} = 0, \tag{3.21}$$

and

$$\begin{aligned} &\frac{1}{c_t} \lambda_f (p_t^*)^{\frac{\lambda_f}{\lambda_f-1}} \left[\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right] s_t + \qquad (3.22) \\ &\beta \xi_p \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}}\right)^{\frac{\lambda_f}{1-\lambda_f}} \left[\frac{1 - \xi_p \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}}}{1 - \xi_p}\right]^{1-\lambda_f} F_{p,t+1} \quad - F_{p,t} \left[\frac{1 - \xi_p \left(\frac{\pi_{t-1}^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{1 - \xi_p}\right]^{1-\lambda_f} = 0 \end{aligned}$$

To conclude, we have $N = 7$ unknowns, $(k, s, l, c, \pi, F_p, p^*)$, in the $N - 1$ equations (3.17), (3.18), (3.19), (3.20), (3.21), (3.22). These equations have been entered into the Dynare file, newsimplemodel.mod, in subdirectory stickypricesonly. We also computed the nominal rate of interest, by including the following equation:

$$E_t \left[-\frac{1}{c_t} + \beta (1 + R_t) \frac{1}{c_{t+1} \pi_{t+1}} \right] = 0.$$

3.5. Analysis of Ramsey Equilibrium

Suppose we start in period 0, in a steady state, when $\pi_{-1} = \bar{\pi}$ and $p_{-1}^* = 1$. Consider a monetary policy which results in $\pi_t = \bar{\pi}$ for $t = 0, 1, \ldots$. Then, (3.20) implies

$$p_0^* = 1 = \left[(1 - \xi_p) + \xi_p \left(p_{-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}} \right]^{\frac{1-\lambda_f}{\lambda_f}}.$$

Substituting this into (3.20) for $t = 1, 2, 3, \ldots$, produces the result, $p_t^* = 0$ for $t = 0, 1, \ldots$. This policy minimizes the distortion in the resource constraint, because (it can be shown) $p_t^* \leq 1$ for all t. Thus, the resource constraint is:

$$c_t + \bar{k}_t - (1 - \delta) \bar{k}_{t-1} = \epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi. \tag{3.23}$$

Under the stated monetary policy, we can combine (3.21) and (3.22) to obtain:

$$\frac{1}{c_t}\left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\epsilon_t\bar{k}_{t-1}^{\alpha}h_t^{1-\alpha}-\phi\right]=\frac{1}{c_t}\lambda_f\left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\epsilon_t\bar{k}_{t-1}^{\alpha}h_t^{1-\alpha}-\phi\right]s_t,$$

or,

$$s_t=\frac{1}{\lambda_f},$$

for all t. Combining this with (3.17), we infer

$$\psi_L h_t^{\sigma_L}c_t=\frac{(1-\alpha)\,\epsilon_t\left(\frac{h_t}{k_{t-1}}\right)^{-\alpha}}{\lambda_f}. \tag{3.24}$$

That is, the marginal rate of substitution between consumption and leisure equals the marginal product of labor, divided by the markup. The final equation of our equilibrium is the intertemporal Euler equation, (3.18), which becomes

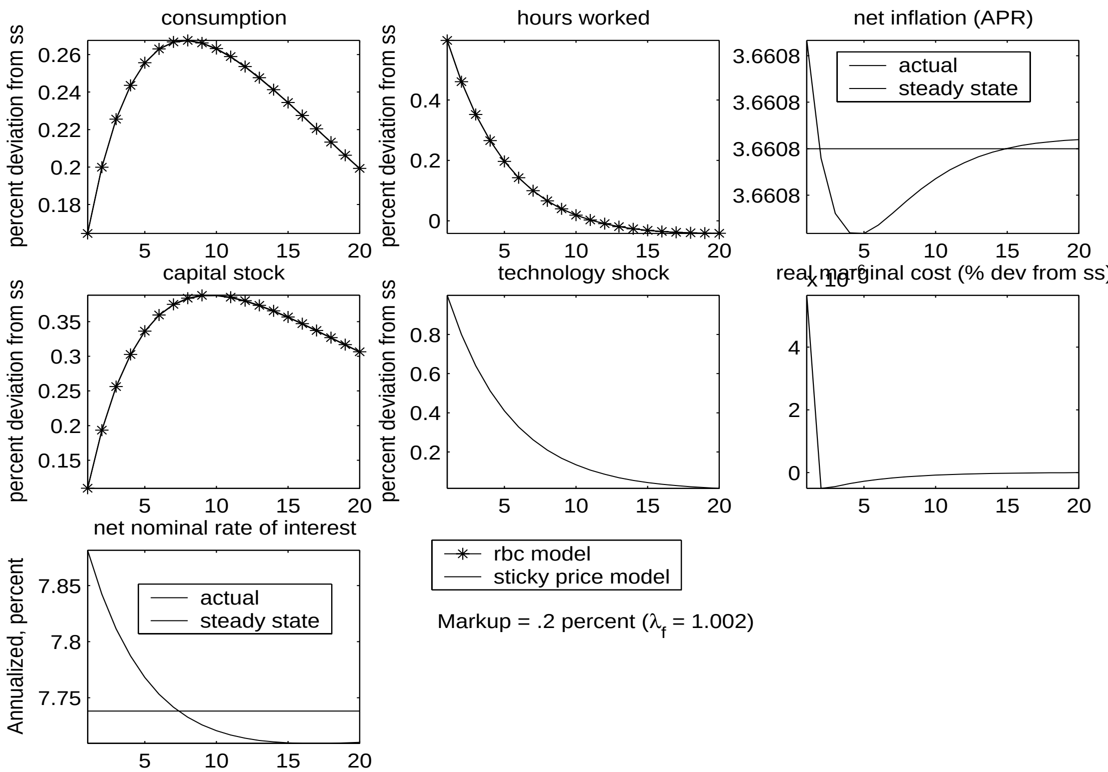
$$-\frac{1}{c_t}+\frac{\beta}{c_{t+1}}\left[\alpha\epsilon_{t+1}\left(\frac{h_{t+1}}{k_t}\right)^{1-\alpha}\frac{1}{\lambda_f}+(1-\delta)\right]=0. \tag{3.25}$$

Note that if $\lambda_f=1$, then (3.23), (3.24) and (3.25) characterize the efficient allocations for the economy with the preferences and technology that we assume. Thus, if λ_f is nearly unity, the constant inflation monetary policy is nearly optimal. However, if λ_f is substantially above unity, then we cannot expect the constant monetary policy to be optimal. If there is a positive markup in steady state, then it is possible that with inflation allowed to vary in the right way, the markup can fall in response to a shock, and this would improve utility relative to the scenario in which inflation is held constant. To see this, note that in (3.24) and (3.25), there appear wedges between marginal rates of substitution in preferences and marginal rates of technical transformation that are different from unity. If instead s_t appeared and could be made to rise, then welfare would increase. Of course, this involves complicated tradeoffs. In this economy, flucttutations in s_t are induced by fluctuations in inflation and these in turn introduce an inefficiency wedge, p_t^*, in the resource constraint. In addition, if welfare rises as s_t rises in response to a shock, then when the shock takes on the opposite sign, s_t will fall and reduce welfare. All these considerations have to be balanced off against each other to determine the optimal response of inflation to a shock.

We illustrate these observations with an example. Suppose

$$\beta=0.99,\ \psi_L=109.8,\ \lambda_f=1.002,\ \alpha=0.40,\ \delta=0.025,\ \xi_p=0.75,\ \iota_2=0.6,\ \sigma_L=1,\rho_\varepsilon=0.8.$$

With this parameterization, we computed the dynamic response of $c_t,\ h_t,\ \pi_t,\ k_t,\ \epsilon_t,\ s_t,\ R_t$ to a one-percent shock in ε_0, 0.01. We computed a linear approximation using Dynare, and so the output of Dynare is a sequence, $\Delta c_t,\ \Delta h_t,\ \Delta\pi_t,\ \Delta k_t,\ \Delta\log\epsilon_t,\ \Delta s_t,\ \Delta R_t$, where Δ denotes deviation from steady state. In the case of $\Delta c_t,\ \Delta h_t,\ \Delta k_t,\ \Delta s_t$, we divided the variable by its corresponding steady state. Because the shock actually fed to Dynare was $\varepsilon_0=1.$, we interpret the output of Dynare as being in percent terms. In the case of inflation, we computed $400\left(\pi+\Delta\pi_t/100-1\right)$, where π is the steady state inflation rate. Thus, we report the net inflation rate, expressed at an annual rate. Similarly, in the case of the nominal interest rate, we computed $400\,(R+\Delta R_t/100)$, where R is the steady state nominal rate of interest. Thus, the nominal rate of interest is reported in net terms, at an annual percent rate. The response of the state of technology, $\Delta \log \epsilon_t$, is expressed in percent deviation from steady state and is not further adjusted. The results are displayed in the following figure:

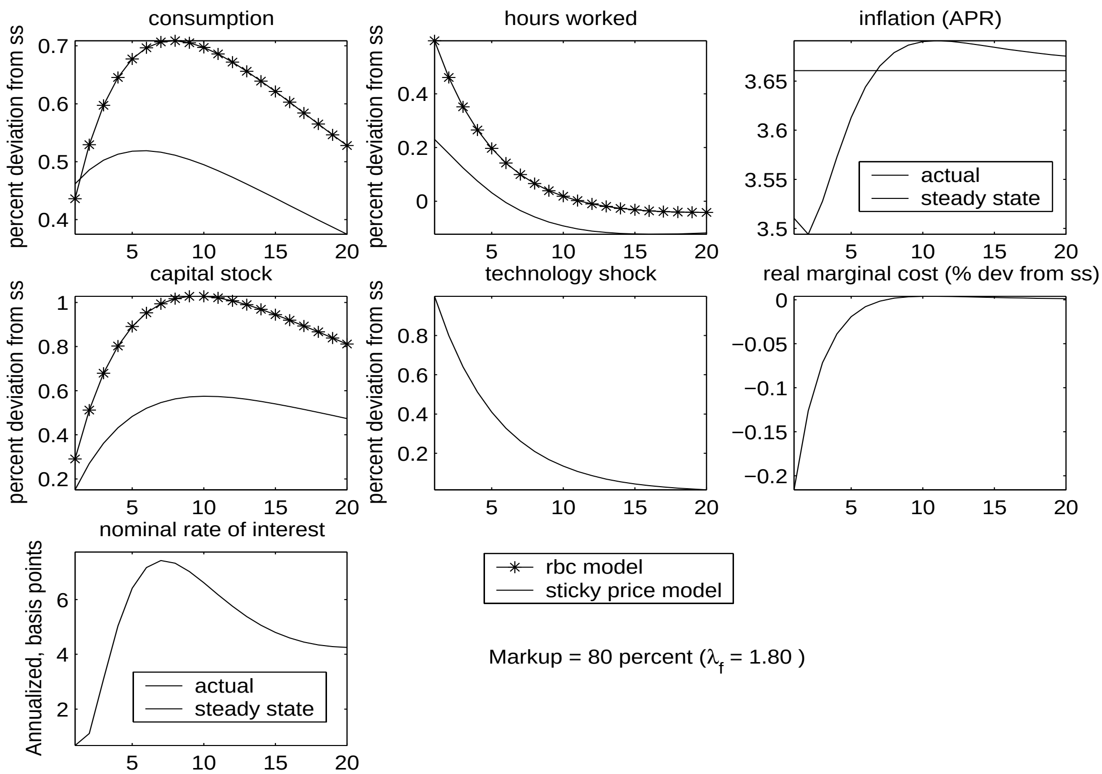


This graph displays both the response of the variables in the sticky price model, as well as in the real business cycle model characterized by (3.23)-(3.25).[2] Note that the quantity allocations in the RBC model and the sticky price model are essentially the same. Also, there is essentially no response in inflation or the markup in the Ramsey equilibrium, as expected.

Next, we consider a very high markup of 80 percent, or $\lambda_f = 1.80$. The response to the

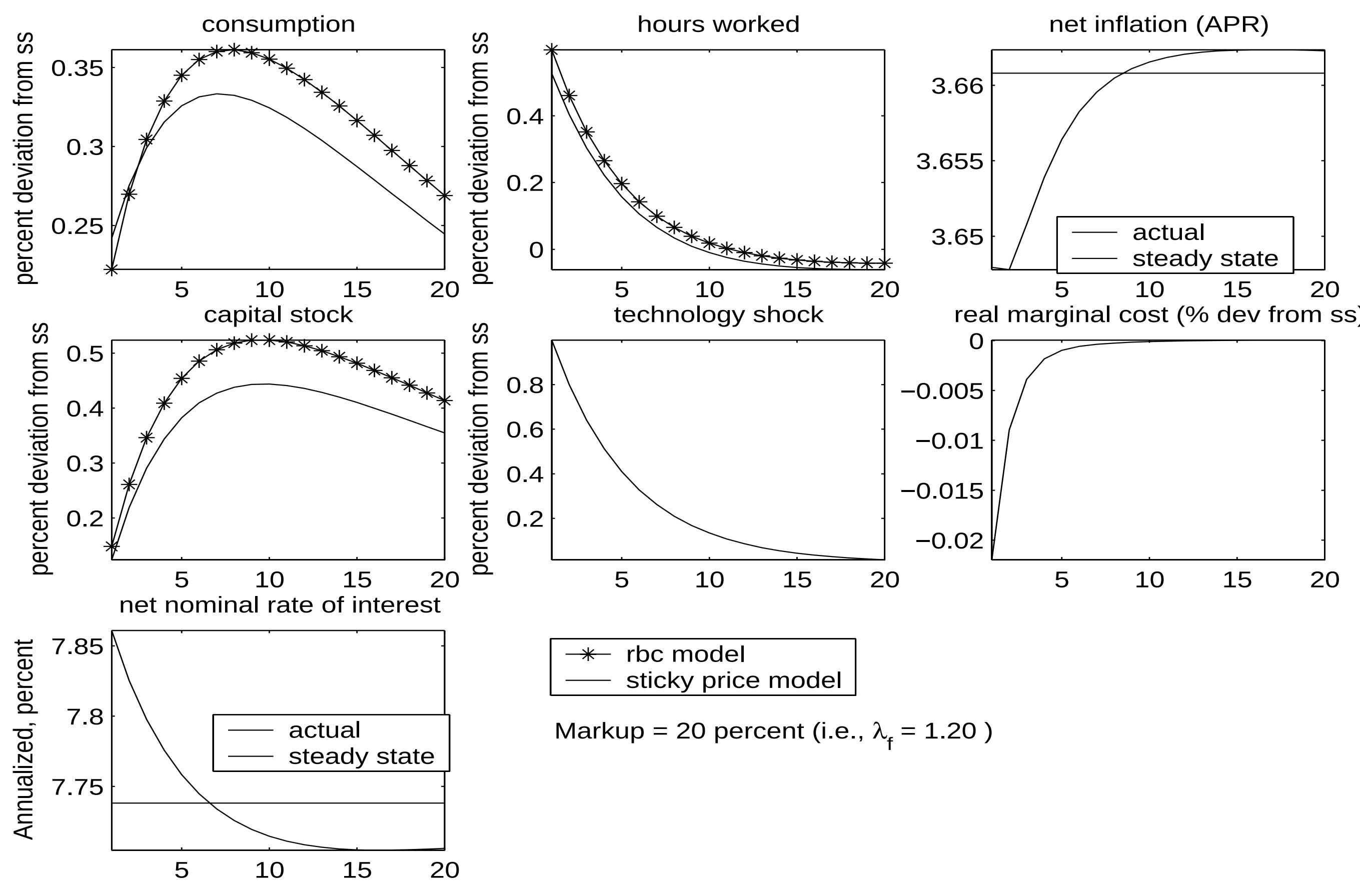
[2]The subdirectory, stickypricesonly, also contains the mod file, rbcmodel.mod, which was used to compute the dynamic response of the variables in the RBC model.

same technology shock is as follows



First, note how the quantities respond very differently in the Ramsey and in the RBC economy. Also, there is now a non-negligible deviation from constant inflation and constant marginal cost. Marginal cost, s_t drops, implying a rise in the markup. This may seem puzzling at first, since a rise in the markup is clearly not directly welfare-increasing. However, recall that this is the response to an unexpected positive shock to technology. With equal probability, the shock would have the other sign. So, to evaluate the welfare consequence of the non-constancy of inflation, one has to trade off the welfare loss of the rise in the markup with the positive technology shock against the welfare gain of the fall in the markup with the negative technology shock. Presumably, the gain associated with a fall in technology outweighs the loss associated with the rise in technology.

Now consider a more 'normal' setting for the markup, $\lambda_f = 1.20$, or 20 percent. With this setting, we obtain the following impulse response function:



These responses resemble those in the low markup economy, though the differences are not completely insignificant.

4. Adding Money to the Model with Calvo Sticky Prices

We now consider the model of the previous section, with money in the utility function.

4.1. Equilibrium Conditions

The household's Lagrangian problem is:

$$E_t^j \sum_{l=0}^{\infty} \beta^l \{ u(c_{t+l} - bc_{t+l-1}) - \psi_L \frac{l_t^{1+\sigma_L}}{1+\sigma_L} - \upsilon \frac{\left(\frac{P_{t+l}c_{t+l}}{M^d_{t+l+1}}\right)^{1-\sigma_q}}{1-\sigma_q}$$

$$+\lambda_t \left[W_t l_t + P_t r_t^k k_t + (1+R_{t-1}) T_t + D_t + X_t - \left(P_t (c_t + i_t) + M^d_{t+1} - M^d_t + T_{t+1} \right) \right]$$

$$+\mu_t \left[(1-\delta) k_t + \left(1 - S\left(\frac{i_t}{i_{t-1}} \right) \right) i_t - k_{t+1} \right] \},$$

where D_t denotes profits and X_t is a transfer from the government:

$$X_t = M_{t+1} - M_t,$$

and in equilibrium it must be that $M_t^d = M_t$. Note that we have introduced money in the utility function, and investment adjustment costs habit persistence in consumption have also been introduced.

The first order conditions for consumption is:

$$P_t \lambda_t = u'\left(c_t - bc_{t-1}\right) - \beta b u'\left(c_{t+1} - bc_t\right) - \upsilon \left(\frac{P_t}{M_t^d}\right)^{1-\sigma_q} c_t^{-\sigma_q},$$

or,

$$\lambda_{z,t} = \frac{1}{c_t - bc_{t-1}} - \beta b \frac{1}{c_{t+1} - bc_t} - \upsilon \left(\frac{\pi_t}{m_t}\right)^{1-\sigma_q} c_t^{-\sigma_q},$$

where $m_t = M_t / P_{t-1}$.

The first order condition for l_t is:

$$\psi_L l_t^{\sigma_L} = \lambda_t W_t = \lambda_{z,t} w_t,$$

where $w_t = W_t / P_t$.

The first order condition for T_{t+1} is:

$$\lambda_t = \beta \lambda_{t+1} \left(1 + R_t\right),$$

or,

$$\lambda_{z,t} = \beta \lambda_{z,t+1} \frac{1 + R_t}{\pi_{t+1}}, \;\; \pi_{t+1} = \frac{P_{t+1}}{P_t}.$$

The first order condition for i_t is:

$$\lambda_t P_t = \mu_t \left[1 - S\left(\frac{i_t}{i_{t-1}}\right) - S'\left(\frac{i_t}{i_{t-1}}\right) \frac{i_t}{i_{t-1}}\right] + \beta \mu_{t+1} S'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2.$$

Define

$$P_{k',t} = \frac{\mu_t}{\lambda_t P_t},$$

so that

$$1 = P_{k',t} \left[1 - S\left(\frac{i_t}{i_{t-1}}\right) - S'\left(\frac{i_t}{i_{t-1}}\right) \frac{i_t}{i_{t-1}}\right] + \beta \frac{\lambda_{z,t+1}}{\lambda_{z,t}} P_{k',t+1} S'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2,$$

The first order condition M_{t+1}^d :

$$\lambda_t = \beta \lambda_{t+1} + \beta \upsilon \left(P_{t+} c_{t+1}\right)^{1-\sigma_q} \left(M_{t+1}^d\right)^{\sigma_q - 2},$$

or, after multiplying by P_t :

$$\lambda_{z,t} = \beta \frac{\lambda_{z,t+1}}{\pi_{t+1}} + \pi_{t+1}^{1-\sigma_q} \beta \upsilon c_{t+1}^{1-\sigma_q} m_{t+1}^{\sigma_q - 2}.$$

The first order condition associated with k_{t+1} is:

$$\mu_t = \beta \left[\lambda_{t+1} P_{t+1} r^k_{t+1} + \mu_{t+1} (1 - \delta) \right],$$

or, after dividing by $\lambda_t P_t$:

$$P_{k',t} = \frac{\lambda_{z,t+1}}{\lambda_{z,t}} \beta \left[r^k_{t+1} + P_{k',t+1} (1 - \delta) \right].$$

The monetary policy rule is:

$$R_t = \rho_i R_{t-1} + (1 - \rho_i) \left[\frac{\bar{\pi}}{\beta} - 1 + \alpha_\pi \left[E_t (\pi_{t+1}) - \bar{\pi} \right] + \alpha_y \log \left(\frac{c_t + i_t}{Y^+} \right) \right] + \varepsilon_t.$$

Putting all the equations together, and including the equations related to sticky prices:

$$
\begin{aligned}
(1)\frac{1}{c_t - bc_{t-1}} - \beta b \frac{1}{c_{t+1} - bc_t} - \upsilon \left(\frac{\pi_t}{m_t}\right)^{1-\sigma_q} c_t^{-\sigma_q} &= \lambda_{z,t} \\
(2) P_{k',t}\left[1 - S\left(\frac{i_t}{i_{t-1}}\right) - S'\left(\frac{i_t}{i_{t-1}}\right)\frac{i_t}{i_{t-1}}\right] + \beta\frac{\lambda_{z,t+1}}{\lambda_{z,t}} P_{k',t+1} S'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2 &= 1 \\
(3)\frac{\lambda_{z,t+1}}{\lambda_{z,t}}\beta\left[r_{t+1}^k + P_{k',t+1}(1-\delta)\right] &= P_{k',t} \\
(4)\beta\frac{\lambda_{z,t+1}}{\pi_{t+1}} + \upsilon\beta\left(\pi_{t+1}c_{t+1}\right)^{1-\sigma_q} m_{t+1}^{\sigma_q - 2} &= \lambda_{z,t} \\
(5)\beta\frac{\lambda_{z,t+1}}{\pi_{t+1}}(1+R_t) &= \lambda_{z,t} \\
(6)\,(1-\delta)k_t + \left(1 - S\left(\frac{i_t}{i_{t-1}}\right)\right) i_t &= k_{t+1} \\
(7)\;\; (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}}\left[\epsilon_t (K_t)^\alpha l_t^{1-\alpha} - \Phi\right] &= c_t + i_t \\
(8)\;\; \lambda_{z,t} s_t \epsilon_t (1-\alpha)\left(\frac{k_t}{l_t}\right)^\alpha &= \psi_L l_t^{\sigma_L} \\
(9)\;\; s_t\epsilon_t\alpha\left(\frac{l_t}{k_t}\right)^{1-\alpha} &= r_t^k \\
(10)\quad \left[(1-\xi_p)\left(\frac{1-\xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right)^{\lambda_f} + \xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]^{\frac{1-\lambda_f}{\lambda_f}} &= p_t^* \\
(11)\;\; E_t\left[\lambda_{z,t}Y_t + \left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}}\beta\xi_p F_{p,t+1} - F_{p,t}\right] &= 0 \\
(12)\;\; E_t\left[\lambda_f\lambda_{z,t}Y_t s_t + \beta\xi_p\left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{\frac{\lambda_f}{1-\lambda_f}} K_{p,t+1} - K_{p,t}\right] &= 0 \\
(13)\;\; \frac{1-\xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{(1-\xi_p)} &= \left(\frac{K_{p,t}}{F_{p,t}}\right)^{\frac{1}{1-\lambda_f}} \\
(14)\;\; \rho_i R_{t-1} + (1-\rho_i)\left[\frac{\bar{\pi}}{\beta} - 1 + \alpha_\pi\left[E_t(\pi_{t+1}) - \bar{\pi}\right] + \alpha_y \log\left(\frac{c_t + i_t}{Y^+}\right)\right] + \varepsilon_t &= R_t
\end{aligned}
$$

The 14 variables to be solved using these 14 equations are $\lambda_{z,t}$, c_t, m_t, π_t, i_t, $P_{k',t}$, r_t^k, R_t, k_t, p_t^*, l_t, s_t, $F_{p,t}$, $K_{p,t}$.

4.2. Steady State

The equations associated with Calvo sticky prices imply:

$$\begin{aligned} s &= \frac{1}{\lambda_f}, \\ F_p &= \frac{\lambda_z Y}{1-\beta\xi_p} \end{aligned}$$

This is as expected. When there are no price distortions in the steady state, then the firm markup is $1/s = \lambda_f$ in the present case of a constant elasticity demand curve. The resource constraint is

$$c + \delta k = k^{\alpha} l^{1-\alpha} - \Phi.$$

We use the supposition that profits are zero in the steady state to determine a value for Φ. If we think of the Cobb-Douglas part of the production function as the firm's 'production function', then with fixed costs, Φ, the firm which sells Y_{jt} must actually produce $Y_{jt} + \Phi$. Given fixed marginal costs, the firm's total cost associated with selling Y_{jt} is $s_t\left(Y_{jt} + \Phi\right)$, in units of the final good (i.e., scaling by P_t). The firms' revenues are $P_{jt}Y_{jt}$, so its profits in units of final goods are

$$\frac{P_{jt}}{P_t} Y_{jt} - s_t\left(Y_{jt} + \Phi\right).$$

In steady state, $P_{jt} = P_t$, $Y_{jt} = Y_t$ for all j because our assumptions guarantee that prices and resources are not distorted in a steady state. So, the zero profit condition in steady state is

$$Y = \frac{1}{\lambda_f}\left(Y + \Phi\right),$$

or,

$$\left(\lambda_f - 1\right) Y = \Phi.$$

In the steady state, profits are $100\left(\lambda_f - 1\right)$ of output sold, and fixed costs must be equal to this amount if profits are to be zero. Substituting in the production function,

$$\left(\lambda_f - 1\right)\left(k^{\alpha} l^{1-\alpha} - \Phi\right) = \Phi,$$

so that

$$\left(\lambda_f - 1\right) k^{\alpha} l^{1-\alpha} = \lambda_f \Phi$$

Combining this with the resource constraint, we obtain:

$$c + \delta k = k^{\alpha} l^{1-\alpha} - \frac{\lambda_f - 1}{\lambda_f} k^{\alpha} l^{1-\alpha} = \frac{1}{\lambda_f} k^{\alpha} l^{1-\alpha}.$$

Collecting our results, we have that the steady state equations are:

$$\begin{aligned}
P_{k',t} &= 1, \\
\lambda_z &= (1-\beta b)\, u'(c-bc) - \upsilon \left(\frac{\pi}{m}\right)^{1-\sigma_q} c^{-\sigma_q} \\
\frac{1}{\beta} &= r^k + 1 - \delta \\
\frac{\pi}{\beta} &= 1 + R \\
\lambda_z &= \beta \frac{\lambda_z}{\pi} + \upsilon\beta \left(\frac{\pi}{m}\right)^{1-\sigma_q} c^{-\sigma_q} \left(\frac{c}{m}\right) \\
\frac{\psi_L l^{\sigma_L}}{\lambda_z} &= w \\
c + \delta k &= \frac{1}{\lambda_f} k^{\alpha} l^{1-\alpha} \\
s &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(r^k\right)^{\alpha} w^{1-\alpha} \\
s &= \frac{r^k}{\alpha\left(\frac{l}{k}\right)^{1-\alpha}} \\
s &= \frac{1}{\lambda_f},
\end{aligned}$$

which represents 10 equations in 11 unknowns:

$$c,\ s,\ l,\ \lambda_z,\ w,\ r^k,\ \pi,\ R,\ m,\ k,\ P_{k'}.$$

Another equation is provided by the assumption that transfers are made to the household to provide them with money:

$$M_{t+1} - M_t = X_t,$$

and dividing this by M_t :

$$\frac{M_{t+1} - M_t}{M_t} = x_t,$$

so that in steady state,

$$\frac{\pi m - m}{m} = \pi - 1 = x.$$

We will just treat π as an exogenous variable, with the understanding that it is actually x that is exogenous. So, by deleting π from the list of 11 unknowns, we have 10 equations in 10 unknowns.

We solve these equations as follows. The variables, R, s, $P_{k'}$, r^k, l/k, c/k and w are virtually immediate. Thus, $P_{k'} = 1$, $s = 1/\lambda_f$, $R = \pi/\beta - 1$ and

$$r^k = \frac{1}{\beta} - (1-\delta),$$

and

$$r^k = \frac{1}{\lambda_f}\alpha\left(\frac{l}{k}\right)^{1-\alpha},$$

so that

$$l_k \equiv \frac{l}{k} = \left(\frac{r^k\lambda_f}{\alpha}\right)^{\frac{1}{1-\alpha}}.$$

The resource constraint can be written:

$$c_k \equiv \frac{c}{k} = \frac{1}{\lambda_f}l_k^{1-\alpha} - \delta,$$

and the wage rate can be solved using the fact that r^k is known and

$$w = \left[\frac{1}{\lambda_f\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(r^k\right)^{\alpha}}\right]^{\frac{1}{1-\alpha}}.$$

We still require k, λ_z, m. The equations that remain available to us are the following three:

$$\begin{aligned}
\lambda_z &= (1-\beta b)\frac{1}{c_k k(1-b)} - \upsilon\left(\frac{\pi}{m}\right)^{1-\sigma_q}(c_k k)^{-\sigma_q} \\
\lambda_z &= \beta\frac{\lambda_z}{\pi} + \upsilon\beta\left(\frac{\pi}{m}\right)^{1-\sigma_q}(c_k k)^{-\sigma_q}\frac{c_k k}{m} \\
\frac{\psi_L(l_k k)^{\sigma_L}}{\lambda_z} &= w
\end{aligned}$$

which now reduces to two equations in two unknowns, k and m. Multiply the first equation by $c_k k$ and use the expression for λ_z from the household's first order condition for labor:

$$c_k k\frac{\psi_L(l_k k)^{\sigma_L}}{w} = \frac{1-\beta b}{(1-b)} - \upsilon\left(\frac{\pi}{m}\right)^{1-\sigma_q}(c_k k)^{1-\sigma_q}$$

SSubstitute from the labor euler equation into the second of the previous two equations:

$$\frac{\psi_L(l_k k)^{\sigma_L}}{w}\left[1-\frac{\beta}{\pi}\right] = \upsilon\beta\left(\frac{\pi}{m}\right)^{1-\sigma_q}\frac{(c_k k)^{1-\sigma_q}}{m}.$$

or,

$$\begin{aligned}
a_0 k^{1+\sigma_L} + \upsilon a_1\left(\frac{k}{m}\right)^{1-\sigma_q} &= a_2 \\
b_0 k^{1+\sigma_L} &= \upsilon b_1\left(\frac{k}{m}\right)^{2-\sigma_q}.
\end{aligned}$$

for known constants, a_0, a_1, a_2, b_0, b_1 and the parameter, υ :

$$\begin{aligned}
a_0 &= c_k\frac{\psi_L l_k{}^{\sigma_L}}{w},\ a_1 = \pi^{1-\sigma_q}c_k{}^{1-\sigma_q},\ a_2 = \frac{1-\beta b}{1-b}, \\
b_0 &= \frac{\psi_L l_k{}^{\sigma_L}}{w}\left[1-\frac{\beta}{\pi}\right],\ b_1 = \beta\pi^{1-\sigma_q}c_k{}^{1-\sigma_q}.
\end{aligned}$$

From the second expression

$$\frac{m}{k} = \left[\frac{\upsilon b_1}{b_0 k^{1+\sigma_L}} \right]^{\frac{1}{2-\sigma_q}},$$

so that

$$k^{1+\sigma_L} + \upsilon^{\frac{1}{2-\sigma_q}} \frac{a_1}{a_0} \left[\frac{b_0}{b_1} k^{1+\sigma_L} \right]^{\frac{1-\sigma_q}{2-\sigma_q}} = \frac{a_2}{a_0} = \frac{w \frac{1-\beta b}{1-b}}{c_k \psi_L l_k^{\sigma_L}} > 0.$$

Note that the function on the left of the equality is zero at $k = 0$ and strictly increasing and convex. As a result, there is a unique value of k that solves this equation. In the case, $\upsilon = 0$, this value can be found analytically:

$$k = \left(\frac{w \frac{1-\beta b}{1-b}}{c_k \psi_L l_k^{\sigma_L}} \right)^{\frac{1}{1+\sigma_L}}.$$

With k in hand, k/m may be found from the previous expression. Note that when $\upsilon = 0$ then $k/m = \infty$ so that $m = 0$. We would expect that the price level would be infinite when $\upsilon = 0$ because in this case there is no use for money.

4.3. Numerical Analysis of the Model

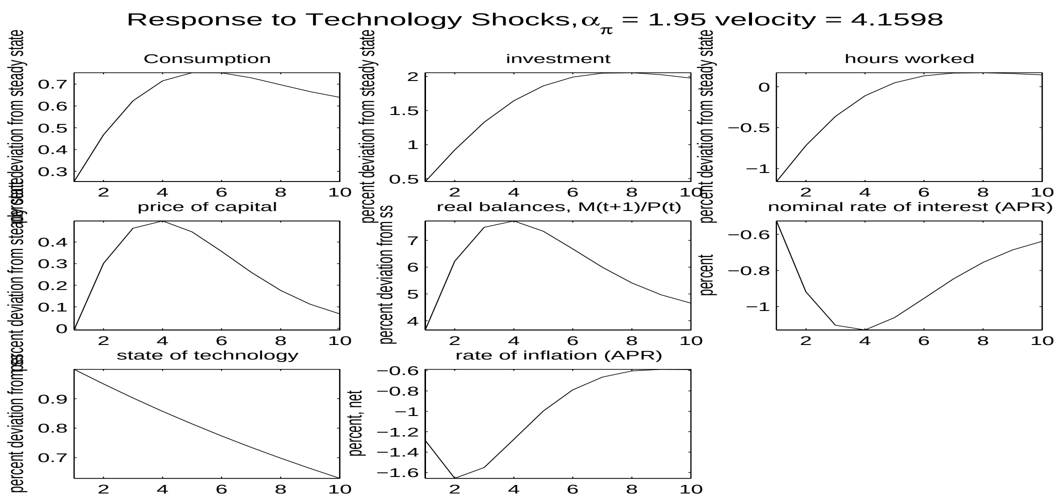
We considered the following parameter values.

$$\beta = 1.03^{-0.25},\ b = 0.63,\ \alpha = 0.36,\ \delta = 0.025,\ \rho = 0.95,$$

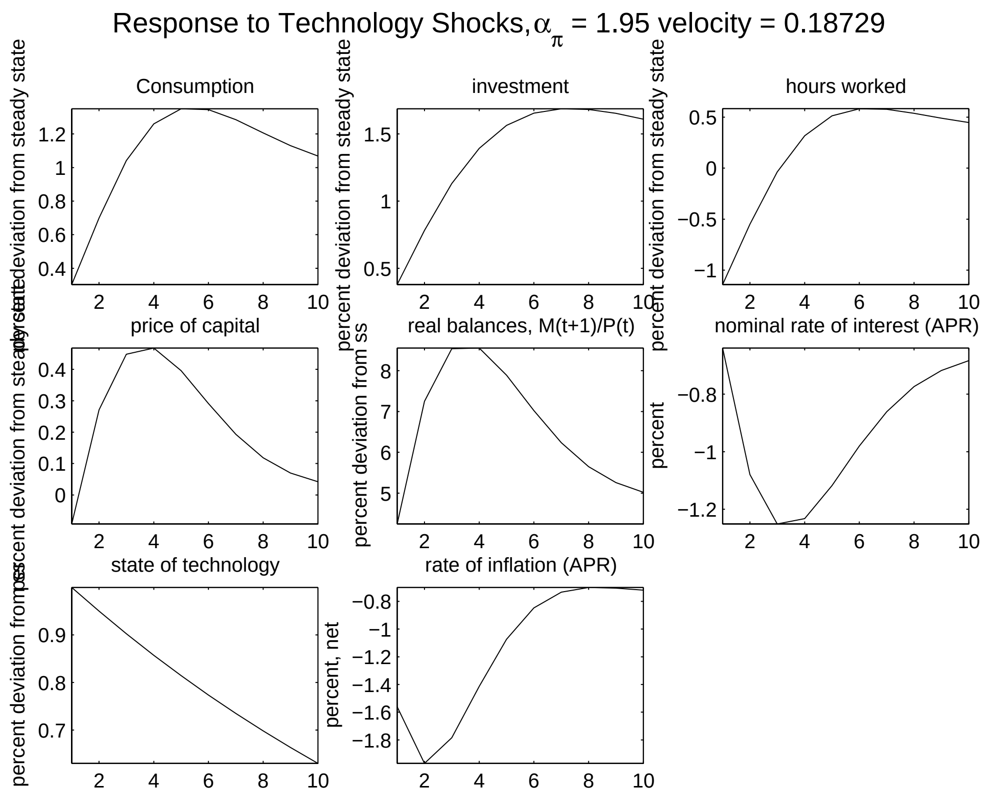
and

$$\begin{aligned} \lambda_f &= 1.20,\ \xi_p = 0.75,\ \iota = 0.84,\ \bar{\pi} = 1 + 0.025/4,\ \upsilon = 0.0005, \\ \rho_i &= 0.81,\ \alpha_\pi = 1.95,\ \alpha_y = 0.18,\ \sigma_L = 1,\ \sigma_q = -1,\ a = 5. \end{aligned}$$

With this parameterization, we have $l = 0.996$, $y/m = 4.16$, $k/y = 11.11$. The impulse response function to



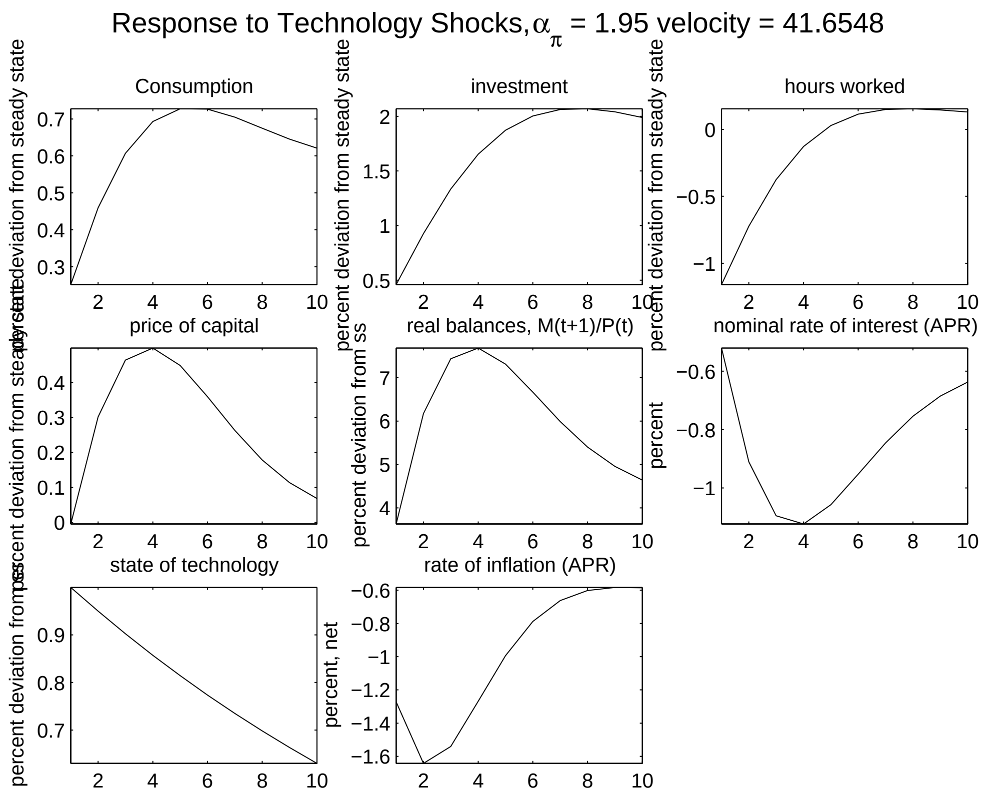
Next, we greatly increase the importance of money, by raising v to 5. In this case, we obtain the following impulse response function to a technology shock:



There is a noticeable difference in the response of consumption, with consumption rising by more here. The response in the other variables is not very different. Presumably, the rise in real balances associated with the fall in the price level is making the marginal utility of consumption rise more, as the real balance effect on consumption is more important.

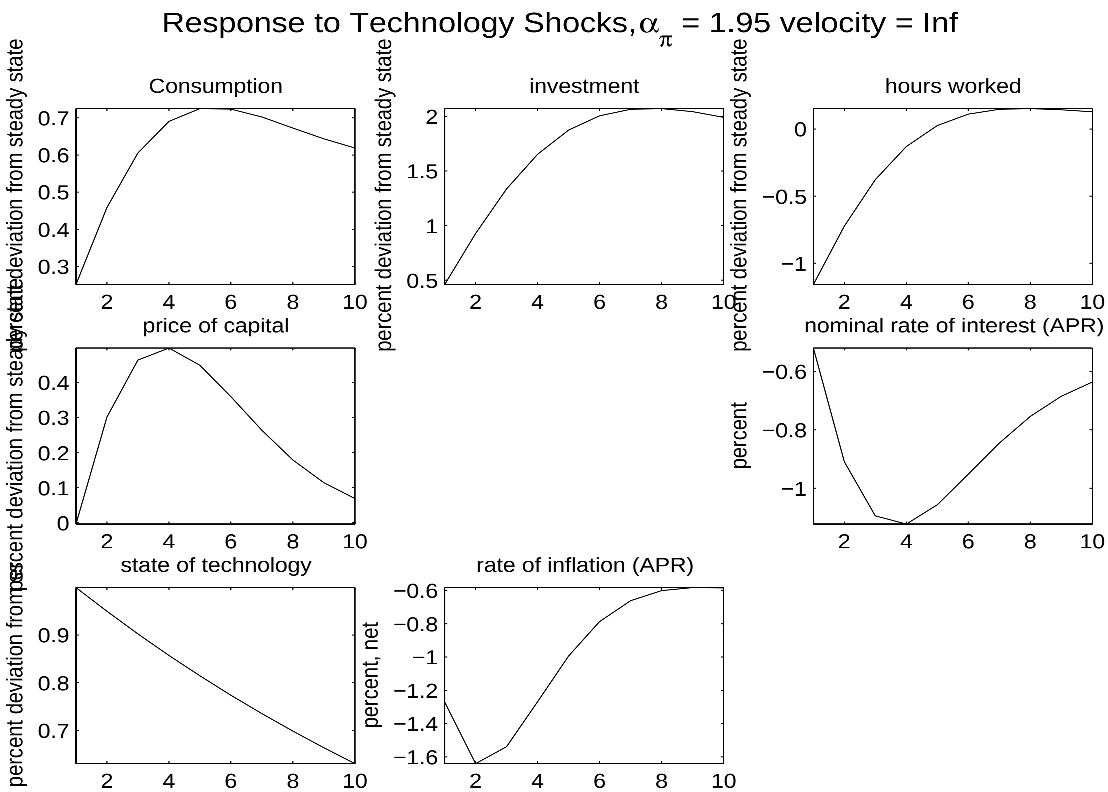
Next, we consider the effect of reducing the importance of money, by reducing v to

0.0000005. The resulting impulse responses are as follows:



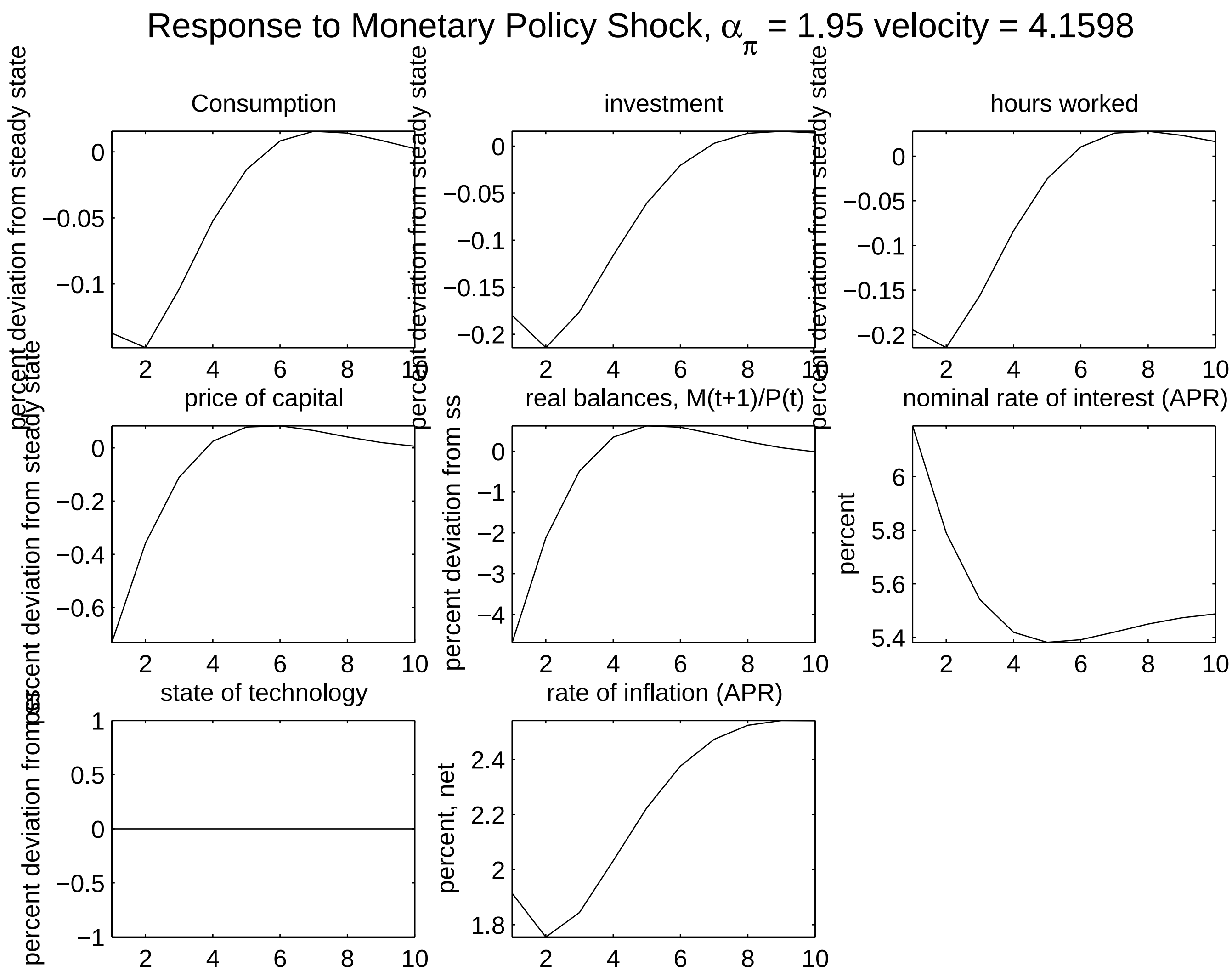
Note that with this ten-fold increase in velocity, the impulse responses have hardly change from our initial, baseline responses.

Next, we set $v = 0$. The steady state algorithm described above works for this case, and produces $m = 0$. In the dynamic equations, we set $v = 0$ in (1) and drop equation (4) and the variable, m_t. With $\alpha_\pi = 1.95$, the following impulse response function was obtained:



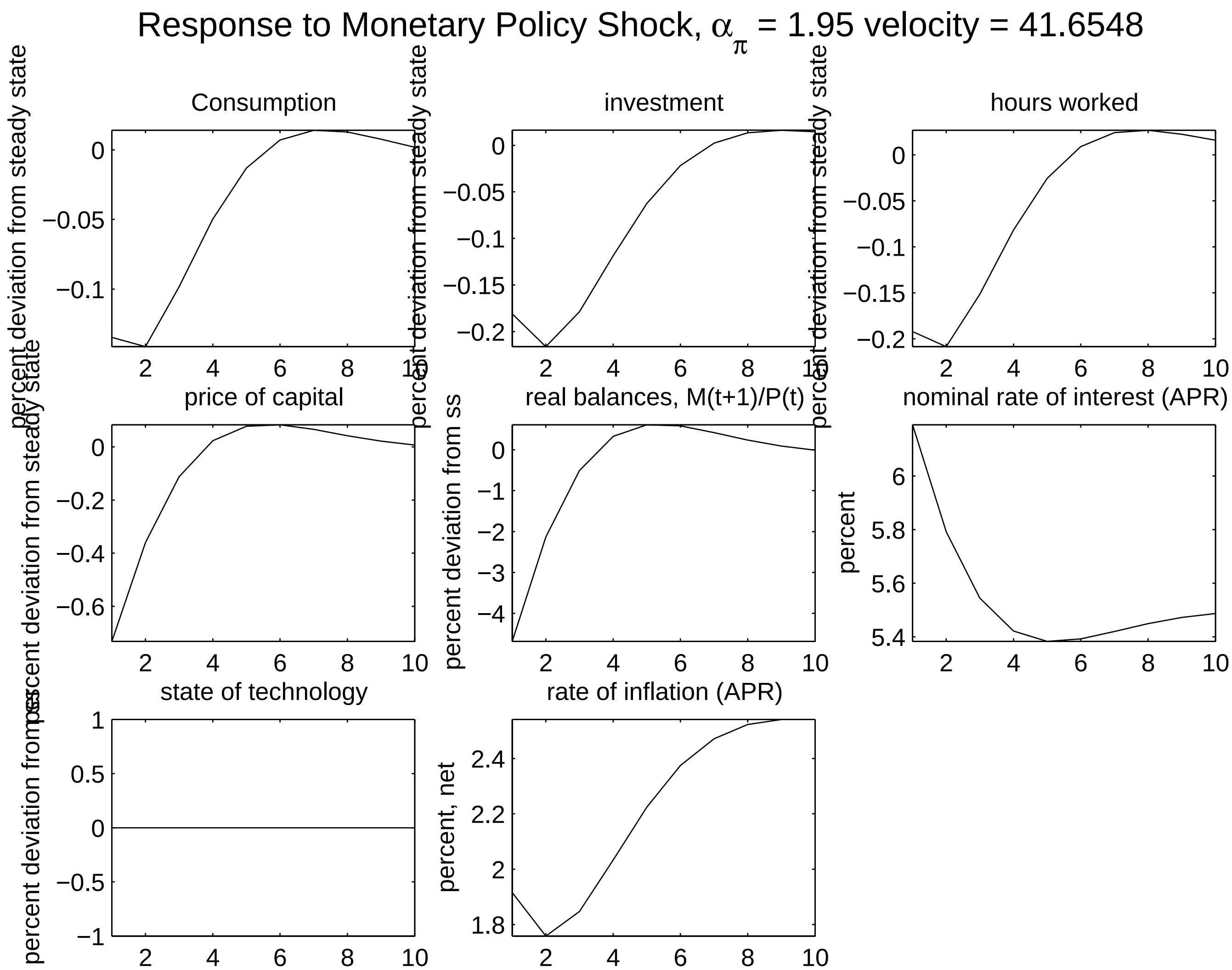
Note that the results are essentially the same. In addition, the range of determinacy is unchanged from before. From this we have to conclude that there must be a mistake somewhere because this model is essentially the standard model with no money.

Next, we consider the response of the system to a monetary policy shock. Here is the response to a 0.0025 shock to monetary policy. This corresponds to a 100 basis point rise in the interest rate, at an annual rate.



Note that to get the interest rate up, they need to reduce the money supply. Also, the interest rate does not rise by the full 100 basis points immediately, because the fall in prospective inflation exerts a countervailing force on the interest rate.

Now v is reduced, and the impulse response function becomes:



This result is the same as before.

Evidently, the size of v does not matter for the responses of the model economy to monetary policy and technology shocks.

5. Adding Wage Frictions to the Model

We now add Calvo-style wage frictions to the model of the previous section, following the analysis of Erceg, Henderson and Levin. We begin by deriving the equations that pertain to household wage setting. After this, we turn to implications for the aggregate resource constraint. We then display the other equilibrium conditions.

5.1. Households

We derive a law of motion for the aggregate real wage ($\tilde{w}_t$) by combining the optimality of the condition of of households which reoptimize their wage, with a cross-household-wage consistency condition.

We suppose there is a continuum of households, $j \in (0,1)$, each of which supplies a differentiated labor service which is aggregated into a homogeneous labor good by perfectly competitive labor contractors using the following constant returns to scale technology:

$$l_t = \left[\int_0^1 (h_{t,j})^{\frac{1}{\lambda_w}} dj \right]^{\lambda_w}, \ 1 \leq \lambda_w < \infty. \tag{5.1}$$

Aggregate labor is sold competitively by the representative labor contractor to intermediate goods producers for wage W_t and the j^{th} household's wage is $W_{j,t}$. The contractor hires $h_{t,j}$, $j \in (0,1)$, in order to maximize profits:

$$\max_{h_{t,j}} W_t \left[\int_0^1 (h_{t,j})^{\frac{1}{\lambda_w}} dj \right]^{\lambda_w} - \int_0^1 W_{t,j} h_{t,j} dj,$$

which leads to the first order condition:

$$W_t l_t^{\frac{\lambda_w - 1}{\lambda_w}} (h_{t,j})^{\frac{1-\lambda_w}{\lambda_w}} = W_{t,j},$$

or,

$$h_{t,j} = l_t \left[\frac{W_{t,j}}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}}. \tag{5.2}$$

The j^{th} household views (5.2) as a demand curve for its specialized labor services. The rules are that if the household posts a wage, $W_{t,j}$, then it must supply the services, $h_{t,j}$, implied by the demand curve.

Thus, the household's problem is to choose its wage rate, $W_{t,j}$. With probability, $1 - \xi_w$, it can optimize its wage rate and with the complementary probability, it cannot. In this case, we suppose that it sets its wage as follows:

$$W_{t,j} = \tilde{\pi}_{w,t-1} W_{t-1,j}, \ \tilde{\pi}_{w,t} \equiv (\pi_{t-1})^{\iota_{w,2}} \bar{\pi}^{1-\iota_{w,2}}. \tag{5.3}$$

The $1 - \xi_w$ households that set their wage optimally in period t all find it optimal to set the same wage, $\tilde{W}_t$. The household which can optimize its wage in period t does so to optimize the following objective:

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \{ -z(h_{j,t+i}) + \lambda_{t+i} W_{j,t+i} h_{j,t+i} \},$$

where λ_{t+i} is the multiplier on the household's budget constraint, (3.7). The household discounts by $\beta \xi_w$ because it is only interested in continuation histories in which it does not reoptimize its period t wage. Here, z indicates the household's disutility of labor:

$$z(h_{j,t}) = \frac{\psi_L}{1+\sigma_L} h_{j,t}^{1+\sigma_L}.$$

Also, λ_{t+i} represents the marginal value of a unit of currency to the household in period $t+i$. Given the utility function

$$\lambda_t = \frac{1}{P_t c_t}.$$

Substituting out for hours worked using the labor demand curve, we obtain:

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \{-z(l_{t+i} \left[\frac{W_{t+i,j}}{W_{t+i}}\right]^{\frac{\lambda_w}{1-\lambda_w}}) + \lambda_{t+i} W_{j,t+i} l_{t+i} \left[\frac{W_{t+i,j}}{W_{t+i}}\right]^{\frac{\lambda_w}{1-\lambda_w}} \}.$$

This can be written

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \{-z(l_{t+i} \left[\frac{W_{t+i,j}}{W_{t+i}}\right]^{\frac{\lambda_w}{1-\lambda_w}}) + \lambda_{t+i} P_{t+i} \frac{W_{t+i}}{P_{t+i}} l_{t+i} \left[\frac{W_{t+i,j}}{W_{t+i}}\right]^{\frac{\lambda_w}{1-\lambda_w}+1} \}.$$

To express the household's objective in terms of $\tilde{W}_t$, it is necessary to express $W_{t+i,j}$ in terms of its period t value. We adopt the following definitions:

$$\tilde{w}_t = \frac{W_t}{P_t},\ w_t = \frac{\tilde{W}_t}{W_t},\ \lambda_{z,t+i} = \lambda_{t+i} P_{t+i}.$$

Then,

$$\frac{W_{t+i,j}}{W_{t+i}} = \frac{\tilde{\pi}_{w,t+i} \cdots \tilde{\pi}_{w,t+1} \tilde{W}_t}{\tilde{w}_{t+i} P_{t+i}} = \frac{\tilde{\pi}_{w,t+i} \cdots \tilde{\pi}_{w,t+1} \tilde{W}_t}{\tilde{w}_{t+i} \pi_{t+i} \cdots \pi_{t+1} P_t} = \frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i},$$

where

$$X_{t,i} \equiv \frac{\tilde{\pi}_{w,t+j} \cdots \tilde{\pi}_{w,t+1}}{\pi_{t+i} \cdots \pi_{t+1}}.$$

Substituting this into the household's objective,

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \{-z(l_{t+i} \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}}) + \lambda_{z,t+i} \tilde{w}_{t+i} l_{t+i} \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}+1} \}.$$

The variable that the household must choose is w_t (note, whether the household is viewed as choosing w_t or $\tilde{W}_t$ makes no difference, since w_t is $\tilde{W}_t$ scaled by a variable over which the household has no control). Maximizing the household's objective with respect to w_t, we obtain:

$$\begin{aligned} E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \{-z'_{t+i} \frac{\lambda_w}{1-\lambda_w} l_{t+i} \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}-1} \frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} & \\ +\lambda_{z,t+i} \tilde{w}_{t+i} l_{t+i} \left(\frac{\lambda_w}{1-\lambda_w}+1\right) \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\} &= 0 \end{aligned}$$

or, after rearranging,

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}} \{\frac{\lambda_{z,t+i}}{\lambda_w} w_t \tilde{w}_t X_{t,i} - z'_{t+i}\} = 0$$

The marginal utility of leisure can be written, after taking into account that labor must always be on the demand curve

$$z'_{t+i} \equiv z'\left(h_{j,t+i}\right) = \psi_L \left(l_{t+i} \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i}\right]^{\frac{\lambda_w}{1-\lambda_w}}\right)^{\sigma_L}$$

Then, the first order condition reduces to

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \, l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}} \{ \frac{\lambda_{z,t+i}}{\lambda_w} w_t \tilde{w}_t X_{t,i} - \psi_L \left(l_{t+i} \left[\frac{w_t \tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}} \right)^{\sigma_L} \} = 0$$

Since w_t is not a random variable at time t, we can multiply through the expectation operator by it or by any power of it. Multiplying by $(w_t)^{-\frac{\lambda_w}{1-\lambda_w}\sigma_L}$,

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \, l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}} \{ \frac{\lambda_{z,t+i}}{\lambda_w} (w_t)^{\left[1-\frac{\lambda_w}{1-\lambda_w}\sigma_L\right]} \tilde{w}_t X_{t,i} - \psi_L \left(l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}} \right)^{\sigma_L} \} = 0$$

or,

$$K_{w,t} = \frac{1}{\psi_L} (w_t)^{\left(1-\frac{\lambda_w}{1-\lambda_w}\sigma_L\right)} \tilde{w}_t F_{w,t}.$$

where

$$\begin{aligned}
K_{w,t} &= E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i (l_{t+i})^{1+\sigma_L} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \\
F_{w,t} &= E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \, l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+i}} X_{t,i} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+i}}{\lambda_w} X_{t,i}.
\end{aligned}$$

Then,

$$w_t = \left[\frac{\psi_L}{\tilde{w}_t} \frac{K_{w,t}}{F_{w,t}} \right]^{\frac{1-\lambda_w}{1-(1+\lambda_w)\sigma_L}}. \tag{5.4}$$

The infinite sums, $K_{w,t}$ and $F_{w,t}$, have a recursive representations. It is crucial to exploit this fact, for computational tractability. Thus

$$\begin{aligned}
F_{w,t} &= l_t \frac{\lambda_{z,t}}{\lambda_w} + (\beta \xi_w) \, l_{t+1} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+1}}{\lambda_w} (X_{t,1})^{\frac{1}{1-\lambda_w}} \\
&+ (\beta \xi_w)^2 \, l_{t+2} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+2}}{\lambda_w} (X_{t+1,1} X_{t,1})^{\frac{1}{1-\lambda_w}} \\
&(\beta \xi_w)^3 \, l_{t+3} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \frac{\tilde{w}_{t+2}}{\tilde{w}_{t+3}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+3}}{\lambda_w} (X_{t+1,2} X_{t,1})^{\frac{1}{1-\lambda_w}} \\
&+ \cdots \\
&+ (\beta \xi_w)^i \, l_{t+i} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \cdots \frac{\tilde{w}_{t+i-1}}{\tilde{w}_{t+i}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+i}}{\lambda_w} (X_{t+1,i} X_{t,1})^{\frac{1}{1-\lambda_w}} \\
&+ \cdot\cdot,
\end{aligned}$$

or,

$$\begin{aligned}
F_{w,t} &= l_t \frac{\lambda_{z,t}}{\lambda_w} + \left[(\beta \xi_w) \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \right]^{\frac{\lambda_w}{1-\lambda_w}} (X_{t,1})^{\frac{1}{1-\lambda_w}} \right] \{ l_{t+1} \frac{\lambda_{z,t+1}}{\lambda_w} \\
&\quad + (\beta \xi_w) \, l_{t+2} \left[\frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+2}}{\lambda_w} (X_{t+1,1})^{\frac{1}{1-\lambda_w}} \\
&\quad (\beta \xi_w)^2 \, l_{t+3} \left[\frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \frac{\tilde{w}_{t+2}}{\tilde{w}_{t+3}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+3}}{\lambda_w} (X_{t+1,2})^{\frac{1}{1-\lambda_w}} \\
&\quad + \cdots \\
&\quad + (\beta \xi_w)^{i-1} \, l_{t+i} \left[\frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} \cdots \frac{\tilde{w}_{t+i-1}}{\tilde{w}_{t+i}} \right]^{\frac{\lambda_w}{1-\lambda_w}} \frac{\lambda_{z,t+i}}{\lambda_w} (X_{t+1,i})^{\frac{1}{1-\lambda_w}} \\
&\quad + \cdots \} \\
&= l_t \frac{\lambda_{z,t}}{\lambda_w} + \left(\beta \xi_w \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \right]^{\frac{\lambda_w}{1-\lambda_w}} X_{t,1}^{\frac{1}{1-\lambda_w}} \right) F_{w,t+1}. \qquad (5.5)
\end{aligned}$$

Note,

$$\pi_{w,t} \equiv \frac{W_t}{W_{t-1}} = \frac{\tilde{w}_t P_t}{\tilde{w}_{t-1} P_{t-1}} = \frac{\tilde{w}_t \pi_t}{\tilde{w}_{t-1}},$$

so that $\tilde{w}_t / \tilde{w}_{t-1} = \pi_{w,t} / \pi_t$. Substituting and rearranging,

$$F_{w,t} = l_t \frac{\lambda_{z,t}}{\lambda_w} + \beta \xi_w \left(\frac{1}{\pi_{w,t+1}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \frac{\tilde{\pi}_{w,t+1}^{\frac{1}{1-\lambda_w}}}{\pi_{t+1}} F_{w,t+1}, \qquad (5.6)$$

which corresponds to the expression we have worked with in the past. (But, (5.5) is simpler!) Now consider $K_{w,t}$:

$$\begin{aligned}
K_{w,t} &= l_t^{1+\sigma_L} + \beta \xi_w (l_{t+1})^{1+\sigma_L} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} X_{t,1} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \\
&\quad + (\beta \xi_w)^2 (l_{t+2})^{1+\sigma_L} \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} \frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} X_{t+1,2} X_{t,1} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \\
&\quad + \cdots \\
&= l_t^{1+\sigma_L} + \beta \xi_w \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} X_{t,1} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \{ (l_{t+1})^{1+\sigma_L} \\
&\quad + \beta \xi_w (l_{t+2})^{1+\sigma_L} \left[\frac{\tilde{w}_{t+1}}{\tilde{w}_{t+2}} X_{t+1,2} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} + \cdots \} \\
&= l_t^{1+\sigma_L} + \beta \xi_w \left[\frac{\tilde{w}_t}{\tilde{w}_{t+1}} X_{t,1} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} K_{w,t+1} \\
&= l_t^{1+\sigma_L} + \beta \xi_w \left[\frac{\tilde{\pi}_{w,t+1}}{\pi_{w,t+1}} \right]^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} K_{w,t+1}. \qquad (5.7)
\end{aligned}$$

In order for (5.5) and (5.7) to be well-defined objects, it is necessary that they be finite. This requires that the 'discount rate' in (5.5) and (5.7) be less than unity in steady state:

$$\beta\xi_w \left(\frac{\tilde{\pi}_{w,t+1}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_w}},\ \beta\xi_w \left(\frac{\tilde{\pi}_{w,t+1}}{\pi_{w,t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} < 1,$$

or, because of (5.3) and that $\pi_{w,t} = \pi_t$ in steady state,

$$\beta\xi_w \left(\frac{\pi}{\bar{\pi}}\right)^{\frac{1-\iota_{w,2}}{\lambda_w-1}},\ \beta\xi_w \left(\frac{\pi}{\bar{\pi}}\right)^{\frac{1-\iota_{w,2}}{\lambda_w-1}\lambda_w(1+\sigma_L)} < 1.$$

We have completed the derivation of the wage rate from the household's first order condition. We now identify a consistency condition that must hold across all household wages, which will allow us to express the real wage, $\tilde{w}_t$, just in terms of aggregate variables. The object, $w_t = \tilde{W}_t/W_t$, will disappear from the analyis. Substituting the demand curve for the j^{th} specialized input, (5.2) into (5.1), we obtain

$$\begin{aligned} l_t &= \left[\int_0^1 \left[l_t \left(\frac{W_{t,j}}{W_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1}{\lambda_w}} dj\right]^{\lambda_w} \\ &= l_t \left(W_t\right)^{\frac{\lambda_w}{\lambda_w-1}} \left[\int_0^1 \left(W_{t,j}\right)^{\frac{1}{1-\lambda_w}} dj\right]^{\lambda_w}, \end{aligned}$$

or,

$$\left(W_t\right)^{\frac{\lambda_w}{1-\lambda_w}} = \left[\int_0^1 \left(W_{t,j}\right)^{\frac{1}{1-\lambda_w}} dj\right]^{\lambda_w}$$

so that the condition across all wages is:

$$\begin{aligned} W_t &= \left[\int_0^1 \left(W_{t,j}\right)^{\frac{1}{1-\lambda_w}} dj\right]^{1-\lambda_w} \\ &= \left[\int_{1-\xi_w} \left(W_{t,j}\right)^{\frac{1}{1-\lambda_w}} dj + \int_{\xi_w} \left(W_{t,j}\right)^{\frac{1}{1-\lambda_w}} dj\right]^{1-\lambda_w}. \end{aligned}$$

In the limits of integration, $1-\xi_w$ refers to the households that reoptimize in period t, while ξ_w refers to the households that do not. Making use of the fact that whether households are selected to optimize or not is determined randomly, we can simplify the previous expression as follows:

$$W_t = \left[\left(1-\xi_w\right)\left(\tilde{W}_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\tilde{\pi}_{w,t} W_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.$$

Divide both sides by W_t,

$$1 = \left[\left(1-\xi_w\right)\left(w_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w},$$

or, after rearranging,

$$w_t = \left[\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right]^{1-\lambda_w}. \tag{5.8}$$

Combining the optimality condition on w_t, (5.4), with the consistency condition, (5.8),

$$\left[\frac{\psi_L}{\tilde{w}_t} \frac{K_{w,t}}{F_{w,t}} \right]^{\frac{1-\lambda_w}{1-(1+\lambda_w)\sigma_L}} = \left[\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{t-1}}{\pi_t} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right]^{1-\lambda_w}$$

or,

$$K_{w,t} = \frac{1}{\psi_L} F_{w,t} \tilde{w}_t \left[\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{t-1}}{\pi_t} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right]^{(1-(1+\lambda_w)\sigma_L)}, \tag{5.9}$$

which is an expression which relates the real wage to aggregate variables only. It is interesting to consider the case, $\xi_w = 0$, when there are no sticky wages. In this case, (5.9) reduces to

$$\psi_L \frac{K_{w,t}}{F_{w,t}} = \tilde{w}_t,$$

or, after substituting out for $K_{w,t}$ and $F_{w,t}$ and rearranging,

$$\lambda_w \frac{\psi_L l_t^{\sigma_L}}{\lambda_{z,t}} = \tilde{w}_t,$$

which says that the real wage in units of the consumption good, $\tilde{w}_t$, is a markup, λ_w, above the household's marginal cost, $\psi_L l_t^{\sigma_L} / \lambda_{z,t}$, also expressed in terms of the consumption good. This is exactly what we would expect.

Sticky wages also have an impact on household utility. Cross-household dispersion in wages lead to cross household dispersion in labor effort and therefore in utility. It can be verified that the cross-sectional average of utility in period t is:

$$\log(c_t) - \frac{\psi_L}{1+\sigma_L} \left(\frac{w_t^*}{w_t^+} \right)^{\frac{\lambda_w(1+\sigma_L)}{\lambda_w - 1}} h_t^{1+\sigma_L},$$

where h_t is the unweighted sum of household hours and w_t^+ satisfies the following equation:

$$w_t^+ = \left[(1-\xi_w) \left(\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right)^{\lambda_w(1+\sigma_L)} + \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} w_{t-1}^+ \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}}. \tag{5.10}$$

For purposes of computing the steady state, it is convenient to wite (5.10) as follows:

$$\left(w_t^+ \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} = (1-\xi_w) \left(\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right)^{\lambda_w(1+\sigma_L)} + \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \left(w_{t-1}^+ \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}}.$$

From this expression, it is evident that existence of a steady state will require

$$\xi_w \left(\frac{\pi_{w,t}}{\tilde{\pi}_{w,t}} \right)^{\frac{\lambda_w(1+\sigma_L)}{\lambda_w-1}} = \xi_w \left(\frac{\tilde{w}_t \pi_t}{\tilde{w}_{t-1}} \frac{1}{(\pi_{t-1})^{\iota_2} \bar{\pi}^{1-\iota_2}} \right)^{\frac{\lambda_w(1+\sigma_L)}{\lambda_w-1}} = \xi_w \left(\frac{\pi}{\bar{\pi}} \right)^{(1-\iota_2)\frac{\lambda_w(1+\sigma_L)}{\lambda_w-1}} < 1,$$

where π denotes the actual steady state inflation rate and $\bar{\pi}$ denotes the constant in the price updating equation. A 'natural' specification might be $\bar{\pi} = 1$. However, note that if we set λ_w small market then the power in the above expression can be quite large. For example, if $\lambda_w = 1.05$, then $\lambda_w/(\lambda_w - 1) = 21$. Then, if ι_2 is small (say, 0.13) π has to be only a little above unity for the condition to be violated. For example, suppose $\pi = 1.0092$ (a 3.7 percent annual inflation rate), $\iota_{w,2} = 0.13$, $\lambda_w = 1.05$, $\sigma_L = 1$, then

$$\xi_w \left(\frac{\pi}{\bar{\pi}} \right)^{(1-\iota_{w,2})\frac{\lambda_w(1+\sigma_L)}{\lambda_w-1}} = 1.16.$$

5.2. Aggregate Resource Constraint

We now develop a relationship linking aggregate homogeneous labor effort in the goods market, l_t, to aggregate household employment, h_t,

$$\begin{aligned} h_t &= \int_0^1 h_{j,t} dj = l_t \int_0^1 \left[\frac{W_{t,j}}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}} dj \\ &= l_t \left(w_t^* \right)^{\frac{\lambda_w}{1-\lambda_w}}, \end{aligned} \tag{5.11}$$

say, where

$$\begin{aligned} \left(w_t^* \right)^{\frac{\lambda_w}{1-\lambda_w}} &= \int_0^1 \left[\frac{W_{t,j}}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}} dj \\ &= \int_{1-\xi_w} \left[\frac{W_{t,j}}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}} dj + \int_{\xi_w} \left[\frac{W_{t,j}}{W_t} \right]^{\frac{\lambda_w}{1-\lambda_w}} dj \\ &= (1-\xi_w) w_t^{\frac{\lambda_w}{1-\lambda_w}} + \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \int_{\xi_w} \left[\frac{W_{t-1,j}}{W_{t-1}} \right]^{\frac{\lambda_w}{1-\lambda_w}} dj \\ &= (1-\xi_w) w_t^{\frac{\lambda_w}{1-\lambda_w}} + \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} w_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}}, \end{aligned}$$

so that

$$w_t^* = \left[(1-\xi_w) w_t^{\frac{\lambda_w}{1-\lambda_w}} + \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} w_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

We substitute out the expression for w_t using (5.8):

$$w_t^* = [(1-\xi_w) \left(\frac{1 - \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} \right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w} \right)^{\lambda_w} + \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}} w_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}}]^{\frac{1-\lambda_w}{\lambda_w}}. \tag{5.12}$$

In order for w_t^* to have a well-defined steady state value, we require that the coefficient on $\left(w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}$ be less than unity:

$$\xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}}\right)^{\frac{\lambda_w}{1-\lambda_w}} < 1$$

or, in steady state:

$$\xi_w \left(\frac{\pi}{\bar{\pi}}\right)^{(1-\iota_{w,2})\frac{\lambda_w}{\lambda_w-1}} < 1.$$

We now can write the resource constraint in terms of aggregate household employment like this:

$$c_t + \bar{k}_t - (1-\delta)\bar{k}_{t-1} = (p_t^*)^{\frac{\lambda_f}{\lambda_f-1}} \left\{ \epsilon_t \bar{k}_{t-1}^{\alpha} \left[(w_t^*)^{\frac{\lambda_w}{\lambda_w-1}} h_t \right]^{1-\alpha} - \phi \right\}, \tag{5.13}$$

where w_t^* satisfies (5.12). Note that when there are no sticky wages, so that $\xi_w = 0$, then $w_t^* = 1$ and no adjustment is made.

Combining (5.6), (5.7), (5.9) and (5.11), we obtain:

$$E_t\left\{ \frac{(w_t^*)^{\frac{\lambda_w}{\lambda_w-1}} h_t}{\lambda_w c_t} + \beta\xi_w \left(\tilde{\pi}_{w,t+1}\right)^{\frac{1}{1-\lambda_w}} \frac{\left(\frac{1}{\pi_{w,t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi_{t+1}} F_{w,t+1} - F_{w,t} \right\} = 0 \tag{5.14}$$

$$\begin{aligned} E_t\Bigg\{ \left[(w_t^*)^{\frac{\lambda_w}{\lambda_w-1}} h_t \right]^{1+\sigma_L} &+ \beta\xi_w \left(\frac{\tilde{\pi}_{w,t+1}}{\pi_{w,t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \frac{1}{\psi_L} \left[\frac{1-\xi_w \left(\frac{\tilde{\pi}_{w,t+1}}{\pi_{w,t+1}}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w} \right]^{1-\lambda_w(1+\sigma_L)} \tilde{w}_{t+1} F_{w,t+1} \\ &- \frac{1}{\psi_L} \left[\frac{1-\xi_w \left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w} \right]^{1-\lambda_w(1+\sigma_L)} \tilde{w}_t F_{w,t} \Bigg\} = 0 \end{aligned} \tag{5.15}$$

5.3. The Other Equilibrium Conditions

Sticky wages has replaced the labor supply curve, (3.11), with the equilibrium conditions in the previous subsection. above. In terms of the equilibrium conditions for the version of the model with just sticky prices, this means that we must replace (3.17) by (3.10). Also, if we are to write the firm efficiency conditions in terms of hours supplied by workers, h_t, then we must use (5.11). Thus, equation (3.10) must be replaced by

$$s_t = \frac{\tilde{w}_t}{(1-\alpha)\,\epsilon_t} \left(\frac{(w_t^*)^{\frac{\lambda_w}{\lambda_w-1}} h_t}{k_{t-1}} \right)^{\alpha} \tag{5.16}$$

In addition, the intertemporal Euler equation for the household, with the rental rate of capital substituted out using the firm marginal cost condition, must be replaced by:

$$-\frac{1}{c_t} + \frac{\beta}{c_{t+1}} \left[\alpha\epsilon_{t+1} \left(\frac{\left(w_{t+1}^*\right)^{\frac{\lambda_w}{\lambda_w-1}} h_{t+1}}{k_t} \right)^{1-\alpha} s_{t+1} + (1-\delta) \right] = 0 \tag{5.17}$$

Finally, the pricing equations need to be modified, given the new definition of $Y_{z,t}$ (see (5.13)).

$$E_t \left\{ \frac{1}{c_t} \left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\epsilon_t \bar{k}_{t-1}^{\alpha} \left(\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h_t \right)^{1-\alpha} - \phi \right] + \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}} \right)^{\frac{1}{1-\lambda_f}} \beta \xi_p F_{p,t+1} - F_{p,t} \right\} = 0, \tag{5.18}$$

and

$$\frac{1}{c_t} \lambda_f \left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\epsilon_t \bar{k}_{t-1}^{\alpha} \left(\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h_t \right)^{1-\alpha} - \phi \right] s_t + \tag{5.19}$$

$$\beta \xi_p \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}} \right)^{\frac{\lambda_f}{1-\lambda_f}} \left[\frac{1 - \xi_p \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}} \right)^{\frac{1}{1-\lambda_f}}}{1 - \xi_p} \right]^{1-\lambda_f} F_{p,t+1} - F_{p,t} \left[\frac{1 - \xi_p \left(\frac{\pi_{t-1}^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_t} \right)^{\frac{1}{1-\lambda_f}}}{1 - \xi_p} \right]^{1-\lambda_f} = 0$$

Note that for $F_{p,t}$ and $K_{p,t}$ to be finite, it is necessary that the relevant discount rates be less than unity in steady state:

$$\left(\frac{\pi}{\bar{\pi}} \right)^{\frac{1-\iota_2}{\lambda_f - 1}} \beta \xi_p, \ \beta \xi_p \left(\frac{\pi}{\bar{\pi}} \right)^{(1-\iota_2)\frac{\lambda_f}{\lambda_f - 1}} < 1.$$

We now have $N = 11$ unknowns, $(k, s, h, c, \pi, F_p, p^*, F_w, \tilde{w}, w^*, w^+)$. The $N-1$ equilibrium conditions are (5.16), (5.17), (5.13), (3.20), (5.18), (5.19), (5.12), (5.14), (5.15) and (5.10). These equilibrium conditions have been entered into the Dynare file, newsimplemodel.mod, in subdirectory stickypriceswages.

5.4. Analysis of the Equilibrium

With both sticky prices and wages, it should be clear that the RBC equilibrium allocations are not attainable. From the point of view of the wedges in the resource constraint, this would require setting π_t and $\pi_{w,t}$ to $\bar{\pi}$, but this would in effect fix the real wage, and the efficient allocations require that the real wage react to shocks.

Consider a case in which we have a hope that the Ramsey equilibrium coincides with the RBC equilibrium. Suppose there are only sticky wages and no sticky prices, so that $\xi_p = 0$, $\xi_w > 0$. Let's see if we can identify an equilibrium that coincides with the equilibrium in the RBC economy. Suppose we set:

$$\tilde{\pi}_{w,t} = \pi_{w,t}. \tag{5.20}$$

By (5.12), this implies that, if $w_{-1}^* = 1$, then $w_t^* = 1$ for all $t \geq 0$. This will be necessary if we're to reproduce the RBC model allocations, because w_t^* appears as a wedge in several places and setting it to unity eliminates those wedges. For convenience, we repeat the definitions of the objects in (5.20):

$$\pi_{w,t} \equiv \frac{W_t}{W_{t-1}} = \frac{\tilde{w}_t P_t}{\tilde{w}_{t-1} P_{t-1}} = \frac{\tilde{w}_t \pi_t}{\tilde{w}_{t-1}}, \ \tilde{\pi}_{w,t} \equiv \left(\pi_{t-1}\right)^{\iota_{w,2}} \bar{\pi}^{1-\iota_{w,2}}.$$

Evidently, achieving $\tilde{\pi}_{w,t} = \pi_{w,t}$ requires that there be no indexing to inflation, i.e., $\iota_{w,2} = 0$ (obviously, wages can't be the same across households - as efficiency requires - if some households link their wages to inflation, which must be time varying to ensure real wage flexibility.) Now, consider (5.14) and (5.15). Imposing (5.20):

$$E_t\{\frac{h_t}{\lambda_w c_t} + \beta\xi_w \frac{\tilde{\pi}_{w,t+1}}{\pi_{t+1}} F_{w,t+1} - F_{w,t}\} = 0$$

$$E_t\{h_t^{1+\sigma_L} + \beta\xi_w \frac{1}{\psi_L}\tilde{w}_{t+1}F_{w,t+1} - \frac{1}{\psi_L}\tilde{w}_t F_{w,t}\} = 0$$

or, using,

$$\frac{\tilde{w}_t \pi_{w,t+1}}{\pi_{t+1}} = \tilde{w}_{t+1},\ \tilde{\pi}_{w,t+1} = \pi_{w,t+1}$$

these reduce to:

$$\frac{h_t}{\lambda_w c_t} = -E_t\left[\beta\xi_w \frac{\tilde{\pi}_{w,t+1}}{\pi_{t+1}} F_{w,t+1} - F_{w,t}\right]$$

$$h_t^{1+\sigma_L} = -\frac{1}{\psi_L}\tilde{w}_t E_t\left[\beta\xi_w \frac{\tilde{\pi}_{w,t+1}}{\pi_{t+1}} F_{w,t+1} - F_{w,t}\right].$$

Equating these two, we obtain:

$$h_t^{1+\sigma_L} = \frac{1}{\psi_L}\tilde{w}_t \frac{h_t}{\lambda_w c_t}, \tag{5.21}$$

or,

$$\psi_L h_t^{\sigma_L} c_t = \frac{\tilde{w}_t}{\lambda_w}.$$

This is just the static efficiency condition for households in the RBC model, if $\lambda_w = 1$. Presumably, efficiency of our proposed policy will require $\lambda_w = 1$.

Now let's pursue the implications of $\xi_p = 0$. By (5.18) and (5.19) (and, that $p_t^* = 1$ by (3.20)),

$$\frac{1}{c_t}\left[\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right] = F_{p,t},$$

and

$$\frac{1}{c_t}\lambda_f\left[\epsilon_t \bar{k}_{t-1}^{\alpha} h_t^{1-\alpha} - \phi\right] s_t = F_{p,t}.$$

Then, equating these two, we obtain:

$$s_t = \frac{1}{\lambda_f}.$$

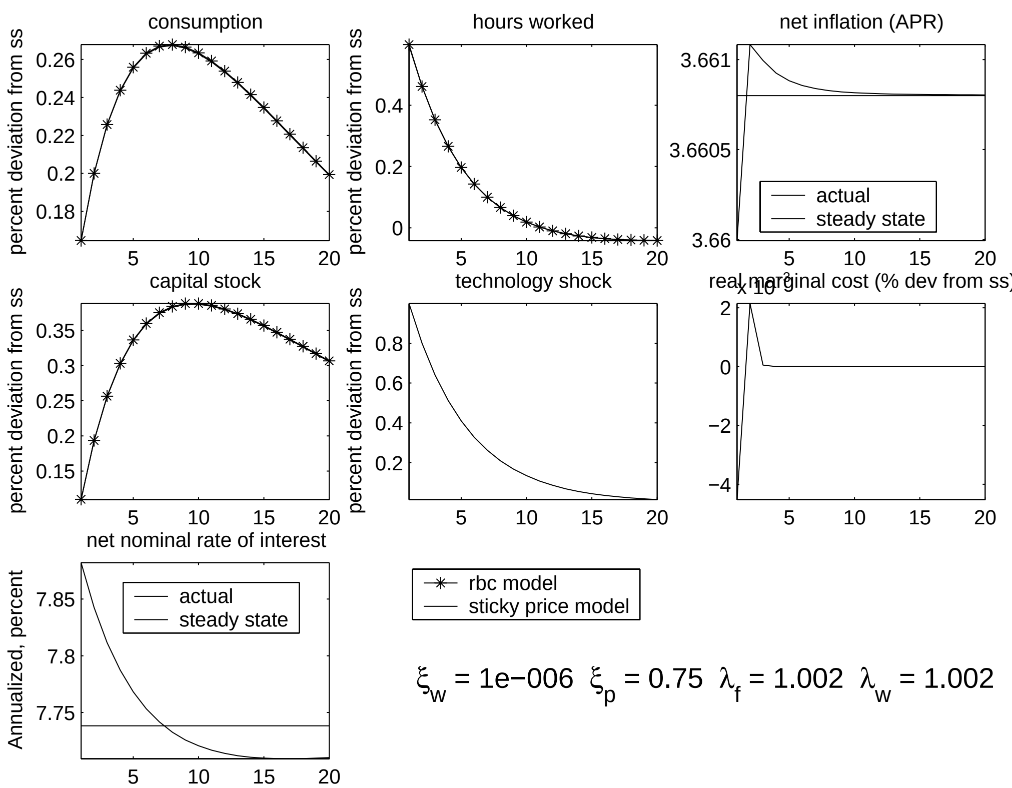
If $\lambda_f = 1$, then (5.21) and (5.16) imply that the intratemporal Euler equation in the RBC model is satisfied. We conclude that if $\lambda_f = \lambda_w = 1$, $\iota_{w,2} = 0$, then an equilibrium in which $\tilde{\pi}_{w,t} = \pi_{w,t} = \bar{\pi}$ for all t has the property that all three efficiency conditions of the RBC model are satisfied: (i) the resource constraint, (5.13), reduces to the RBC resource constraint, (ii) conditions (3.10), (5.21) and $\lambda_f = 1$ imply that the RBC intratemporal efficiency condition hols and (iii) condition (5.17) with $w_t^* = s_t = 1$ corresponds to the RBC model intertemporal equation. Since the three efficiency conditions of the RBC model uniquely (together with a boundedness condition) characterize the best possible allocations given preferences and technology, it follows that under the stated conditions, $\tilde{\pi}_{w,t} = \pi_{w,t} = \bar{\pi}$ is the Ramsey policy. We summarize these results in the form of a proposition

Proposition 5.1. *If $\lambda_f = \lambda_w = 1$, $\xi_p = \iota_{w,2} = 0$, then the Ramsey allocations coincide with those of the RBC economy, and $\pi_{w,t} = \bar{\pi}$.*

We now compare the Ramsey and the RBC allocations. Consider the following parameter values:

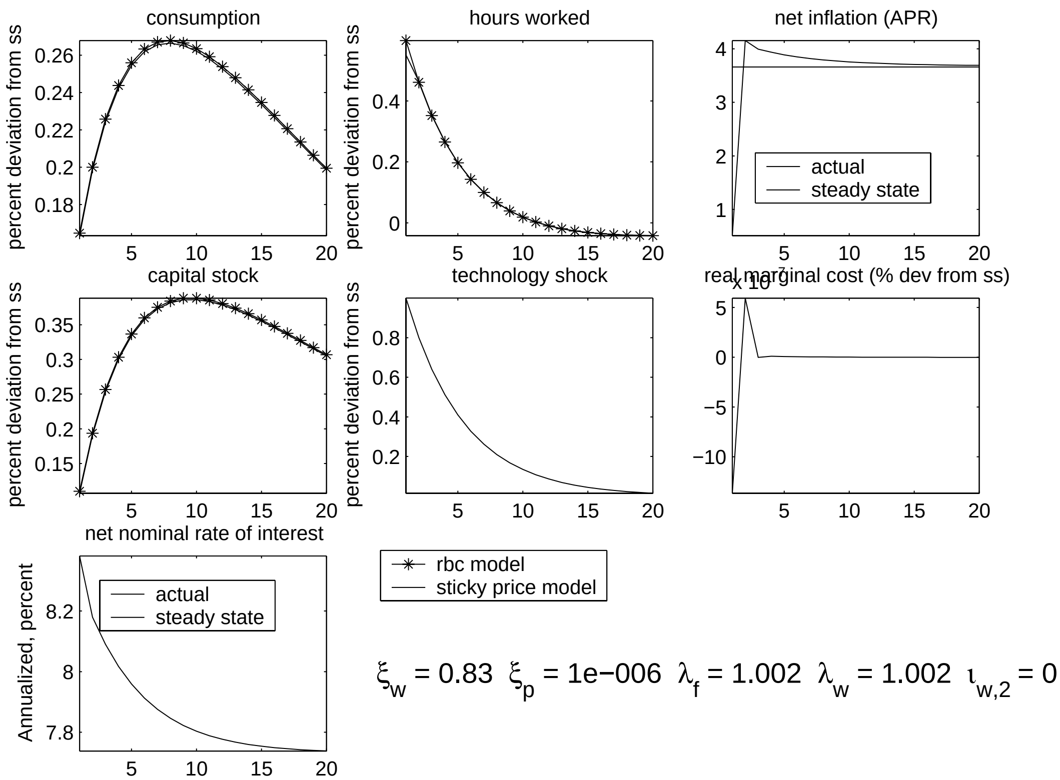
$$\beta = 0.99,\ \psi_L = 109.8,\ \lambda_f = 1.002,\ \alpha = 0.40,\ \delta = 0.025,\ \xi_p = 0.75,\ \iota_2 = 0.6,\ \lambda_w = 1.05,\ \sigma_L = 1.$$

We consider several special cases. We begin with the case, $\xi_w \simeq 0$, $\lambda_f = \lambda_w = 1.002$, when prices are sticky and wages are not, and monopoly power is minimized. The Ramsey and RBC economy responses to a technology shock are presented below.



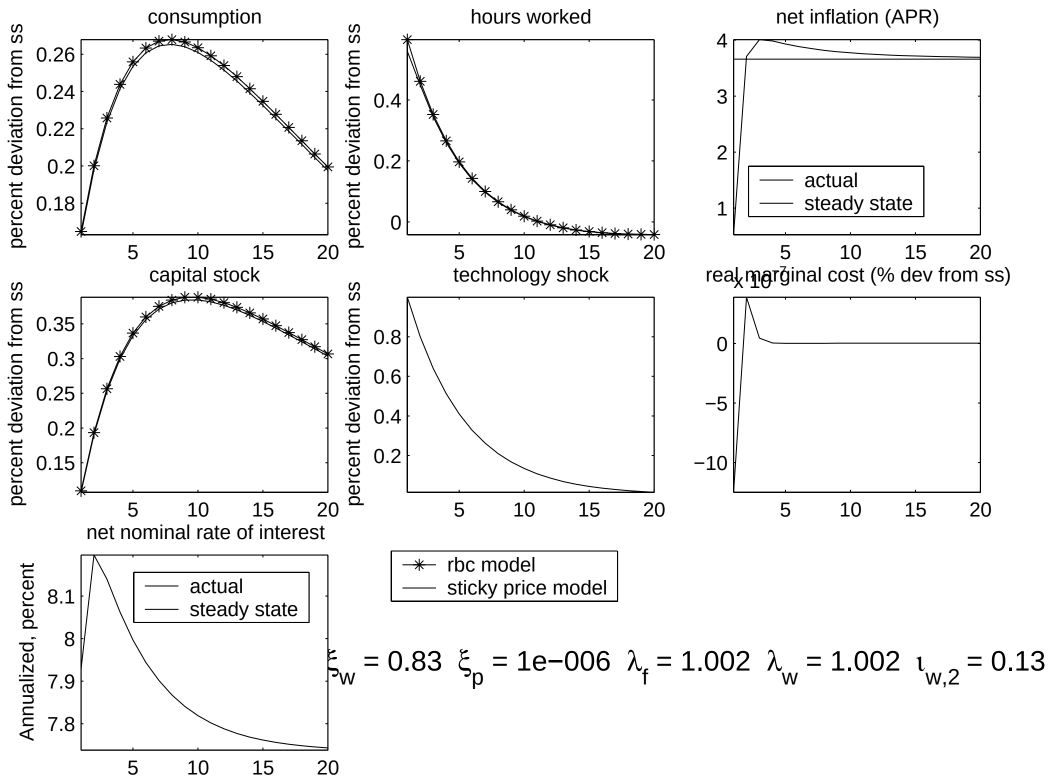
Note that the Ramsey and RBC allocations virtually coincide, as expected

Now we consider the opposite case, in which $\xi_w = 0.83$ and $\xi_p = 0.000001$:

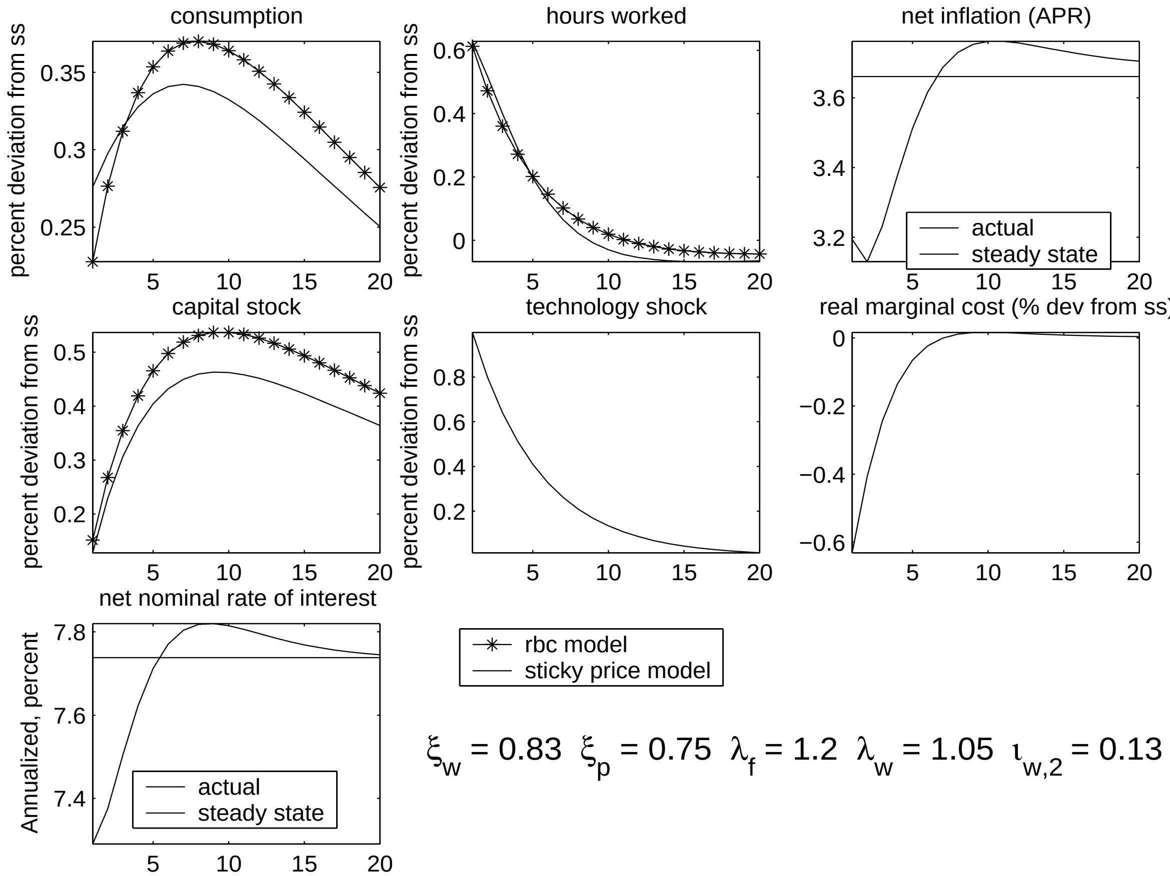


The two allocations are also very similar here, consistent with our proposition. Note now inflation drops sharply with the technology shock, to allow the real wage to rise. Marginal cost is essentially constant.

We now investigate how far from the RBC model you end up when the conditions for RBC=Ramsey are not satisfied. Here is what happens when you set $\iota_{w,2} = 0.13$. It makes virtually no difference, although there is some noticeable impact on R_t.



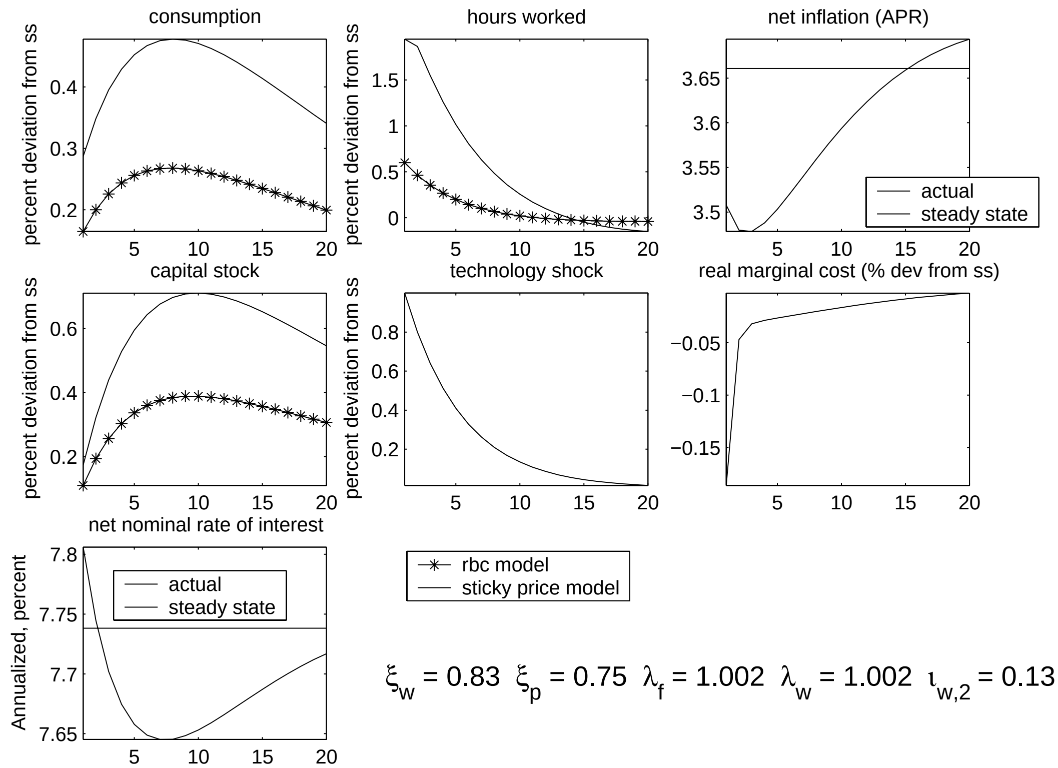
Now consider the case, $\xi_p = 0.75$, $\xi_w = 0.83$, $\lambda_f = 1.20$, $\lambda_w = 1.05$



This change has a more noticeable impact on the interest rate. And, predictably, it reduces the inflation impact of the shock. In terms of the impact on quantity allocations, the effect is rather small.

A surprising result was obtained by perturbing the previous parameterization, and setting

$\lambda_f = \lambda_w = 1.002$ and otherwise leaving the parameters unchanged. We obtained:



Note how much stronger the response of the allocations in the Ramsey equilibrium are!

6. Adding Habit Persistence and Investment Adjustment Costs to the Model

6.1. The Equilibrium Conditions

The marginal utility of consumption, $\lambda_{z,t}$, is now

$$E_t\left[\lambda_{z,t} - \frac{1}{c_t - bc_{t-1}} + b\beta \frac{1}{c_{t+1} - bc_t}\right] = 0. \tag{6.1}$$

With this change, $1/c_t$ must be replaced by $\lambda_{z,t}$ in (5.18), (5.19), and (5.14).

Adjustment costs in investment change the household first order condition for investment, (5.17):

$$-\lambda_{z,t} + \lambda_{z,t+1}\beta \frac{1}{q_t}\left[\alpha\epsilon_{t+1}\left(\frac{\left(w_{t+1}^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h_{t+1}}{k_t}\right)^{1-\alpha} s_{t+1} + q_{t+1}(1-\delta)\right] = 0, \tag{6.2}$$

and add a new first order condition for investment:

$$E_t\left[\lambda_{zt} q_t F_{1,t} - \lambda_{zt} + \beta\lambda_{zt+1} q_{t+1} F_{2,t+1}\right] = 0.$$

We suppose that

$$\begin{aligned} F(I_t, I_{t-1}) &= [1 - S(I_t/I_{t-1})]\, I_t \\ &= \left[1 - \frac{S''}{2}(\frac{I_t}{I_{t-1}} - 1)^2\right] I_t, \end{aligned}$$

so that

$$\begin{aligned} F_{1t} &= 1 - \frac{S''}{2}(\frac{I_t}{I_{t-1}} - 1)^2 - S''(\frac{I_t}{I_{t-1}} - 1)\frac{I_t}{I_{t-1}} \\ F_{2,t+1} &= S''(\frac{I_{t+1}}{I_t} - 1)\left(\frac{I_{t+1}}{I_t}\right)^2. \end{aligned}$$

Substituting this into the first order condition for investment:

$$\begin{aligned} E_t\{\lambda_{zt} q_t \left[1 - \frac{S''}{2}(\frac{I_t}{I_{t-1}} - 1)^2 - S''(\frac{I_t}{I_{t-1}} - 1)\frac{I_t}{I_{t-1}}\right] \\ -\lambda_{zt} + \beta\lambda_{zt+1} q_{t+1} S''(\frac{I_{t+1}}{I_t} - 1)\left(\frac{I_{t+1}}{I_t}\right)^2\} &= 0. \end{aligned} \tag{6.3}$$

With the investment adjustment costs, the resource constraint is changed

$$c_t + I_t = (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left\{\epsilon_t \bar{k}_{t-1}^{\alpha} \left[(w_t^*)^{\frac{\lambda_w}{\lambda_w - 1}} h_t\right]^{1-\alpha} - \phi\right\} \tag{6.4}$$

where

$$\bar{k}_t - (1-\delta)\bar{k}_{t-1} = \left[1 - \frac{S''}{2}(\frac{I_t}{I_{t-1}} - 1)^2\right] I_t \tag{6.5}$$

The equation defining the nominal rate of interest is:

$$E_t\{\beta \frac{1}{\pi_{t+1}} \lambda_{z,t+1}(1 + R_t) - \lambda_{z,t}\} = 0 \tag{6.6}$$

The variables to be determined in equilibrium are the following $N = 15$: R_t, $\lambda_{z,t}$, I_t, π_t, q_t, h_t, c_t, k_{t+1}, s_t, F_p, p^*, F_w, $\tilde{w}$, w^*, w^+. The $N - 1$ equations are (5.18), (5.19), (5.14) with $1/c_t$ replaced with $\lambda_{z,t}$, (5.15), (5.16), (3.20), (5.12), (5.10), (6.1), (6.2), (6.3), (6.6), (6.4) and (6.5).

With some algebra, it can be verified that the average, across all households, of period utility is:

$$\log(c_t - bc_{t-1}) - \frac{\psi_L}{1 + \sigma_L}\left(\frac{w_t^*}{w_t^+}\right)^{\frac{\lambda_w(1+\sigma_L)}{\lambda_w - 1}} h_t^{1+\sigma_L}.$$

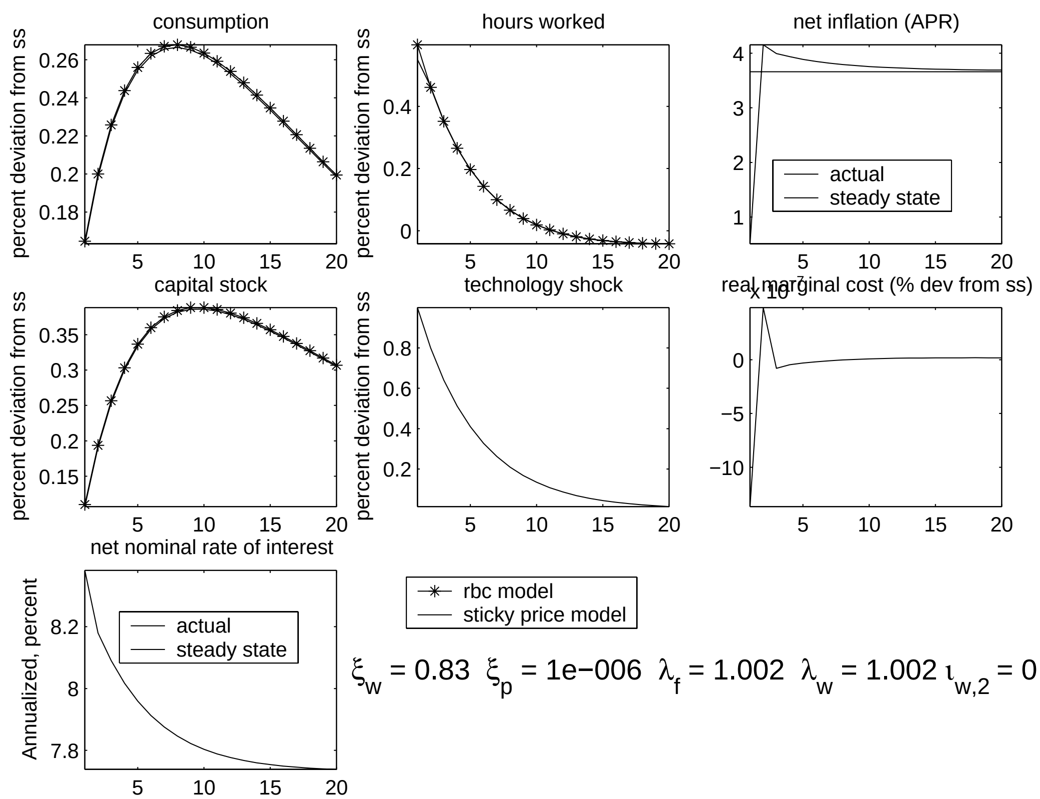
We use this utility function to define the objective in the Ramsey problem. The equilibrium conditions of this version of the model have been entered into the Dynare file, newsimplemodel.mod, in subdirectory sticypriceswageshabitadjust.

6.2. Analysis of Equilibrium

Consider the following parameter values:

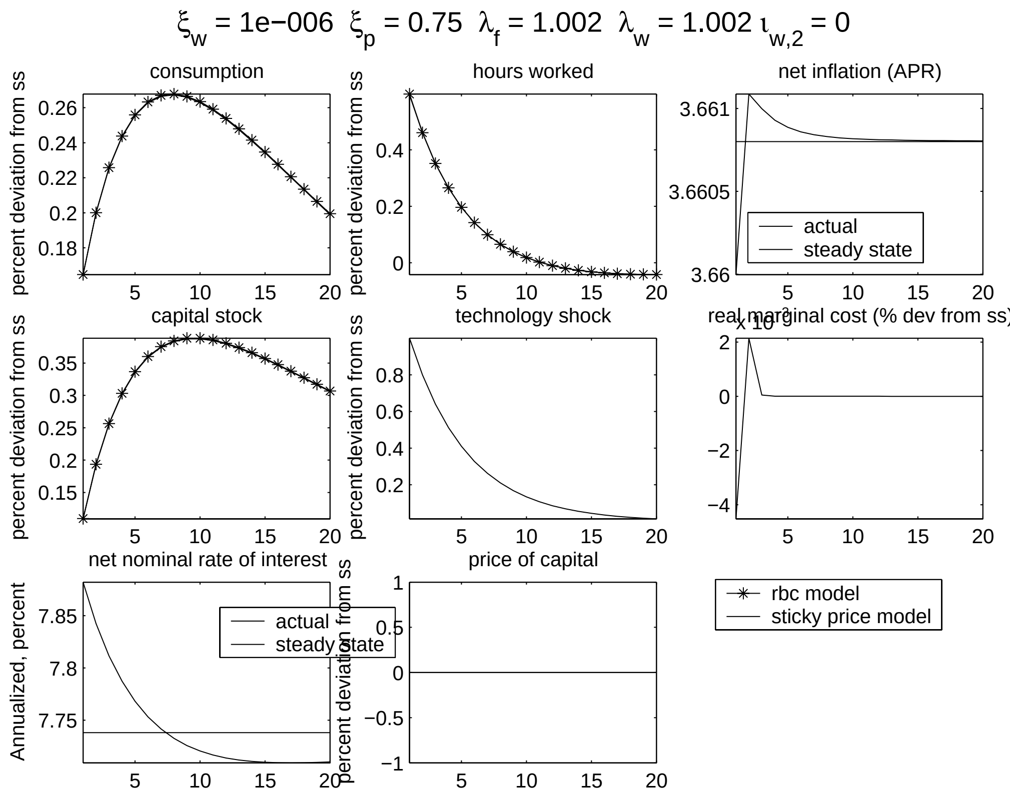
$$\begin{aligned} \beta &= 0.99,\ \psi_L = 109.8,\ \lambda_f = 1.2,\ \lambda_w = 1.05,\ \alpha = 0.40,\ \delta = 0.025, \\ \iota_2 &= 0.6,\ S'' = 5.1,\ \sigma_L = 1,\ b = 0.63,\ \iota_{w,2} = 0.13,\ \xi_w = 0.83,\ \xi_p = 0.75 \end{aligned} \tag{6.7}$$

There are special cases of this model worth considering. Consider, for example, the case, $S'' = 0.000001$, $\xi_w = 0.83$, $\lambda_f = \lambda_w = 1.002$, $b = 0$, $\xi_p = 0.000001$, $\iota_{w,2} = 0$. This corresponds to a case that we considered before, and it's worth verifying that we reproduce it now:



Consider now the 'opposite' from the above case, the one in which $\xi_p = 0.75$, $\xi_w =$

0.000001 :



This reproduces what we had before.

To establish a suitable benchmark for this model with habit persistence, we solve the version of the RBC model with habit persistence and adjustment costs (see rbcmodel.mod). The first order condition for investment is (6.3), which, in steady state implies $q_t = 1$. The intertemporal Euler equation for this model is (5.17) with $w_t^* = s_t = 1$. In steady state, that equation is:

$$\frac{1}{\beta} = \alpha \left(\frac{h}{k}\right)^{1-\alpha} + 1 - \delta,$$

or,

$$\frac{h}{k} = \left[\frac{\frac{1}{\beta} - (1-\delta)}{\alpha}\right]^{\frac{1}{1-\alpha}}.$$

The intratemporal Euler equation is, in steady state:

$$\lambda_z (1-\alpha)\left(\frac{k}{h}\right)^{\alpha} = \psi_L h^{\sigma_L}.$$

The resource constraint is (6.4). This, combined with the capital accumulation equation, (6.5), is, in steady state:

$$c = \left[\left(\frac{h}{k}\right)^{1-\alpha} - \delta\right] k.$$

The marginal utility of consumption is (6.1), which, in steady state is:

$$\lambda_z = \frac{1 - b\beta}{c\,(1-b)}.$$

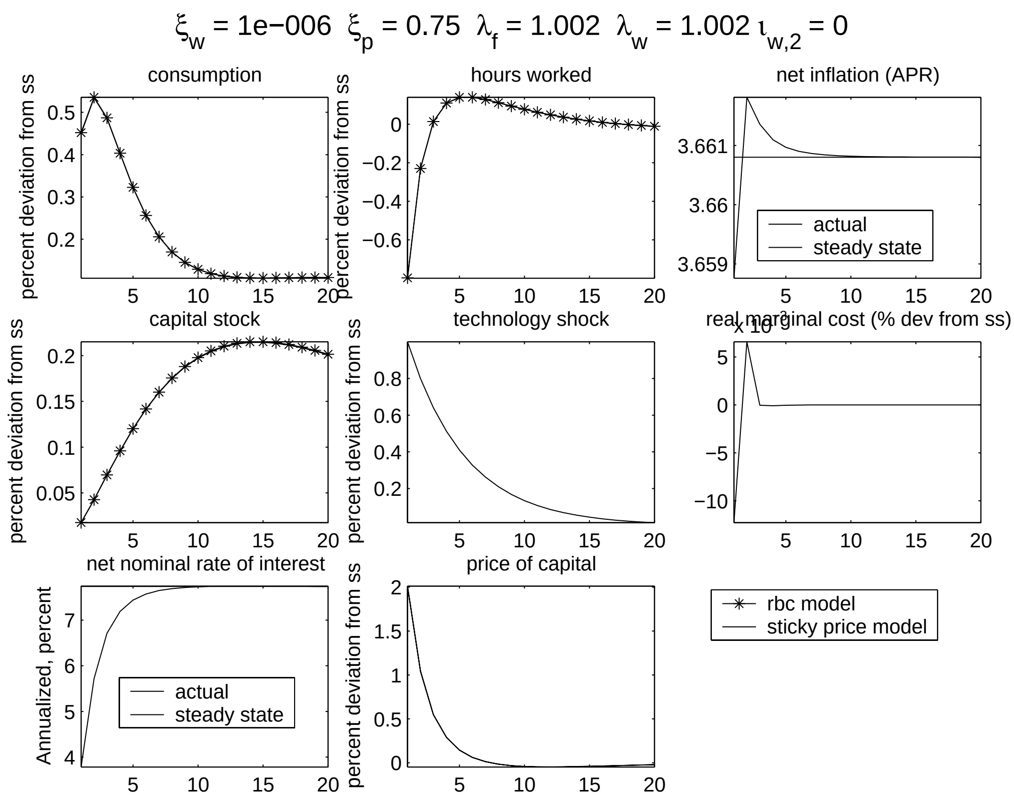
Combining the previous three equations:

$$\frac{(1-b\beta)\,(1-\alpha)}{1-b}\left(\frac{h}{k}\right)^{-\alpha} = \left[\left(\frac{h}{k}\right)^{1-\alpha} - \delta\right] k\psi_L h^{\sigma_L},$$

or,

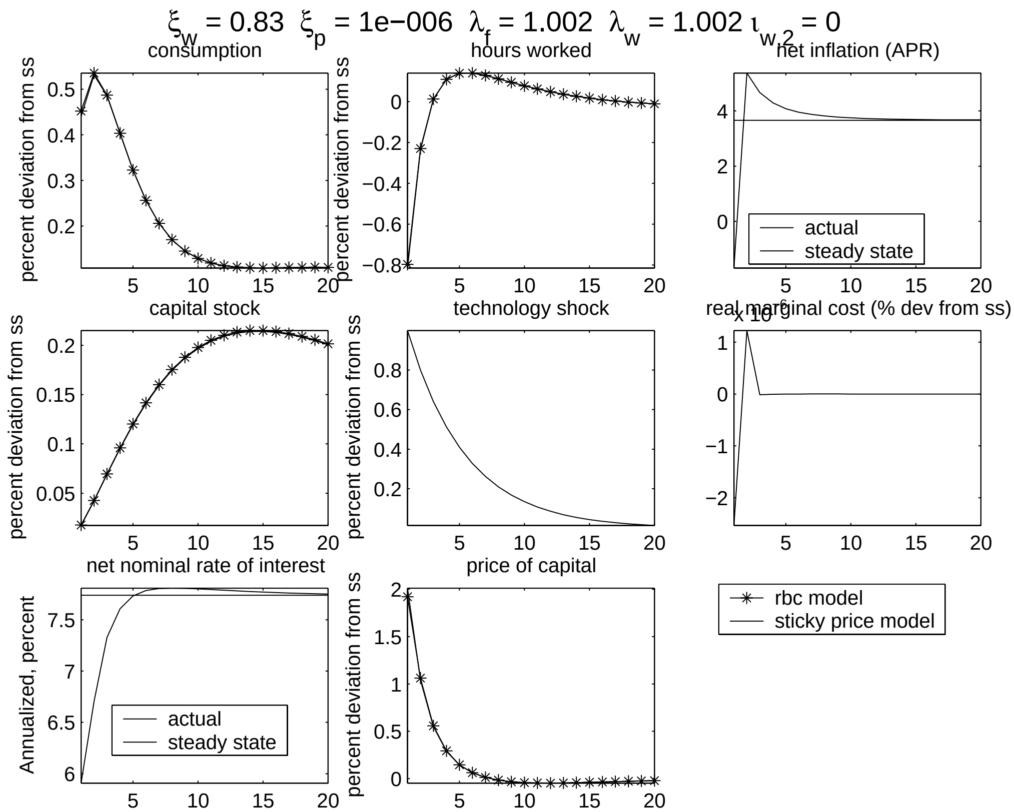
$$h = \left[\frac{\frac{1-b\beta}{1-b}\,(1-\alpha)\left(\frac{h}{k}\right)^{1-\alpha}}{\left[\left(\frac{h}{k}\right)^{1-\alpha} - \delta\right]\psi_L}\right]^{\frac{1}{1+\sigma_L}}$$

Given this RBC model benchmark, we simulated the model with $\lambda_w = \lambda_f = 1.002$, $\xi_w = 0.000001$, $\xi_p = 0.75$, $S'' = 5.1$, $b = 0.63$. We found the following



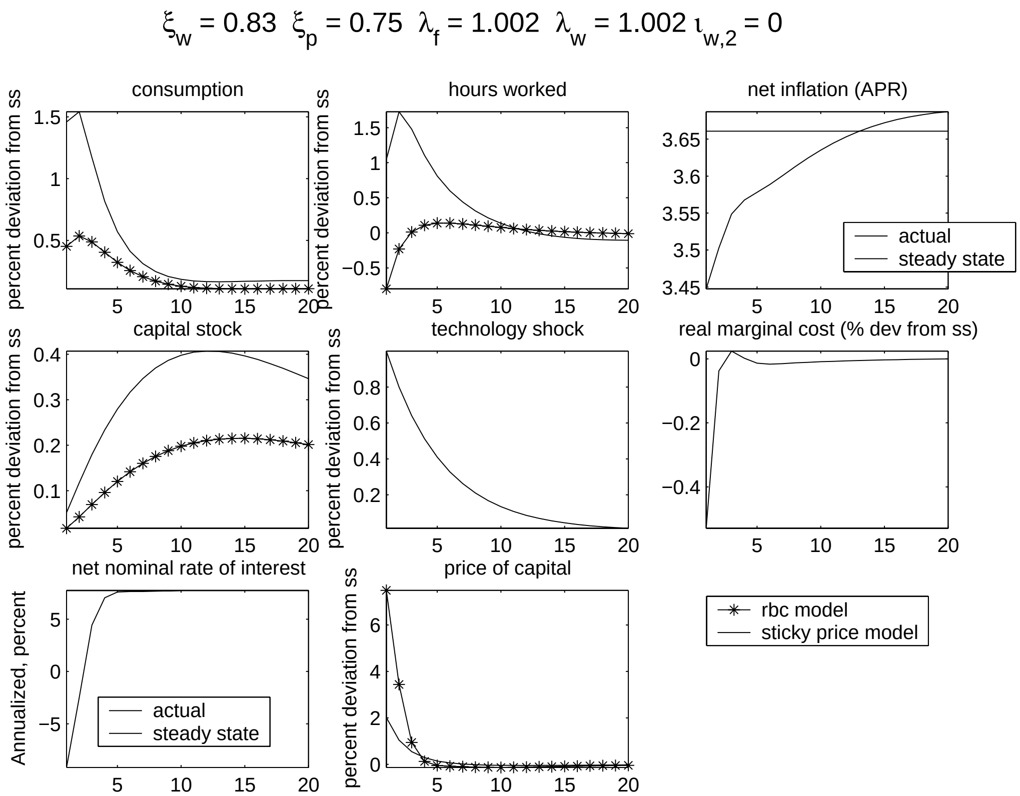
Again, it looks like the Ramsey policy duplicates the RBC model.

Next, we considered the case, $\xi_w = 0.83$, $\xi_p = 0.000001$. The results are consistent with expectations:



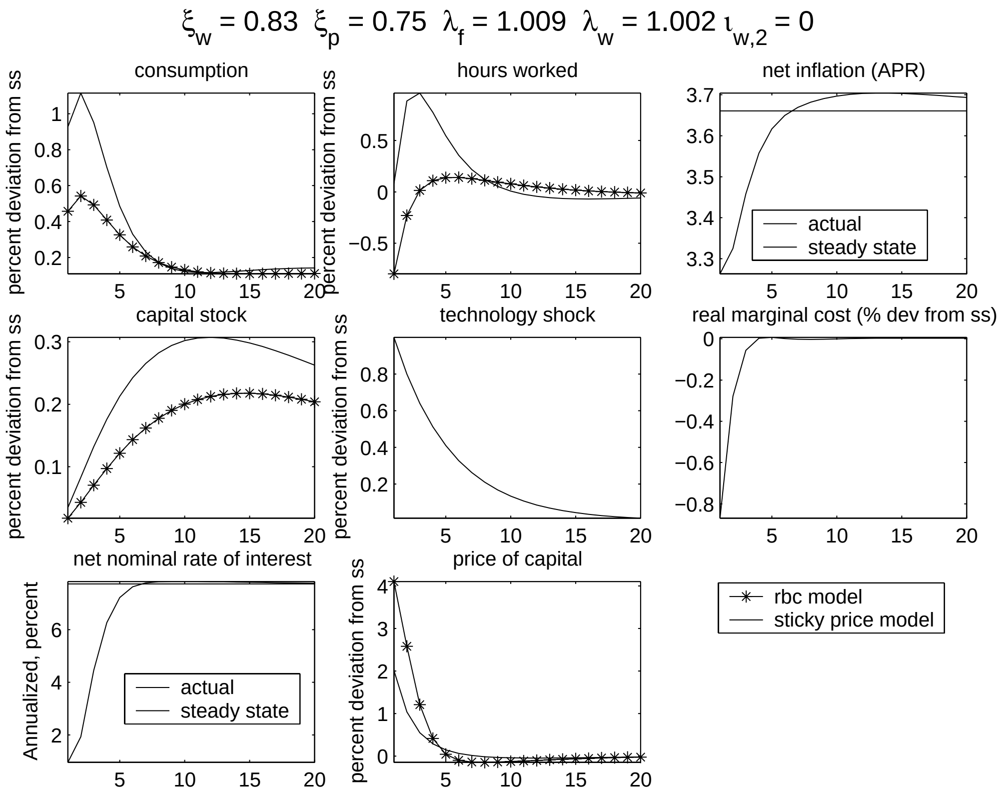
Now let's see how far off things get when there are both sticky prices and sticky wages.

First, we simply set $\xi_w = 0.83$, $\xi_p = 0.75$ and kept $\lambda_f = \lambda_w = 1.002$, $\iota_{w,2} = 0$. Then,



Note - starred line for price of capital is sticky price model and solid line is rbc

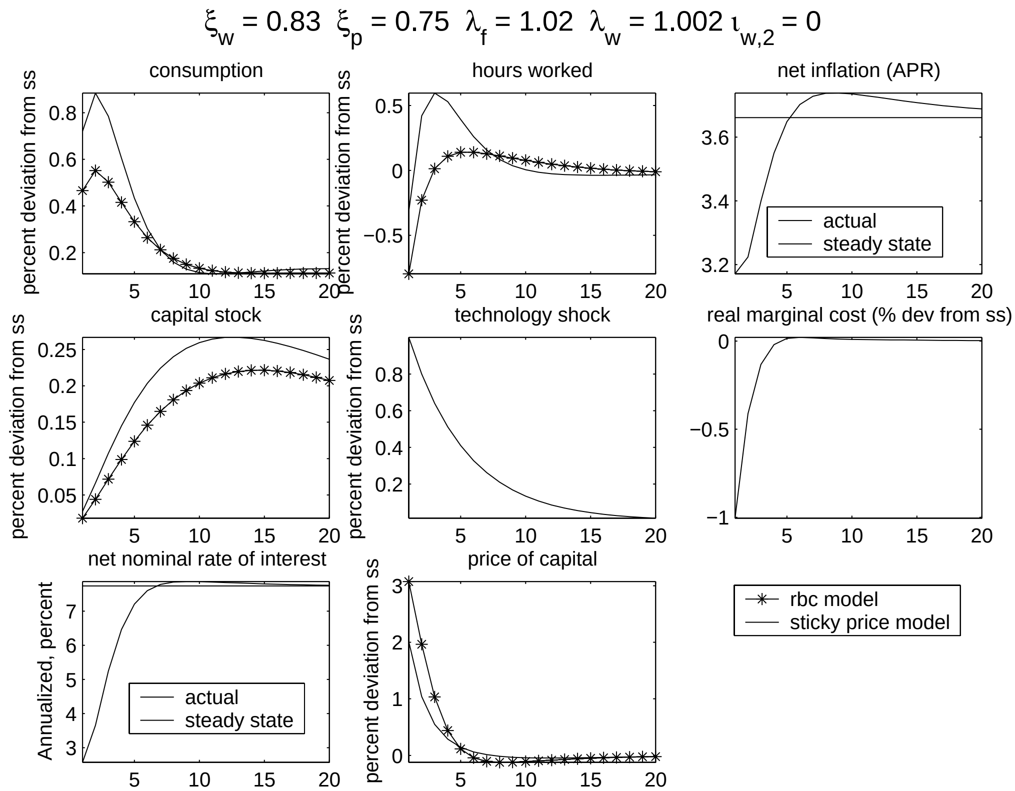
Note that the response of the allocations is now dramatically higher than it is in the RBC model! This is the same as the striking result we observed when we were studying sticky prices and wages alone. When λ_f was changed to 1.009[3]:



Note - starred line for price of capital is sticky price model and solid line is rbc

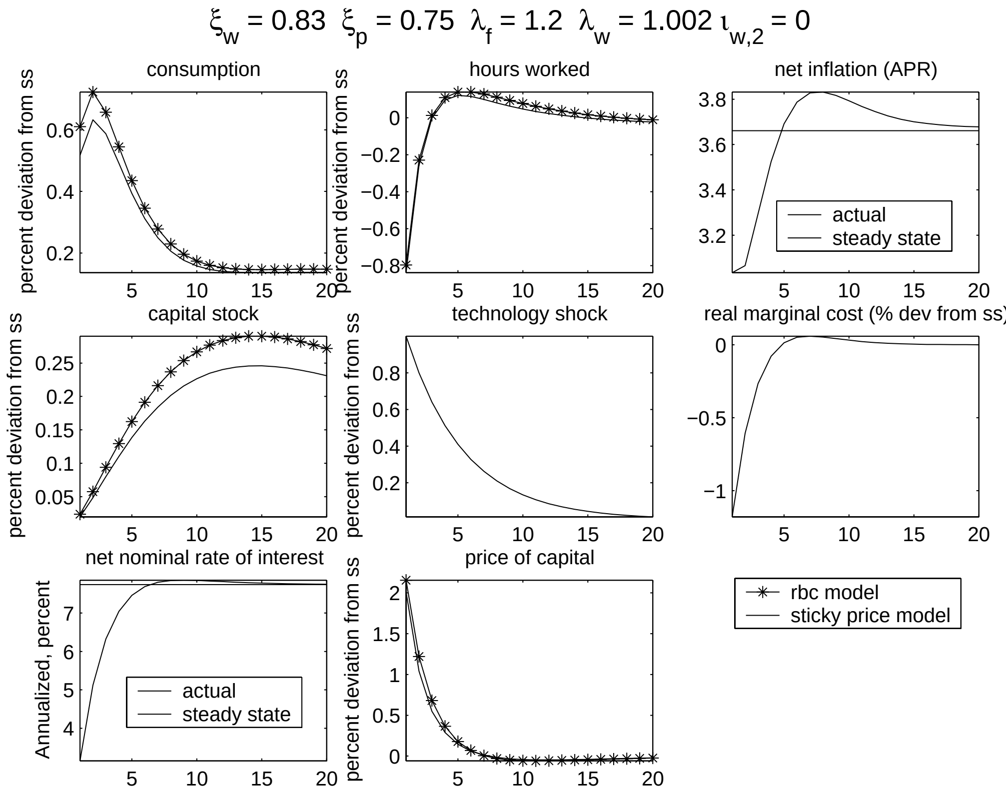
[3] In all cases, we did the calculations using initval in Dynare, so that our initial guess of the steady state was given to Dynare as an initial guess. Typically, since we compute the steady state exactly ourselves in ssnew.m, Dynare simply accepts our steady state. However, in the cases $\lambda_f = 1.009$, and $\lambda_f = 1.02$ Dynare got lost when it started with our initial conditions. In these cases, we manually verified that our steady state is correct, and then forced Dynare to accept our steady state. The results in this figure are based on this type of run. We bypassed Dynare's steady state calculation by replacing line 59 in dynare_solve.m with 'if $5 > 2$'.

When we tried $\lambda_f = 1.02$, we obtained the following result:



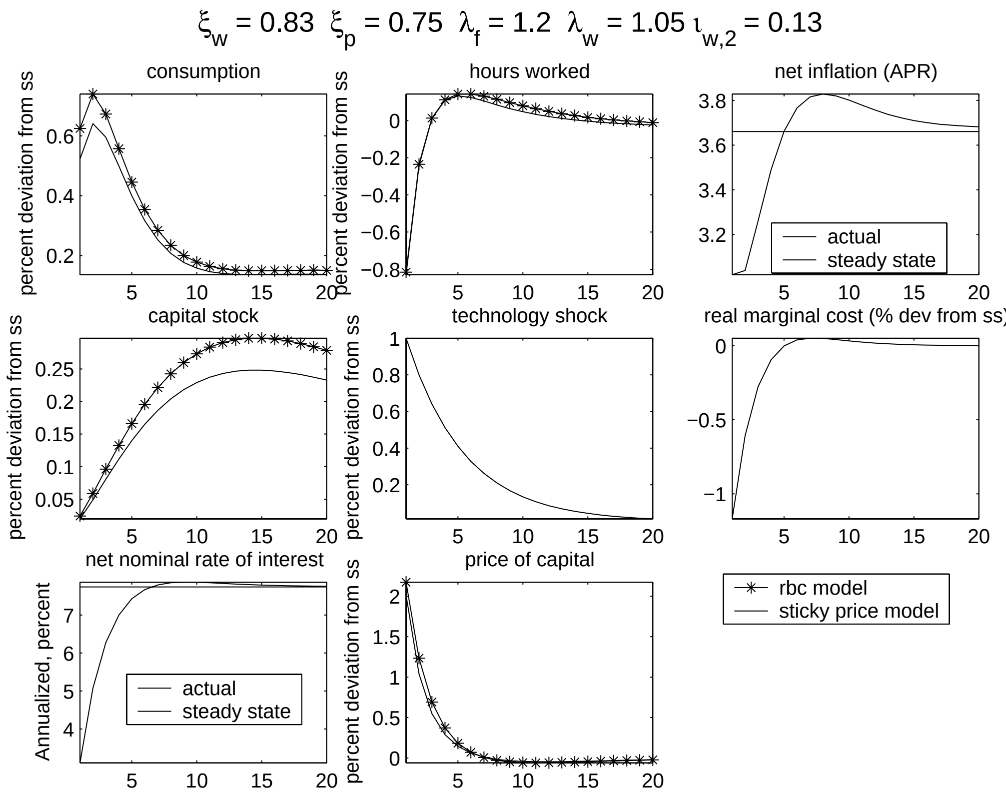
Note - starred line for price of capital is sticky price model and solid line is rbc

Note how the results appear to be continuous in the parameter λ_f, and how the Ramsey allocations are converging to the RBC model calculations. When we tried $\lambda_f = 1.20$, the two almost coincide:



Note - starred line for price of capital is sticky price model and solid line is rbc

We then set all parameters to their benchmark values, $\lambda_w = 1.05$, $\iota_{w,2} = 0.13$, when we obtained



Note - starred line for price of capital is sticky price model and solid line is rbc

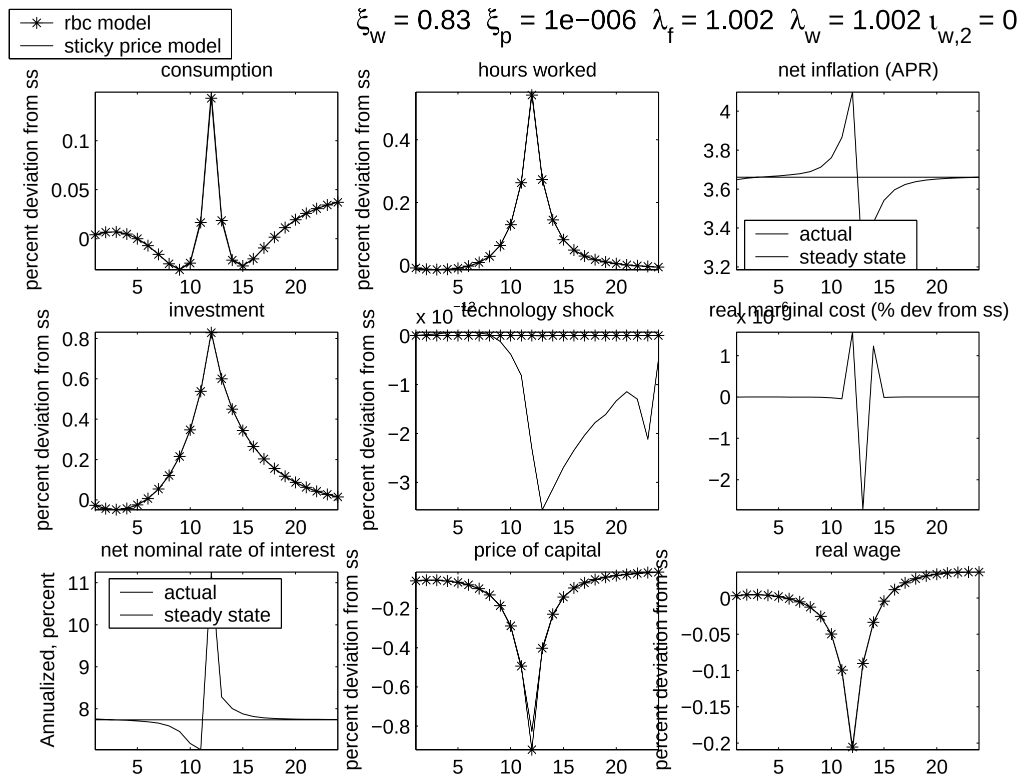
Interestingly, the Ramsey and RBC model allocations almost coincide now.

7. Anticipated Shocks

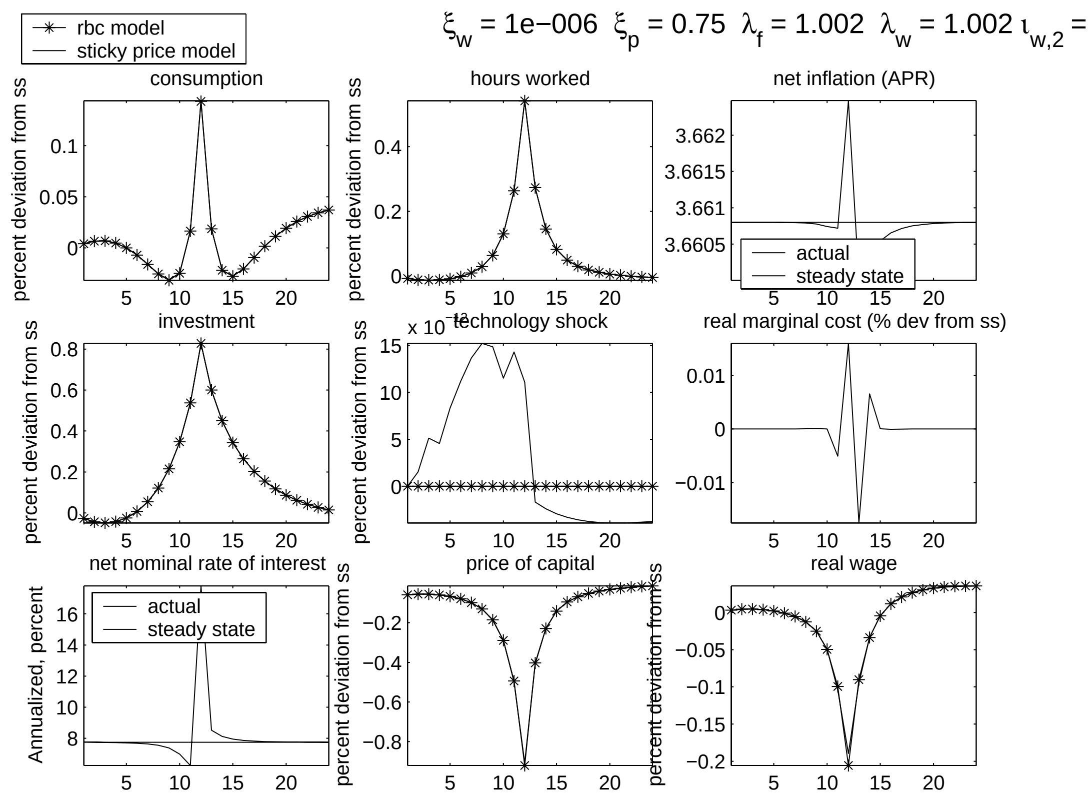
We now consider an alternative representation of the technology shock. We replace the representation (3.16) with

$$\log \epsilon_t = \rho_\varepsilon \log \epsilon_{t-1} + \varepsilon_{t-12} + \xi_t.$$

The code pertaining to this case appears in the subdirectory, 'mirage'. We began by simulating this model under the benchmark parameterization, (6.7), modified so that $\lambda_f = \lambda_w = 1.002$, $\xi_p = 0.000001$, $\xi_w = 0.83$, $\iota_{w,2} = 0$. This is the case of sticky wages and flexible prices. The figure below shows that the Ramsey allocations exactly reproduce the RBC allocations in this case. The experiment is one in which there is an expectation that techology will rise by 1 percent in period 13, an expectation which turns out not to be fulfilled. Note how sharply the real rate of interest rises ultimately with the shock. Note, too, that the real wage falls. This reflects that the real wage must be equal to the marginal product of labor, and the latter must fall. The increase in employment, therefore, reflects a positive labor supply effect.

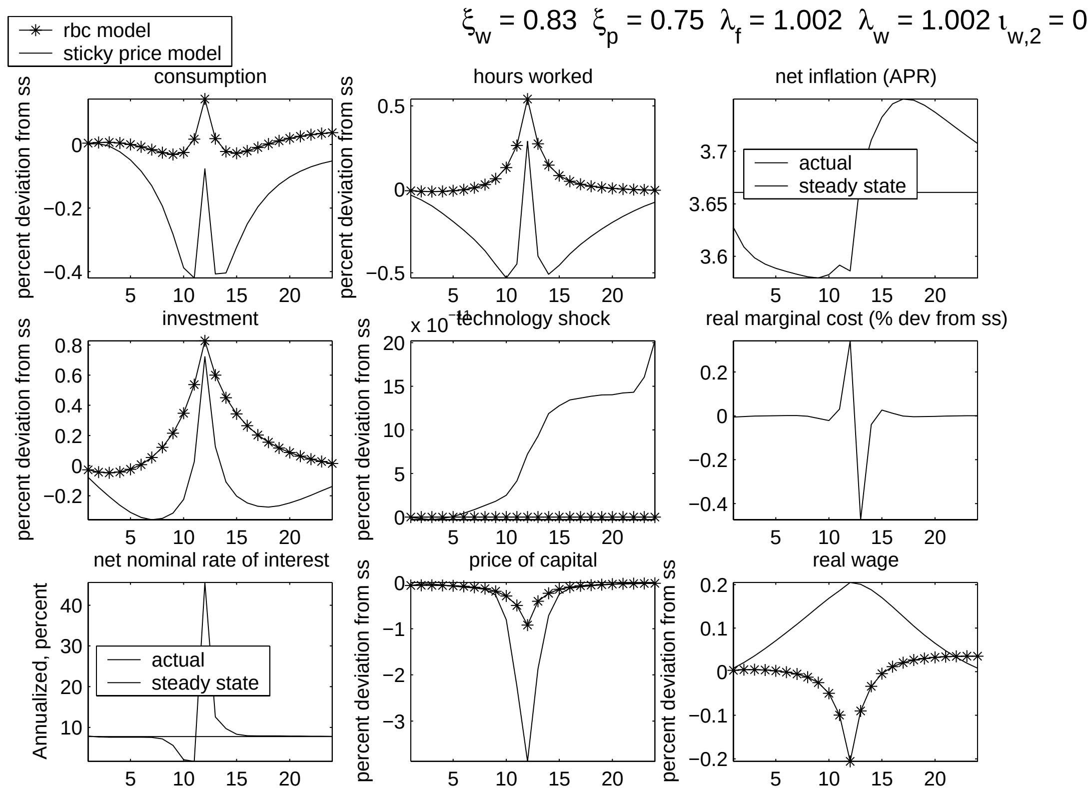


Next, we made prices sticky and wages flexible, $\xi_p = 0.75$, $\xi_w = 0.000001$. We obtained the following results:



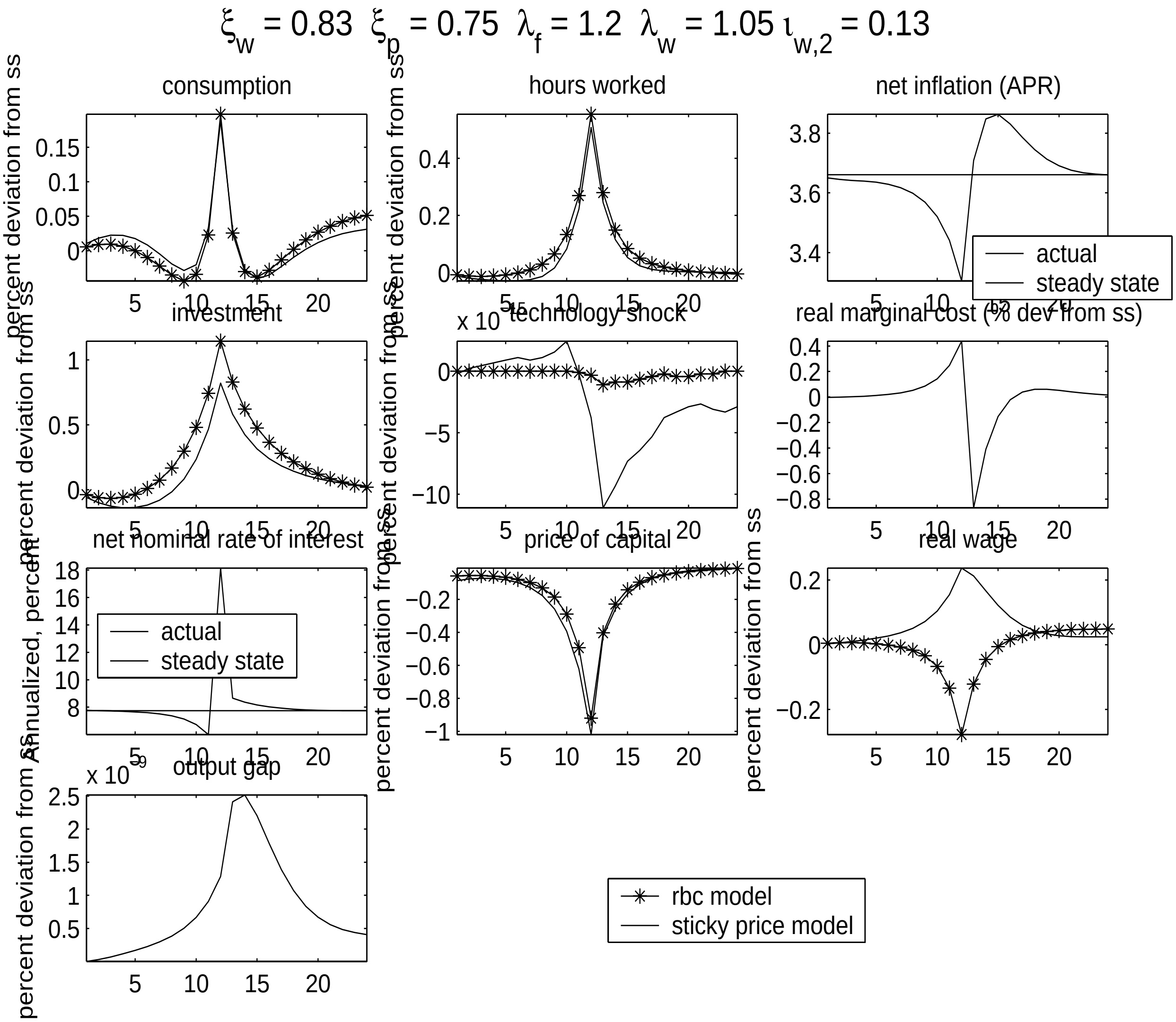
Note how the inflation rate is essentially constant now, so that the fall in the real wage is accomplished by a fall in the nominal wage rate.

Next, we raised ξ_w to 0.83, and obtained the following results:



We have seen this sort of result several times now. Apparently, when there are sticky wages and prices, then low markups cause the Ramsey allocations to look very different from the RBC allocations. Note, for example, that the real wage now increases.

Finally, we consider our benchmark parameter values, (6.7):

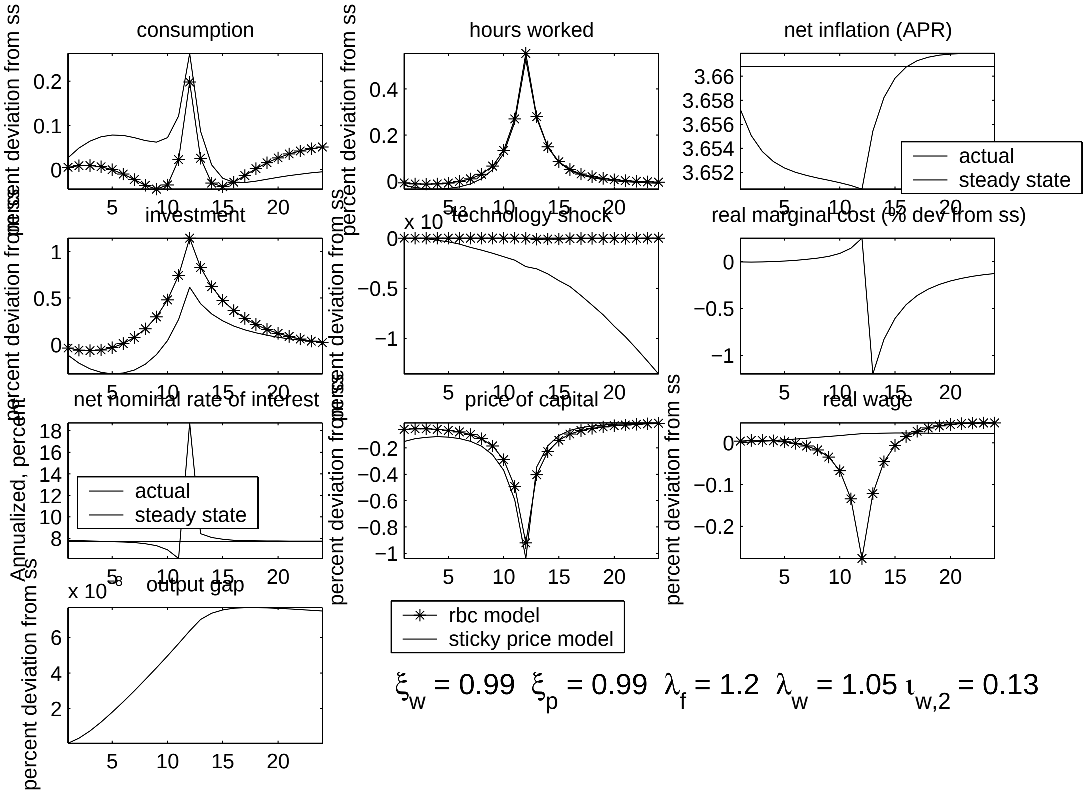


These results are very interesting in several ways. First, the Ramsey quantity allocations are very similar to the RBC allocations. Second, note that the Ramsey real wage is now rising rather than falling. This is an important finding, and needs further explanation. We have also reported the output gap

$$gap_t = (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \, w_t^{*\frac{\lambda_w}{\lambda_w - 1}(1-\alpha)}.$$

This quantity is extraordinarily small, its maximum is 2.5×10^{-9}. This is in percent deviation from steady state.

We investigated what happens when wages and prices are virtually completely sticky, so that the real wage cannot change, by setting $\xi_p = \xi_w = 0.99$ and keeping all other parameters at their benchmark values. We obtained essentially the same results:



8. Pulling All the Equilibrium Conditions Together and Adding Growth

We now consider the case, $\mu_z > 0$, in (3.3). With this change, the economy follows a deterministic growth path in steady state. All variables should be interpreted as scaled by z_t. This causes the dynamic equilibrium conditions in the model to acquire growth rate adjustments. We repeat all the equilibrium conditions of the previous section, except the growth adjustments have been incorportated. The equations pertaining to prices are:

$$p_t^* - \left[(1-\xi_p) \left(\frac{1 - \xi_p \left(\frac{\pi_{t-1}^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_t} \right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p} \right)^{\lambda_f} + \xi_p \left(\frac{\pi_{t-1}^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_t} p_{t-1}^* \right)^{\frac{\lambda_f}{1-\lambda_f}} \right]^{\frac{1-\lambda_f}{\lambda_f}} = 0 \qquad (8.1)$$

and

$$E_t \left\{ \lambda_{z,t} \left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\epsilon_t \left(\frac{k_{t-1}}{\mu_z} \right)^{\alpha} \left(\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h_t \right)^{1-\alpha} - \phi \right] + \left(\frac{\pi_t^{\iota_2} \bar{\pi}^{1-\iota_2}}{\pi_{t+1}} \right)^{\frac{1}{1-\lambda_f}} \beta \xi_p F_{p,t+1} - F_{p,t} \right\} = 0, \qquad (8.2)$$

and

$$\lambda_{z,t}\lambda_f\left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\epsilon_t\left(\frac{k_{t-1}}{\mu_z}\right)^{\alpha}\left(\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w-1}}h_t\right)^{1-\alpha}-\phi\right]s_t+ \tag{8.3}$$

$$\beta\xi_p\left(\frac{\pi_t^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi_{t+1}}\right)^{\frac{\lambda_f}{1-\lambda_f}}\left[\frac{1-\xi_p\left(\frac{\pi_t^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f}F_{p,t+1}-F_{p,t}\left[\frac{1-\xi_p\left(\frac{\pi_{t-1}^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f} = 0.$$

In the price equations, the only required adjustment is in the production function, where k_{t-1} needs to be adjusted.

Now, consider the wage equations. These are (5.14), (5.15), (5.12), and (5.10). In adjusting these equations, it is to be born in mind that the wage updating term, $\tilde{\pi}_{w,t+1}$, does not need to be adjusted, since in t he model with technological growth we suppose that the wages of non-optimizing households must be augmented by the growth rate of technology as follows:

$$W_{j,t}=\tilde{\pi}_{w,t}\mu_z W_{j,t-1},\ \tilde{\pi}_{w,t}\equiv\left(\pi_{t-1}\right)^{\iota_{w,2}}\bar{\pi}^{1-\iota_{w,2}}.$$

At the same time the definition of nominal wage growth, $\pi_{w,t}$, must be interpreted as

$$\pi_{w,t}=\frac{\tilde{w}_t\mu_z\pi_t}{\tilde{w}_{t-1}},$$

where $\tilde{w}_t$ is the real wage, scaled by z_t. The version of (5.14) with technical growth is derived elsewhere. Using the derivation there, we find that the adjusted equation is:

$$E_t\{\lambda_{z,t}\frac{\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w-1}}h_t}{\lambda_w}+\beta\xi_w\tilde{\pi}_{w,t+1}^{\frac{1}{1-\lambda_w}}\frac{\left(\frac{\tilde{w}_t}{\tilde{w}_{t+1}\pi_{t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi_{t+1}}F_{w,t+1}-F_{w,t}\}=0 \tag{8.4}$$

$$E_t\{\left[\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w-1}}h_t\right]^{1+\sigma_L} \tag{8.5}$$

$$+\beta\xi_w\left(\frac{\tilde{\pi}_{w,t+1}}{\tilde{w}_{t+1}\pi_{t+1}}\tilde{w}_t\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}\frac{1}{\psi_L}\left[\frac{1-\xi_w\left(\frac{\tilde{\pi}_{w,t+1}}{\tilde{w}_{t+1}\pi_{t+1}}\tilde{w}_t\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right]^{1-\lambda_w(1+\sigma_L)}\tilde{w}_{t+1}F_{w,t+1}$$

$$-\frac{1}{\psi_L}\left[\frac{1-\xi_w\left(\frac{\tilde{\pi}_{w,t}}{\tilde{w}_t\pi_t}\tilde{w}_{t-1}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right]^{1-\lambda_w(1+\sigma_L)}\tilde{w}_tF_{w,t}\} = 0$$

and no adjustment is required. Similarly, no adjustment is required for the following equation:

$$w_t^*=[(1-\xi_w)\left(\frac{1-\xi_w\left(\frac{\tilde{\pi}_{w,t}}{\tilde{w}_t\pi_t}\tilde{w}_{t-1}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right)^{\lambda_w}+\xi_w\left(\frac{\tilde{\pi}_{w,t}}{\tilde{w}_t\pi_t}\tilde{w}_{t-1}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}]^{\frac{1-\lambda_w}{\lambda_w}}. \tag{8.6}$$

Similarly, no adjustment is required to the following equation:

$$w_t^+ = \left[(1-\xi_w) \left(\frac{1-\xi_w \left(\frac{\tilde{\pi}_{w,t}}{\tilde{w}_t \pi_t} \tilde{w}_{t-1} \right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w} \right)^{\lambda_w(1+\sigma_L)} + \xi_w \left(\frac{\tilde{\pi}_{w,t}}{\tilde{w}_t \pi_t} \tilde{w}_{t-1} w_{t-1}^+ \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}} . \tag{8.7}$$

In the definition of marginal cost, an adjustment is required:

$$s_t = \frac{\tilde{w}_t}{(1-\alpha)\epsilon_t} \left(\frac{\mu_z (w_t^*)^{\frac{\lambda_w}{\lambda_w - 1}} h_t}{k_{t-1}} \right)^{\alpha} \tag{8.8}$$

A similar adjustment is required in the resource constraint:

$$c_t + I_t = (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left\{ \epsilon_t \left(\frac{k_{t-1}}{\mu_z} \right)^{\alpha} \left[(w_t^*)^{\frac{\lambda_w}{\lambda_w - 1}} h_t \right]^{1-\alpha} - \phi \right\} \tag{8.9}$$

where

$$k_t - (1-\delta)\mu_z^{-1} k_{t-1} = \left[1 - \frac{S'' \mu_z^2}{2} (\frac{I_t}{I_{t-1}} - 1)^2 \right] I_t \tag{8.10}$$

Here, we interpret I_t as investment scaled by z_t.

The equation defining the nominal rate of interest must be adjusted:

$$E_t\{\beta \frac{1}{\pi_{t+1}\mu_z} \lambda_{z,t+1} (1+R_t) - \lambda_{z,t}\} = 0 \tag{8.11}$$

The adjustment to the equation for $\lambda_{z,t}$ is:

$$E_t \left[\lambda_{z,t} - \frac{\mu_z}{c_t \mu_z - b c_{t-1}} + b\beta \frac{1}{\mu_z c_{t+1} - b c_t} \right] = 0. \tag{8.12}$$

The adjustment for the investment equation is:

$$-\lambda_{z,t} + \lambda_{z,t+1} \beta \frac{1}{\mu_z q_t} \left[\alpha \epsilon_{t+1} \left(\frac{\mu_z (w_{t+1}^*)^{\frac{\lambda_w}{\lambda_w - 1}} h_{t+1}}{k_t} \right)^{1-\alpha} s_{t+1} + q_{t+1}(1-\delta) \right] = 0, \tag{8.13}$$

The adjustment for the investment equation is:

$$\begin{aligned} E_t\{\lambda_{zt} q_t \left[1 - \frac{S'' \mu_z^2}{2} (\frac{I_t}{I_{t-1}} - 1)^2 - S'' \mu_z^2 (\frac{I_t}{I_{t-1}} - 1) \frac{I_t}{I_{t-1}} \right] \\ -\lambda_{zt} + \beta \lambda_{zt+1} q_{t+1} S'' \mu_z^2 (\frac{I_{t+1}}{I_t} - 1) \left(\frac{I_{t+1}}{I_t} \right)^2 \} &= 0. \end{aligned} \tag{8.14}$$

The equations for the steady state of the RBC version of the model are:

$$\begin{aligned} \frac{\mu_z h}{k} &= \left[\frac{\frac{\mu_z}{\beta} - (1-\delta)}{\alpha} \right]^{\frac{1}{1-\alpha}} \\ h &= \left(\frac{\frac{\mu_z - b\beta}{(\mu_z - b)} (1-\alpha) \left(\frac{\mu_z h}{k} \right)^{1-\alpha}}{\psi_L \left[\left(\frac{\mu_z h}{k} \right)^{1-\alpha} - [\mu_z - (1-\delta)] \right]} \right)^{\frac{1}{1+\sigma_L}} \end{aligned}$$

9. Introducing Monetary Policy

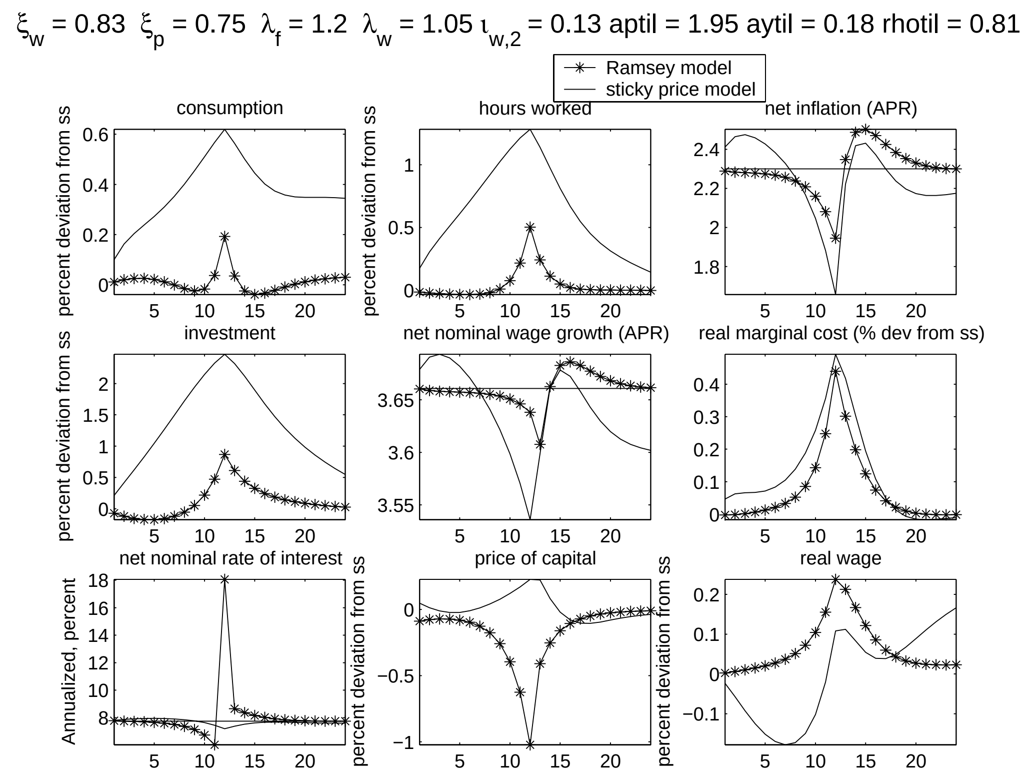
To the 14 private sector equilibrium conditions we add the following monetary policy rule:

$$\log R_t = (1-\rho)\log(R) + \rho \log R_{t-1} + \frac{1}{R}(1-\rho)\tilde{a}_p \pi \log \frac{\pi_{t+1}}{\pi} + (1-\rho)\tilde{a}_y \frac{1}{4R}\log\frac{y_t}{y} + \frac{1}{400R}x_t^p, \tag{9.1}$$

where x_t^p is an iid monetary policy shock and y_t denotes output, $c_t + I_t$. The parameters values are (6.7) and

$$\tilde{a}_p = 1.95,\ \tilde{a}_y = 0.18,\ \rho = 0.81.$$

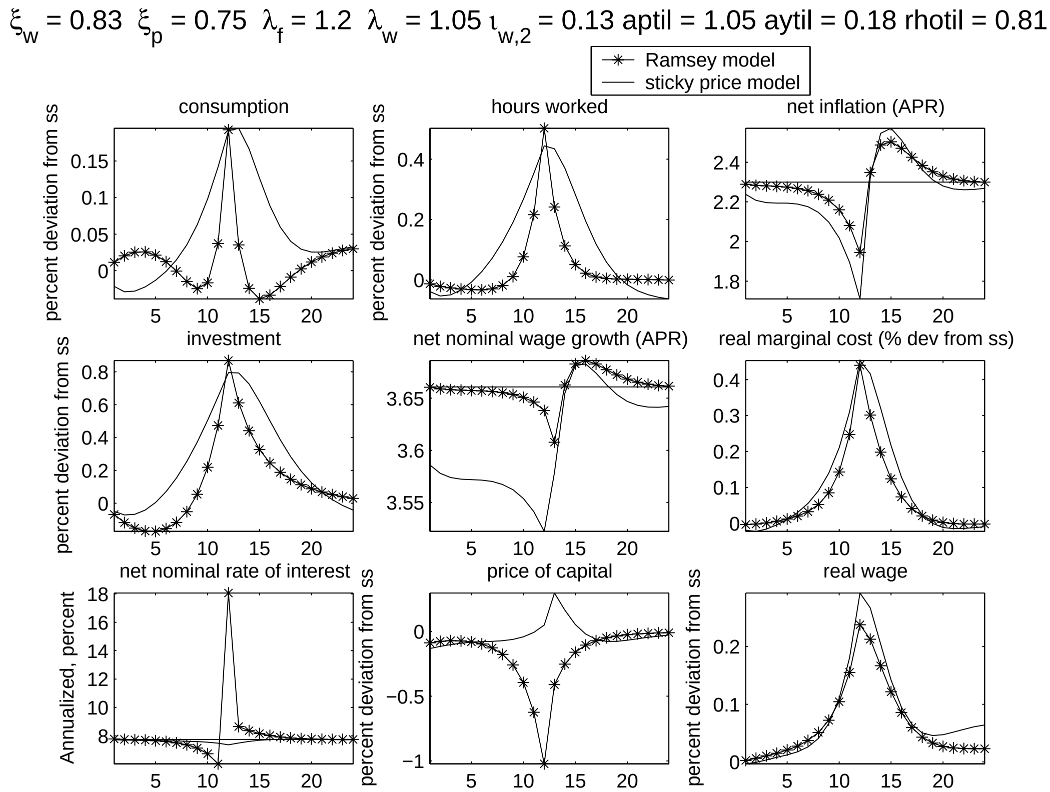
Following compares the impulse responses associated with this model, with the corresponding Ramsey allocations:



There are several things of interest here. First, the real wage in the exogenous monetary policy equilibrium is lower than it is in the Ramsey equilibrium. This seems to reflect a rise in inflation in the immediate aftermath of the signal shock. Wage inflation also increases, but by an order of magnitude less than the change in inflation. If policy allowed inflation to fall more, this would produce the rise in the real wage necessary to achieve the optimum sought by the Ramsey allocations. Second, the rise in marginal cost is the same across the two equilibria. Presumably, this reflects the much greater rise in employment in the Ramsey equilibrium. Third, and most important, the stock price rises and falls, and the real quantity allocations rise and fall too.

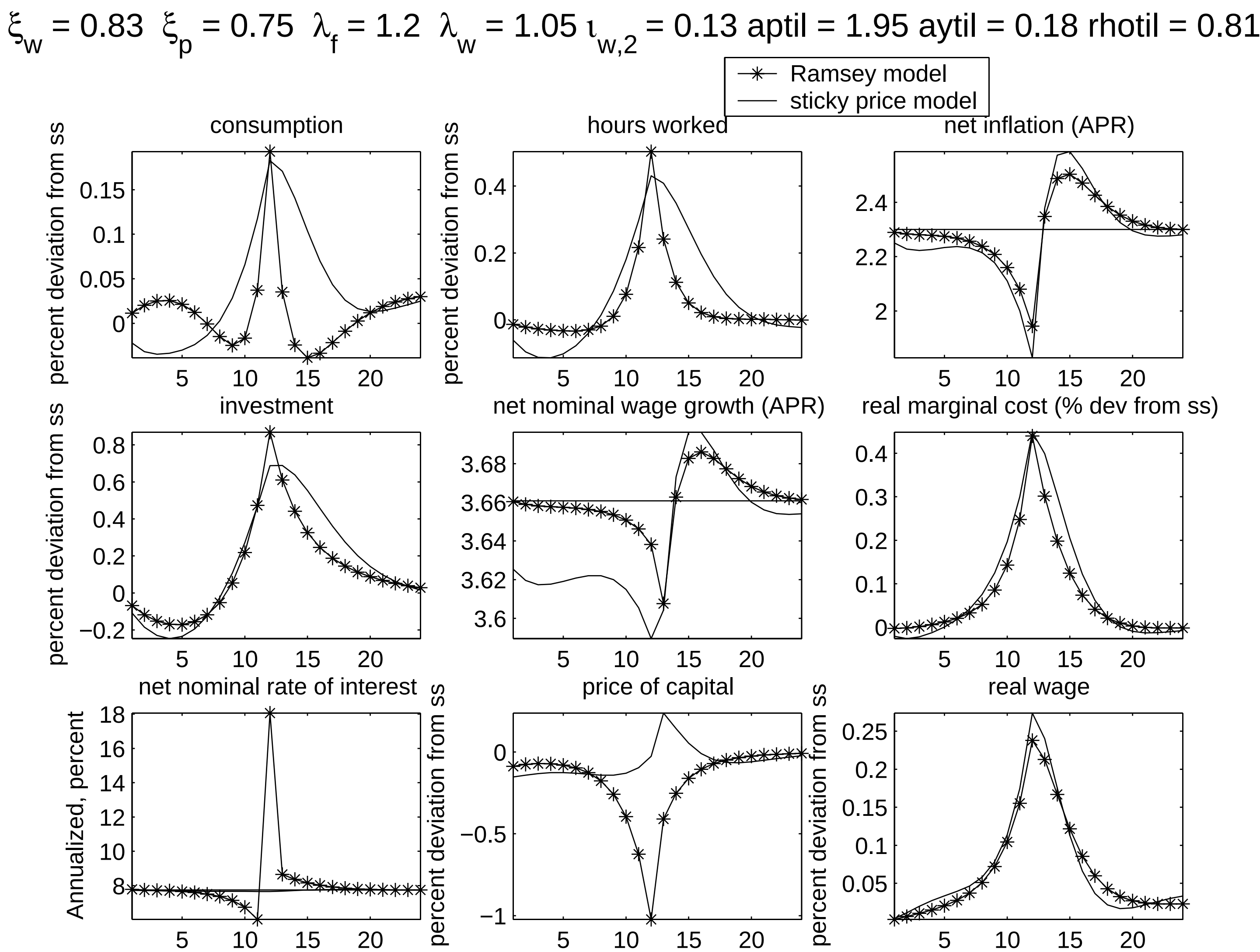
To explore the issue of what would be a better policy, we simulated the system with a change in the monetary policy rule, in which the coefficient on inflation was reduced to 1.05

from 1.95. The result was as follows:



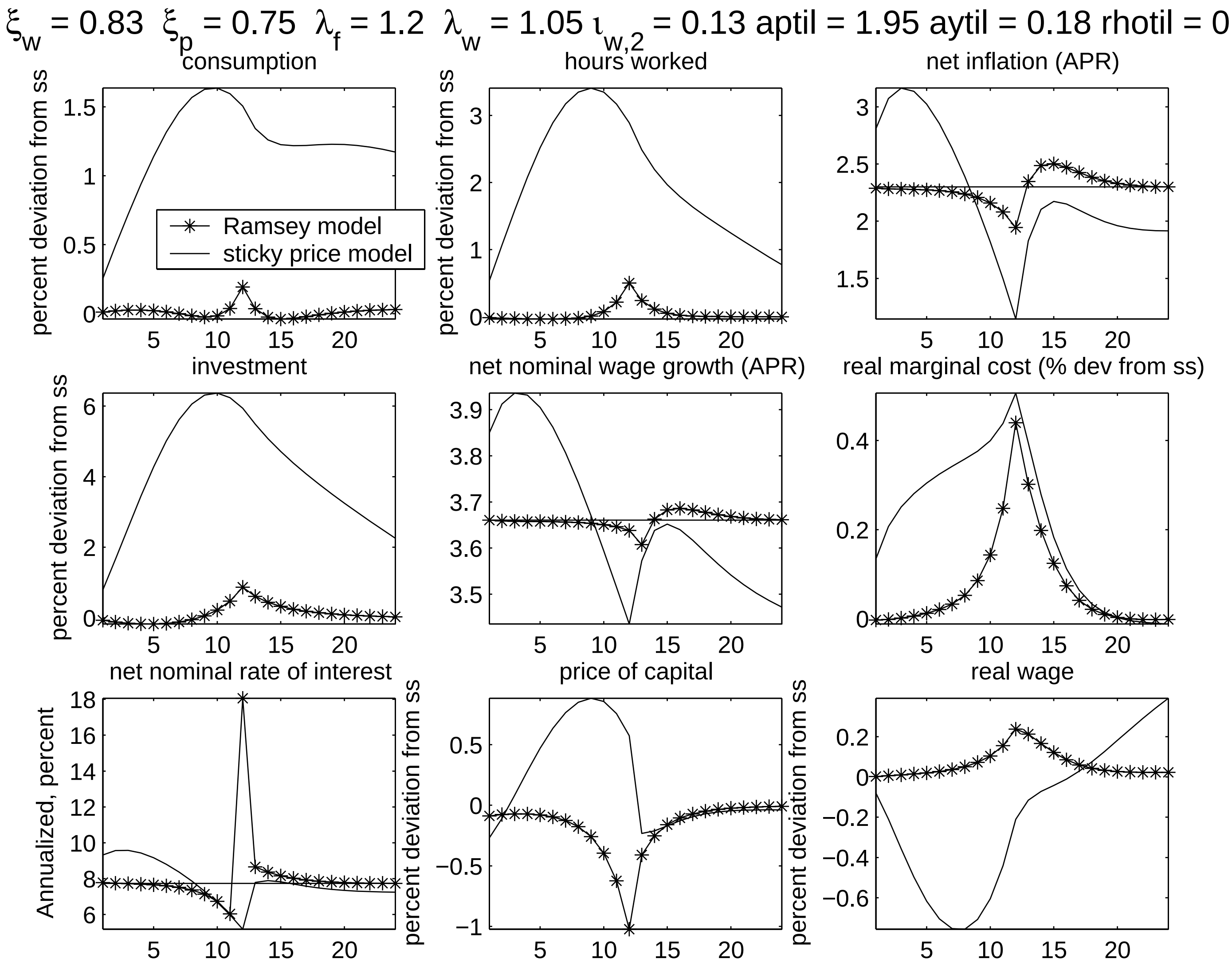
Notice how consumption, hours, inflation, investment real marginal cost and the real wage now more nearly resemble the corresponding Ramsey allocations. Also, the behavior of the real wage now more closely resembles what it is in the Ramsey equilibrium. By reducing the emphasis on inflation targetting, policy has been able to come closer to the ideal policy. This has happened primarily via a drop in inflation. Wage growth has fallen too, but by a much smaller amount.

We explored an alternative idea. We replaced π_{t+1}/π with $\pi_{w,t+1}/(\pi\mu_z)$ in the monetary policy rule, (9.1). We then obtained the following results



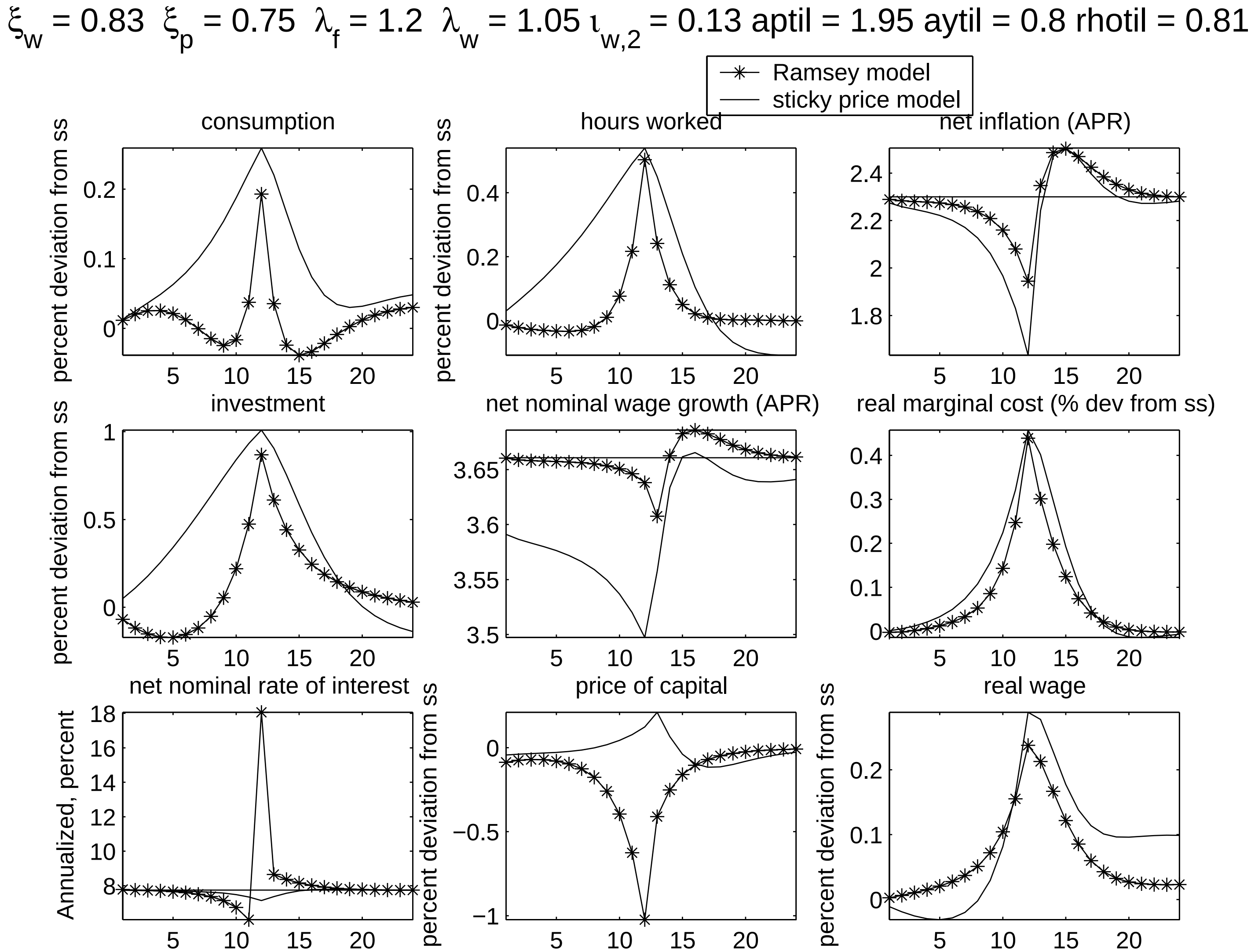
Interestingly, we obtained virtually the same results as when we reduced the coefficient on inflation in the Taylor rule.

Next, we set the coefficient on the lagged interest rate, ρ, equal to zero. We then obtained the following result



The output effects are even stronger now. They are associated with an even bigger fall in the real wage. This is consistent with the idea that the 'problem' with the estimated monetary policy is that it allows the real wage to drop too much. Again, the growth rate of the nominal wage rate exhibits substantial gyrations. However, these are much smaller than the variations in inflation.

Next, we considered the possibility of raising the coefficient on output. We obtained these results:



This also works quite well.

10. Adding Financial Frictions

We now add the financial frictions proposed by Bernanke, Gertler and Gilchrist. This introduces 3 new relations: the optimality condition associated with the standard debt contract offered to entrepreneurs, a zero profit condition on banks and the law of motion for entrepreneurial net worth.

10.1. The Addional Equilibrium Conditions

The optimality condition associated with standard debt contracts is:

$$E_t\left\{[1-\Gamma(\bar{\omega}_{t+1})]\frac{1+R^k_{t+1}}{1+R_t}+\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1})-\mu G'(\bar{\omega}_{t+1})}\left[\frac{1+R^k_{t+1}}{1+R_t}\left(\Gamma(\bar{\omega}_{t+1})-\mu G(\bar{\omega}_{t+1})\right)-1\right]\right\}=0 \tag{10.1}$$

Here, the rate of return on capital, R^k_{t+1}, is defined as follows:

$$R^k_{t+1}=\frac{r^k_{t+1}+(1-\delta)q_{t+1}}{q_t}\pi_{t+1}-1,$$

and the rental rate of capital is

$$r^k_{t+1}=\alpha\epsilon_{t+1}\left(\frac{\mu_z\left(w^*_{t+1}\right)^{\frac{\lambda_w}{\lambda_w-1}}h_{t+1}}{k_t}\right)^{1-\alpha}s_{t+1}.$$

Also, in (10.1),

$$\begin{aligned}\Gamma(\bar{\omega}) &= \bar{\omega}\left[1-F(\bar{\omega})\right]+G(\bar{\omega}) \qquad (10.2)\\ G(\bar{\omega}_t) &= \int_0^{\bar{\omega}_t}\omega dF(\omega)\end{aligned}$$

The zero profit condition on banks is:

$$\Gamma(\bar{\omega}_{t+1})-\mu G(\bar{\omega}_{t+1})=\frac{1+R_t}{1+R^k_{t+1}}\left(1-\frac{n_{t+1}}{q_tk_{t+1}}\right), \tag{10.3}$$

and the law of motion for net worth is:

$$n_{t+1}=\frac{\gamma}{\pi_t\mu^*_z}\left\{R^k_t-R_{t-1}-\mu\int_0^{\bar{\omega}_t}\omega dF(\omega)\left(1+R^k_t\right)\right\}k_tq_{t-1}+w^e+\gamma\left(\frac{1+R_{t-1}}{\pi_t}\right)\frac{1}{\mu^*_z}n_t. \tag{10.4}$$

Some existing equations need to be changed. The resource constraint becomes:

$$d_t+c_t+I_t+\Theta\frac{1-\gamma}{\gamma}\left[n_{t+1}-w^e\right]=\left(p^*_t\right)^{\frac{\lambda_f}{\lambda_f-1}}\left\{\epsilon_t\left(\frac{k_t}{\mu^*_z}\right)^{\alpha}\left[\left(w^*_t\right)^{\frac{\lambda_w}{\lambda_w-1}}L_t\right]^{1-\alpha}-\phi\right\} \tag{10.5}$$

Here, $[n_{t+1}-w^e]/\gamma$ denotes the assets of entrepreneurs before they have received their real transfer, w^e, and before it is determined if they are to be selected to exit. The fraction that exit is $(1-\gamma)$ times this amount, and they consume Θ of their assets, with the other $1-\Theta$ being transferred to households. Also, d_t denotes the resources used up in monitoring:

$$d_t=\frac{\mu G(\bar{\omega}_t)\left(1+R^k_t\right)q_{t-1}k_t}{\mu^*_z}\frac{1}{\pi_t}.$$

In the modified economy, entrepreneurs rather than households accumulate capital. This means that the household intertemporal equation, (6.2), must be deleted. So, we have added three new equations, (10.1), (10.3) and (10.4) and deleted one. The net increase in the number of equations is two. We increase the number of endogenous variables by two, $\bar{\omega}_{t+1}$ and n_{t+1}.

10.2. The Steady State

To solve this model, we need to develop an algorithm for computing its steady state. In our analysis, we distinguish between steady state inflation, π, and the quantity appearing in the price and wage updating equations, $\bar{\pi}$. Equation (8.1) in steady state, is:

$$p^* = \left[\frac{(1-\xi_p)\left(\frac{1-\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right)^{\lambda_f}}{1-\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{\lambda_f}{1-\lambda_f}}} \right]^{\frac{1-\lambda_f}{\lambda_f}}.$$

Note that, if $\pi = \bar{\pi}$ then $p^* = 1$. Equation (8.2):

$$F_p = \frac{\lambda_z (p^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\left(\frac{k}{\mu_z}\right)^{\alpha} \left((w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h\right)^{1-\alpha} - \phi \right]}{1 - \left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}} \beta\xi_p},$$

assuming

$$\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}} \beta\xi_p < 1.$$

Equation (8.3) in steady state is:

$$F_p = \frac{\lambda_z\lambda_f (p^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\left(\frac{k}{\mu_z}\right)^{\alpha} \left((w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h\right)^{1-\alpha} - \phi \right] s}{\left[\frac{1-\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f} \left[1 - \beta\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]}$$

Equating the preceding two equations:

$$s = \frac{1}{\lambda_f} \frac{\left[\frac{1-\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f} \left[1 - \beta\xi_p\left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]}{1 - \left(\frac{\pi^{\iota_2}\bar{\pi}^{1-\iota_2}}{\pi}\right)^{\frac{1}{1-\lambda_f}} \beta\xi_p}. \tag{10.6}$$

In the case, $\pi = \bar{\pi}$, $s = 1/\lambda_f$. Equation (8.4) in steady state is:

$$F_w = \frac{\lambda_z \frac{(w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h}{\lambda_w}}{1 - \beta\xi_w \tilde{\pi}_w^{\frac{1}{1-\lambda_w}} \frac{\left(\frac{1}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi}},$$

as long as the condition,

$$\beta\xi_w \tilde{\pi}_w^{\frac{1}{1-\lambda_w}} \frac{\left(\frac{1}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi} < 1,$$

is satisfied. Also

$$\tilde{\pi}_w = (\pi)^{\iota_{w,2}} \bar{\pi}^{1-\iota_{w,2}}.$$

Equation (**??**) is

$$F_w = \frac{\left[(w^*)^{\frac{\lambda_w}{\lambda_w-1}} h\right]^{1+\sigma_L}}{\frac{1}{\psi_L}\left[\frac{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right]^{1-\lambda_w(1+\sigma_L)} \tilde{w}\left[1-\beta\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}\right]},$$

as long as

$$\beta\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} < 1.$$

Equating the two expressions for F_w, we obtain:

$$\tilde{w} = W\lambda_w \frac{\psi_L h^{\sigma_L}}{\lambda_z}, \tag{10.7}$$

where

$$W = (w^*)^{\frac{\lambda_w}{\lambda_w-1}\sigma_L}\left[\frac{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right]^{\lambda_w(1+\sigma_L)-1} \frac{1-\beta\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\beta\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}}.$$

In steady state, (8.6) reduces to:

$$w^* = \left[\frac{(1-\xi_w)\left(\frac{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right)^{\lambda_w}}{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}}}\right]^{\frac{1-\lambda_w}{\lambda_w}} \tag{10.8}$$

According to the wage equation, the wage is a markup, $W\lambda_w$, over the household's marginal cost. Note that the magnitude of the markup depends on the degree of wage distortions in the steady state. These will be important to the extent that $\tilde{\pi}_w \neq \pi_w$.

Equation (5.10) reduces to:

$$w_t^+ = \left[\frac{(1-\xi_w)\left(\frac{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right)^{\lambda_w(1+\sigma_L)}}{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}}}\right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}},$$

as long as

$$\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} < 1.$$

The marginal cost equation, (8.8) implies:

$$s = \frac{\tilde{w}}{(1-\alpha)} \left(\frac{\mu_z \left(w^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h}{k} \right)^{\alpha},$$

where w^* is determined by (10.8). The steady state rental rate of capital is:

$$r^k = \alpha \left(\frac{\mu_z \left(w^*\right)^{\frac{\lambda_w}{\lambda_w - 1}} h}{k} \right)^{1-\alpha} s. \tag{10.9}$$

In steady state, the capital accumulation equation, (8.10), is

$$\left[1 - (1-\delta)\mu_z^{-1}\right] k = I.$$

In steady state, the equation for the nominal rate of interest, (8.11), reduces to:

$$1 + R = \frac{\pi \mu_z}{\beta} - 1. \tag{10.10}$$

In steady state, the marginal utility of consumption, (8.12), is

$$\lambda_z = \frac{1}{c} \frac{\mu_z - b\beta}{\mu_z - b}. \tag{10.11}$$

Finally, the euler equation for investment, (8.14), reduces to

$$q = 1.$$

We proceed as follows. First, fix the nominal rate of interest according to (10.10). Now, fix a value for r^k. Solve (10.9) for h/k :

$$\frac{h}{k} = \frac{1}{\mu_z} \left(w^*\right)^{\frac{\lambda_w}{1-\lambda_w}} \left(\frac{r^k}{\alpha s} \right)^{\frac{1}{1-\alpha}},$$

where s is determined by (10.6). Then,

$$R^k = \left[r^k + (1-\delta)\right] \pi - 1.$$

Then, solve

$$\left[1 - \Gamma(\bar{\omega})\right] \frac{1+R^k}{1+R} + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} \left[\frac{1+R^k}{1+R} \left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1 \right] = 0.$$

for $\bar{\omega}$. Then, find n/k which solves (10.3):

$$\frac{n}{k} = 1 - \frac{1+R^k}{1+R} \left[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right]$$

In steady state, (10.4) is

$$n = \frac{\gamma}{\pi \mu_z^*} \left\{ R^k - R - \mu \int_0^{\bar{\omega}} \omega dF(\omega) \left(1 + R^k\right) \right\} \left(\frac{k}{n}\right) n + w^e + \gamma \left(\frac{1+R}{\pi \mu_z^*}\right) n,$$

so that

$$\begin{aligned}
n &= \frac{w^e}{1 - \frac{\gamma}{\pi \mu_z^*} \left\{R^k - R - \mu G(\bar{\omega})(1 + R^k)\right\} \left(\frac{k}{n}\right) - \gamma \left(\frac{1+R}{\pi \mu_z^*}\right)}, \\
k &= \left(\frac{k}{n}\right) n \\
h &= \left(\frac{h}{k}\right) k \qquad (10.12) \\
I &= \left[1 - (1 - \delta)\mu_z^{-1}\right] k,
\end{aligned}$$

where $G(\bar{\omega})$ is obtained from (10.2).

We now need to solve the resource constraint for consumption. But, first we require ϕ. We compute ϕ to guarantee that firm profits are zero in a steady state where $\pi = \bar{\pi}$. Let h^{ss} and k^{ss} denote hours worked and capital in such a steady state. Also, let F^{ss} denote gross output of the final good in that steady state. Write sales of final good firm as $F^{ss} - \phi$. Real marginal cost in this steady state is $s^{ss} = 1/\lambda_f$. Since this is a constant, the total costs of the firm are $s^{ss} F^{ss}$. Zero profits requires $s^{ss} F^{ss} = F^{ss} - \phi$. Thus, $\phi = (1 - s^{ss}) F^{ss} = F^{ss}(1 - 1/\lambda_f)$, or,

$$\phi = \left(\frac{k^{ss}}{\mu_z^*}\right)^{\alpha} (h^{ss})^{1-\alpha} \left(1 - \frac{1}{\lambda_f}\right).$$

Solve the steady state version of the resource constraint, (10.5), for c :

$$d + c + I + \Theta \frac{1-\gamma}{\gamma} \left[n - w^e\right] = (p^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left(\frac{k}{\mu_z^*}\right)^{\alpha} \left[(w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h\right]^{1-\alpha} - \phi.$$

Compute the steady state real wage using (8.8):

$$\tilde{w} = s(1 - \alpha) \left[\frac{\mu_z (w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h}{k}\right]^{-\alpha}.$$

Then, solve the labor supply equation, (10.7), for h :

$$h = \left[\frac{\lambda_z}{W \lambda_w \psi_L} \tilde{w}\right]^{\frac{1}{\sigma_L}},$$

where λ_z is obtained using (10.11). These calculations began by fixing a value for r^k. Adjust r^k until the value of h obtained from the above expression coincides with the value implied by (10.12).

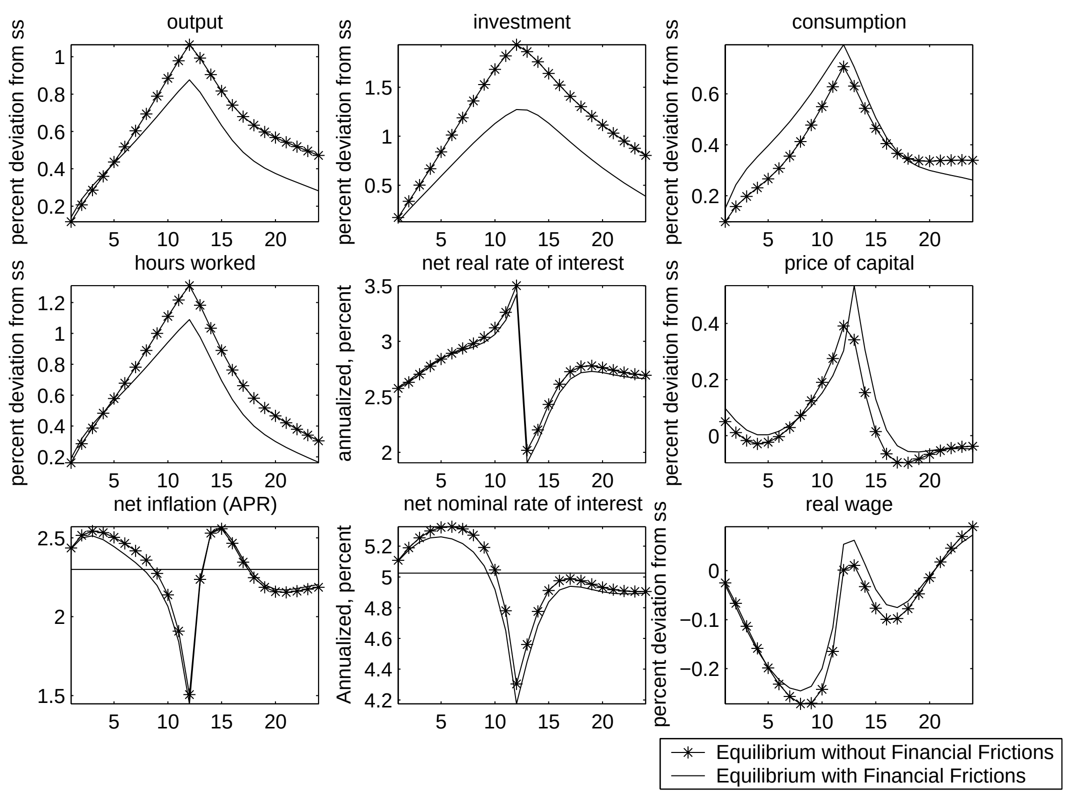
10.3. Simulation Results

We considered the following parameter values:

$$
\begin{aligned}
\lambda_w &= 1.05,\ \sigma_L = 1,\ \beta = 1.01358^{-0.25},\ b = 0.63,\ \psi_L = 109.82,\ \iota_{w,2} = 0.13, \\
\xi_w &= 0.81,\ S'' = 12.3,\ \sigma_a = 1000000,\ F(\bar{\omega}) = 0.013,\ \mu = 0.33,\ \gamma = 1 - 0.0238, \\
w^e &= 0.009,\ \Theta = 0.1,\ \mu_z = 1.0136^{0.25},\ \lambda_f = 1.20,\ \alpha = 0.40,\ \delta = 0.025,\ \xi_p = 0.63, \\
\iota_2 &= 0.84,\ \bar{\pi} = (1 + 0.009152)/\mu_z,\ \alpha_\pi = 1.95,\ \alpha_y = 0.18,\ \rho_i = 0.81,\ \rho = 0.83.
\end{aligned}
$$

We then simulated the response of the economy with the monetary policy rule to a signal shock, and compared it to the response of the economy without the financial frictions. Note that qualitatively, things have not changed much. Interestingly, the financial frictions reduce the response of the economy somewhat. An exception is the price of capital, which has a slightl

Figure 9: Simple Monetary Model and Associated Ramsey Equilibrium



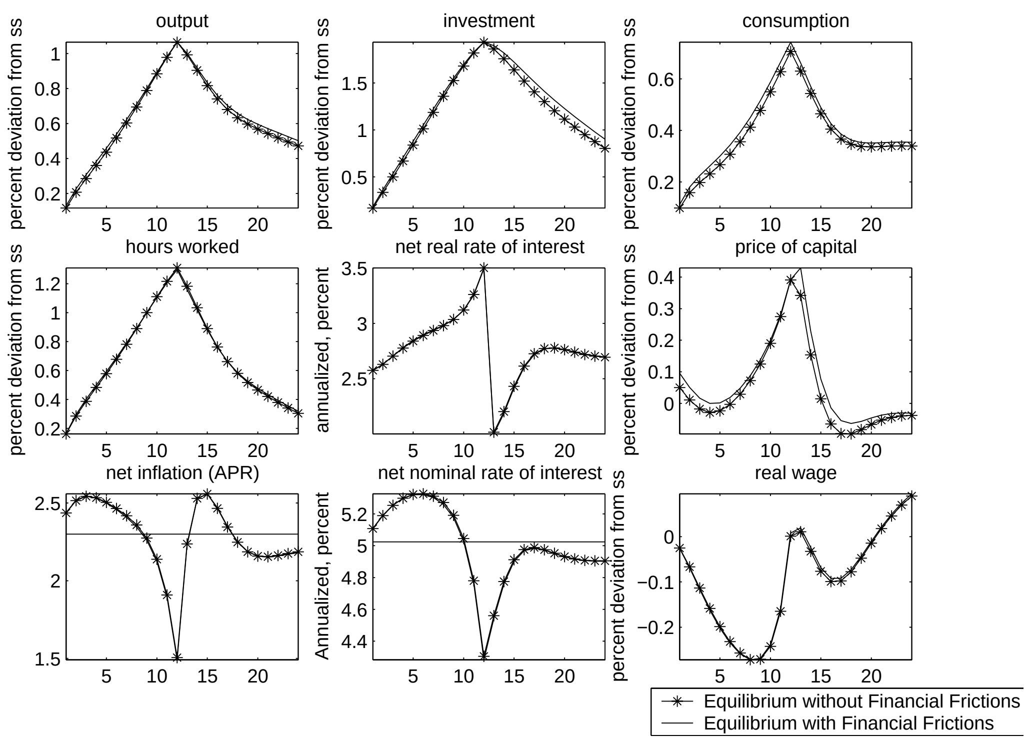
It is interesting to compare the steady state of the economies with and without financial frictions:

$$\frac{y^f}{y} = 0.72,\ \frac{h^f}{h} = 0.94,\ \frac{k^f}{k} = 0.51,$$

where the superscript, f, indicates the economy with financial frictions. Evidently, the impact of financial frictions on the average values of the variables is quite substantial. Ouput (here defined as $c + I$) is reduced by 28 percent, hours by 6 percent and the stock of capital by nearly 50 percent. As a check on the calculations, we repeated the above calculations setting

$\mu = 0.033$. The results are as follows:

Figure 9: Simple Monetary Model and Associated Ramsey Equilibrium

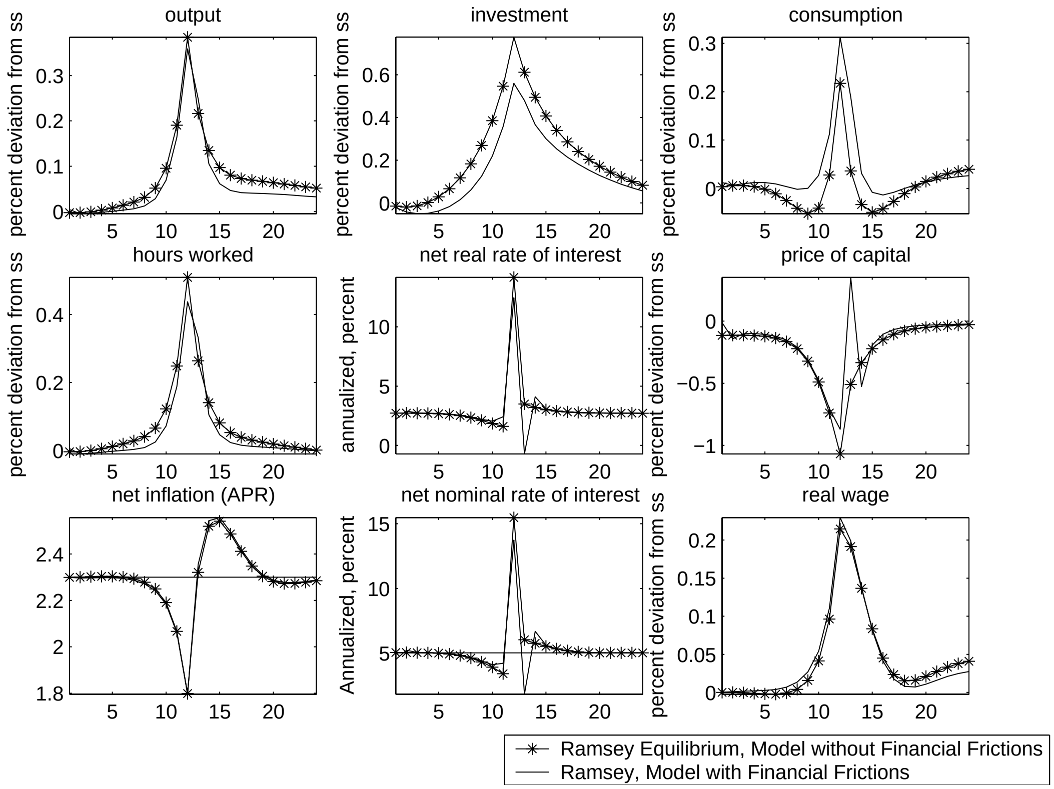


Note that, as expected, the two responses are now vitually the same. Similarly, the steady states are more similar too:

$$\frac{y^f}{y} = 0.93, \ \frac{h^f}{h} = 0.99, \ \frac{k^f}{k} = 0.87.$$

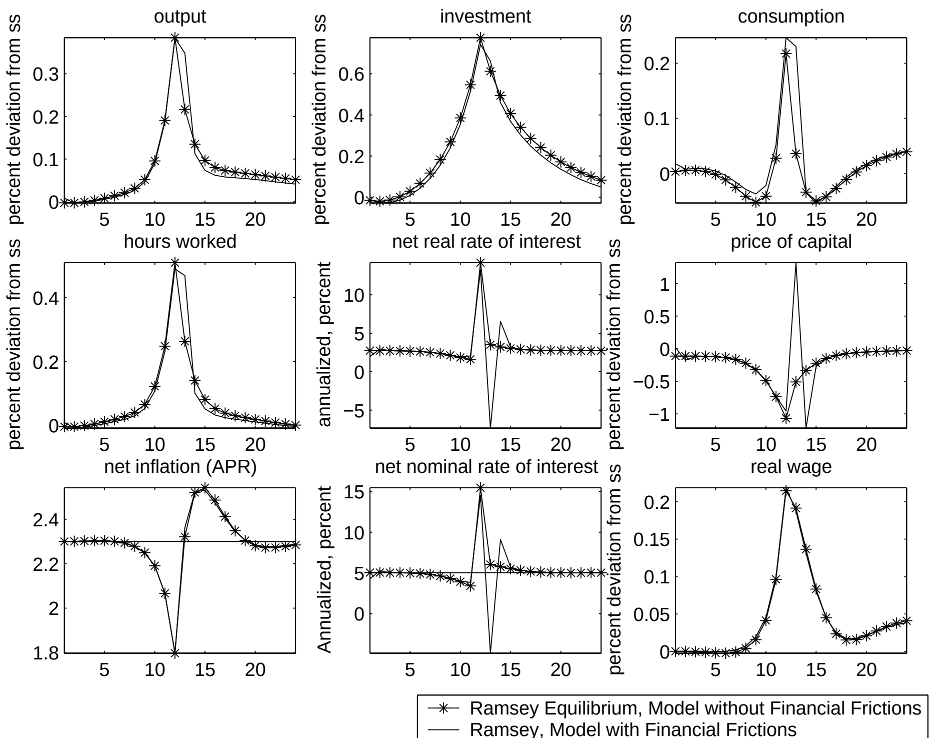
Next, we turn to the Ramsey equilibria. Following is the Ramsey equilibrium of the model

Figure 9: Simple Monetary Model and Associated Ramsey Equilibrium



Note how very similar these paths are. We also considered the case, $\mu = 0.033$. The results are as follows:

Figure 9: Simple Monetary Model and Associated Ramsey Equilibrium



The quantities seem to converge to the $\mu = 0$ case, but the financial variables, the real and nominal rate of interest and the price of capital, seem to not to.

11. Getting the Ramsey Policy Rule

Suppose the endogenous variables are in the $m \times 1$ vector, z_t, and multipliers are in the $n \times 1$ vector, λ_t. The solution to the system can be written:

$$\begin{pmatrix} \lambda_t \\ z_t \end{pmatrix} = A_1 \lambda_{t-1} + A_2 z_{t-1} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} s_t,$$

where A_1 is an $(m+n)\times n$ matrix and A_2 is $(m+n)\times m$. The $(m+n)\times(m+n)$ orthonormal matrix, Q, and the $(m+n)\times n$ upper triangular matrix R denote the elements of the QR decomposition of A_1 :

$$QR = A_1.$$

Then,

$$Q' \begin{pmatrix} \lambda_t \\ z_t \end{pmatrix} = R\lambda_{t-1} + Q' A_2 z_{t-1} + Q' \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} s_t,$$

where

$$R = \begin{bmatrix} \underbrace{\tilde{R}}_{n \times n} \\ \underbrace{0}_{m \times n} \end{bmatrix},$$

where $\tilde{R}$ is upper triangular and square. Write this out more carefully,

$$\begin{bmatrix} Q'_{11} & Q'_{21} \\ Q'_{12} & Q'_{22} \end{bmatrix} \begin{pmatrix} \lambda_t \\ z_t \end{pmatrix} = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} \lambda_{t-1} + Q' A_2 z_{t-1} + Q' \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} s_t$$

Then, the second set of equations says:

The task is to use the second equation to express λ_t as a function of other stuff. Then, use the result to substitute out for λ_{t-1} in an expression involving z_t. Now you've got a system that involves only z_t and s_t.

References

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