Alternative Representation of Technology Shock

• Idea:

  – what actually happens is that technology takes off and people are overoptimistic about how long it will keep going.

  – when their expectations are wrong, they only learn about it slowly.
Alternative Representation of Technology Shock ...

- Time series representation:
  \[ a_t = a_t^T + a_t^P, \]
  where
  \[ a_t^P = \theta_1 a_{t-1}^P + \theta_2 a_{t-2}^P + \varepsilon_t^P \]
  \[ a_t^T = \phi_1 a_{t-1}^T + \phi_2 a_{t-2}^T + \phi_3 a_{t-3}^T + \varepsilon_t^T \]
  \[ S_t = \varepsilon_t^P + \text{meas error}_t \]

- Agents only see \( a_t, S_t \).
- False optimism
  \[ S_t \text{ big (suggesting shock to } a_t^P \text{) } \]
  \[ \text{in fact, the shock is to } a_t^T. \]
- Need (for estimation and forecasting) a law of motion that allows us to forecast \( a_t^T, a_t^P, a_t \) given observations on \( a_t, S_t \).
Alternative Representation of Technology Shock ...

- Set up in state-space/observer form

\[
\xi_t \equiv \begin{pmatrix}
a_t^T \\
ad_t^T \\
ad_{t-1}^T \\
ad_{t-2}^T \\
a_t^P \\
ad_{t-1}^P \\
\varepsilon_t^P 
\end{pmatrix} = \begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_1 & \theta_2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \begin{pmatrix}
a_{t-1}^T \\
ad_{t-2}^T \\
ad_{t-3}^T \\
ad_{t-1}^P \\
ad_{t-2}^P \\
\varepsilon_{t-1}^P 
\end{pmatrix} + \begin{pmatrix}
\varepsilon_t^T \\
0 \\
0 \\
\varepsilon_t^P \\
0 \\
\varepsilon_t^P 
\end{pmatrix}
\]

\[
\xi_t = F\xi_{t-1} + \varepsilon_t, \ E\varepsilon_t\varepsilon_t' = Q = \begin{bmatrix}
\sigma^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^P & 0 & \sigma^P \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^P & 0 & \sigma^P 
\end{bmatrix}
\]

\[
X_t = \begin{pmatrix}
S_t \\
a_t
\end{pmatrix}, \ X_t = H'_{2 \times 6} \underline{\xi_t}_{6 \times 1} + w_t, \ Ew_t w_t' = R = \begin{bmatrix}
\sigma^{\text{meas error}} & 0 \\
0 & 0
\end{bmatrix}.
\]
Alternative Representation of Technology Shock ...

- State space/observer form:
  \[
  \xi_t = F\xi_{t-1} + \varepsilon_t, \quad E\varepsilon_t\varepsilon'_t = Q
  \]
  \[
  X_t = H'\xi_t + w_t, \quad Ew_tw'_t = R.
  \]

- Notation
  \[
  \xi_{t+j|t} \equiv P \left[ \underbrace{\xi_{t+j|t}}_{\text{linear formula}} \bigg| X_t, X_{t-1}, \ldots, X_1 \right]
  \]
  \[
  \xi_{t+j|t} = F^j\xi_{t|t}, \quad X_{t+j|t} = H'\xi_{t+j|t}, \quad j > 0
  \]

- Recursive formula for updating estimate of \( \xi_t \) when \( X_t \) arrives:
  \[
  \hat{\xi}_{t|t} = \hat{\xi}_{t|t-1} + P \left[ \underbrace{\xi_t - \hat{\xi}_{t|t-1}}_{\text{linear projection}} \bigg| X_t - H'\xi_{t|t-1} \right]
  \]
  \[
  = \xi_{t|t-1} + K_t \left[ X_t - H'\xi_{t|t-1} \right]
  \]
Alternative Representation of Technology Shock ...

- The Kalman gain solves the following ‘regression first order conditions’:

\[
0 = E \left[ \xi_t - \xi_{t|t-1} - K_t \left(X_t - H'\xi_{t|t-1}\right) \right] \left(X_t - H'\xi_{t|t-1}\right)'
\]

\[
\rightarrow K_t = P_{t|t-1}H \left[H'P_{t|t-1}H + R\right]^{-1}
\]

\[
P_{t|t-1} \equiv E \left(\xi_t - \xi_{t|t-1}\right) \left(\xi_t - \xi_{t|t-1}\right)'.
\]

- If the first date, \( t = 1 \), is sufficiently far in the past, then, 'steady state Kalman gain'

\[
P_{t|t-1} \rightarrow P, \ K_t \rightarrow \hat{K} \equiv PH \left[H'PH + R\right]^{-1}
\]

\[
P = FPF' + Q - FPH \left[H'PH + R\right]^{-1} H'PF'
\]

- Recursive representation of \( \xi_{t|t} \)

\[
\xi_{t|t} = F\xi_{t-1|t-1} + K \left[X_t - H'F\xi_{t-1|t-1}\right]
\]

\[
= M\xi_{t-1|t-1} + KX_t.
\]

- Note: if eigenvalues of \( M \) inside unit circle, then

\[
\xi_{t|t} = KX_t + MKX_{t-1} + M^2KX_{t-2} + ...
\]
Alternative Representation of Technology Shock ...

• Continue to rewrite:

\[
\xi_{t|t} = M \xi_{t-1|t-1} + K \left( H' [F \xi_{t-1} + \varepsilon_t] + w_t \right) \\
= M \xi_{t-1|t-1} + KH' F \xi_{t-1} + K \left( H' \varepsilon_t + w_t \right).
\]

• Denote

\[
\tilde{\xi}_t \equiv \begin{pmatrix} \xi_t \\ \xi_{t|t} \end{pmatrix}.
\]

• The law of motion of expectations and actuals, \( \tilde{\xi}_t \), is

\[
\tilde{\xi}_t = \tilde{F} \tilde{\xi}_{t-1} + \tilde{\varepsilon}_t, \quad E\tilde{\varepsilon}_t\tilde{\varepsilon}'_t = \tilde{Q},
\]

where

\[
\tilde{F} = \begin{bmatrix} F & 0 \\ KH'F & M \end{bmatrix}, \quad \tilde{\varepsilon}_t = \begin{pmatrix} \varepsilon_t \\ \varepsilon_t \left( KH' \varepsilon_t + w_t \right) \end{pmatrix}, \\
\tilde{Q} = \begin{bmatrix} Q & QHK' \\ KH'Q & KH'QH + R \end{bmatrix} K'.
\]
Alternative Representation of Technology Shock ...

• Example

\[ \sigma_P^2 = \sigma_T^2 = 1, \ \sigma_{\text{meas error}}^2 = 0.1. \]

\[ \phi_1 = 1.90, \ \phi_2 = -1.06, \ \phi_3 = 0.144 \]

\[ \theta_1 = 1.90, \ \theta_2 = -0.9025. \]

• Note:

– the single shock leads to a long sequence of forecast errors
– it takes a long time (over six years) to catch on to what has happened.
– the optimal forecast tends to react to a surprise (after the first period) as though it is partially due to a temporary shock, so the response resembles the ‘flip’ of the temporary response.
CGG Model

• Natural equilibrium:
  – absence of price distortions induces cross-industry efficiency:
    \[ N_{i,t} = N_t \text{ all } i \]
    so that aggregate production relation:
    \[ Y_t = A_t N_t, \ y_t = a_t + n_t \] (1)

  – Labor market efficiency (in logs):
    \[ \log \frac{MRS_t}{c_t + \varphi n_t + \tau_t} = \log \frac{MP_{L,t}}{a_t} \] (2)

  – Combine (1) and (2):
    \[ a_t = y_t + \varphi (y_t - a_t) + \tau_t \]

  – natural level of output and employment:
    \[ y_t^* = a_t, \ n_t^* = y_t^* - a_t = 0. \]
CGG Model ...

– Interest rate in the ‘natural’ equilibrium steers households to choose efficient levels of employment and consumption.
– Household intertemporal Euler equation:

\[ C_{t}^{-1} = \beta E_t C_{t+1}^{-1} R_t / \bar{\pi}_{t+1}. \]

* in logs:

\[ -c_t = r_t - \text{rr} - E_t c_{t+1} - E_t \pi_{t+1} \]

\[ \text{rr} \equiv - \log \beta \]

* substitute ‘natural’ output, \( y_t^* \), and inflation, \( \pi_t = 0 \), into household Euler equation:

\[ \frac{y_t^*}{a_t} = - [rr_t^* - \text{rr}] + E_t(\frac{y_{t+1}^*}{a_{t+1}}) \]

high current technology implies lower natural rate \hspace{1cm} high future technology implies higher natural rate

\[ rr_t^* - \text{rr} = - \frac{y_t^*}{a_t} + \frac{y_{t+1}^*}{a_{t+1}|_t} \]
CGG Model ...

• Actual equilibrium:

Actual equilibrium intertemporal equation: \[ y_t = - \left[ r_t - \pi_{t+1|t} - rr \right] + y_{t+1|t} \]

Natural equilibrium intertemporal equation: \[ y^*_t = - \left[ rr^*_t - rr \right] + y^*_{t+1|t} \]

– subtract:

\[ \underbrace{x_t}_{\text{gap}} = - \left[ r_t - \pi_{t+1|t} - rr^*_t \right] + x_{t+1|t} \]

Employment in actual equilibrium equals the gap

\[ x_t \equiv y_t - y^*_t = a_t + n_t - a_t = \underbrace{n_t}_{\text{gap}} \]

– price-setting with calvo frictions:

\[ \hat{\pi}_t = \frac{(1 - \beta \theta) (1 - \theta)}{\theta} (1 + \varphi) x_t + \beta \hat{\pi}_{t+1|t}, \]
Equations of Taylor rule Equilibrium ($\psi = 0$)

The equations are summarized as follows:

\[ rr^*_t - rr + a_t - a_{t+1|t} = 0 \] (natural (Ramsey) rate)

\[ \tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-1} + \tilde{\varepsilon}_t, \quad E\tilde{\varepsilon}_t\tilde{\varepsilon}'_t = \tilde{Q} \]

\[ \beta \pi_{t+1|t} + \frac{(1 - \theta)(1 - \beta \theta)}{\theta} x_t - \pi_t = 0 \] (Calvo pricing equation)

\[ x_t + [r_t - \pi_{t+1|t} - rr^*_t] - x_{t+1|t} = 0 \] (intertemporal equation)

\[ r_t - \phi_\pi \pi_{t+1|t} - \phi_x x_t = 0 \] (policy rule)
Solving the Sticky Price Model

- Define:

\[ z_t = \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ r^*_t \\ a_t \end{pmatrix} . \]

- We seek a solution of the form:

\[ z_t = A z_{t-1} + B \xi_{t|t} . \]

- \( A, B \) determined by requirement that equilibrium conditions are satisfied.
Solving the Sticky Price Model ... 

- The equilibrium conditions are expressed in matrix form as follows:

\[
E_t \{ \begin{bmatrix}
\beta & 0 & 0 & 0 & 0 \\
\frac{1}{\sigma} & 1 & 0 & 0 & 0 \\
(1 - \alpha)\phi_\pi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\pi_{t+1} \\
x_{t+1} \\
r_{t+1} \\
rr_{t+1} \\
a_{t+1} \\
r_{t+1} - \pi_{t+2}
\end{bmatrix}
\}
\]

\[
+ \begin{bmatrix}
-1 & \kappa & 0 & 0 & 0 \\
0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} & 0 \\
0 & (1 - \alpha)\phi_x & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\pi_t \\
x_t \\
r_t \\
rr_t \\
a_t \\
r_t - \pi_{t+1}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\pi_{t-1} \\
x_{t-1} \\
r_{t-1} \\
rr_{t-1} \\
a_{t-1} \\
r_{t-1} - \pi_t
\end{bmatrix}
\]

\[
H'^t \xi_{t+1|t} + H'^t \xi_{t|t} = 0
\]
Solving the Sticky Price Model ...

• The full system:

\[
\alpha_0 z_{t+1|t} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 H' F \xi_{t|t} + \beta_1 H' \xi_t = 0
\]

\[
\xi_t = \tilde{F} \xi_{t-1} + \tilde{\epsilon}_t.
\]

• Evidently:

\[
z_{t+1|t} = A z_t + B F \xi_{t|t}
\]

\[
z_t = A z_{t-1} + B \xi_{t|t}
\]

• after substituting,

\[
\alpha_0 \left[ A (A z_{t-1} + B \xi_{t|t}) + B F \xi_{t|t} \right] + \alpha_1 \left( A z_{t-1} + B \xi_{t|t} \right) + \alpha_2 z_{t-1} + [\beta_0 H' F + \beta_1 H'] \xi_{t|t} = 0
\]

\[
\left[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 \right] z_{t-1} + \left[ \alpha_0 (AB + BF') + \alpha_1 B + \beta_0 H' F + \beta_1 H' \right] \xi_{t|t} = 0
\]

which implies:

\[
\alpha (A) = 0
\]

\[
\alpha_0 \left( A \begin{bmatrix} B \\ 5 \times 6 \end{bmatrix} + BF \right) + \alpha_1 B + \beta_0 H' F + \beta_1 H' = \begin{bmatrix} 0 \\ 5 \times 6 \end{bmatrix}
\]
Solving the Sticky Price Model ...

- $A = 0$ solves $\alpha (A) = 0$

- Recall:

\[
vec (A_1 A_2 A_3) = (A'_3 \otimes A_1) vec (A_2)
\]

so that,

\[
0 = vec [\alpha_0 (AB + BF)] + \alpha_1 B + \beta_0 H' F + \beta_1 H'
\]

\[
= vec [\alpha_0 AB] + vec [\alpha_0 BF] + vec [\alpha_1 B] + vec [\beta_0 H' F] + vec [\beta_1 H'].
\]

- note

\[
vec [\alpha_0 AB] = (I \otimes (\alpha_0 A)) vec (B), \quad vec [\alpha_0 BF] = (F' \otimes \alpha_0) vec (B), \\
vec [\alpha_1 B] = (I \otimes \alpha_1) vec (B),
\]
Solving the Sticky Price Model ...

• need to solve for \( B \):

\[
\alpha_0 BF + \alpha_1 B + \beta_0 H'F + \beta_1 H' = 0,
\]

– or,

\[
[(F' \otimes \alpha_0) + (I \otimes \alpha_1)] \, vec(B) = -vec[\beta_0 H'F] - vec[\beta_1 H'].
\]

– or,

\[
Z \delta = d,
\]

– where

\[
Z = (F' \otimes \alpha_0) + (I \otimes \alpha_1)
\]
\[
d = -vec[\beta_0 H'F] - vec[\beta_1 H']
\]
\[
\delta = vec(B).
\]

– so, the solution is found:

\[
\delta = Z^{-1}d.
\]
Solving the Sticky Price Model ...

- Model solution:

\[ z_t = Az_{t-1} + BD\tilde{\xi}_t \]

\[ \tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-1} + \tilde{\varepsilon}_t, \quad E\tilde{\varepsilon}_t\tilde{\varepsilon}_t' = \tilde{Q} \]

\[ \tilde{\xi}_t \equiv \begin{pmatrix} \xi_t \\ \xi_{t|t} \end{pmatrix} \]

\[ D = \begin{pmatrix} 0 & I \end{pmatrix}. \]

- Can use this as a basis for simulation and/or estimation.
Solving the Sticky Price Model ...

- Example #1

\[ \phi_x = 0, \phi_\pi = 1.5, \beta = 0.99, \varphi = 1, \theta = 0.75. \]
\[ \phi_1 = 1.90, \phi_2 = -1.06, \phi_3 = 0.144 \text{ (roots: 0.9, 0.8, 0.2)} \]

\[ \theta_1 = 1.9, \theta_2 = -0.9025 \text{ (roots: 0.95, 0.95)} \]

inflation is always positive (bad!), hours eventually converges from below (bad!).
Solving the Sticky Price Model ...

- Example #2
  - transitory part is purely iid

\[ \phi_1 = \phi_2 = \phi_3 = 0 \]

- hours work strong (good!) inflation high (bad!).
Solving the Sticky Price Model ...

- Example #3: like previous example, except $\theta$ raised to 0.95

- rise in gap massive, rise in inflation smaller.
Example #4. Set $\beta = 0.999$ (raising it only by 0.009).

- Now, both actual and expected inflation drop (good!). Output gap is gigantic.