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416, Spring 2008

Homework 3, due May 2.

1. Suppose the equilibrium of a model satisfies the following difference equation:

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0, \quad z_t \sim n \times 1.$$

where z_{-1} is given at time 0. Define

$$Y_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix},$$

and the first order representation of Y_t is

$$aY_{t+1} + bY_t = 0,$$

where

$$a_{2n \times 2n} = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \quad b_{2n \times 2n} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ I & 0 \end{bmatrix}.$$

Suppose that α_0 is invertible, so that a is too. Suppose $\pi \equiv -a^{-1}b$ has the following decomposition:

$$\pi = T\Lambda T^{-1}, \quad T = \begin{bmatrix} T_1 & \cdots & T_{2n} \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} \tilde{T}_1 \\ \vdots \\ \tilde{T}_{2n} \end{bmatrix},$$

where Λ is a diagonal matrix with the eigenvalues of π on the diagonal, T_i is $2n \times 1$ and $\tilde{T}_i$ is $1 \times 2n$, for $i = 1, \dots, 2n$. A ‘solution’ is a sequence, $\{Y_t\}$, which satisfies the difference equation and is consistent with z_{-1} . The space of solutions can be represented as a point in R^n . A ‘minimal state space solution’ (MSV) is a solution that has the property that $DY_t = 0$ for all t , where D is $n \times 2n$. A ‘convergent solution’ is a solution having the property, $Y_t \rightarrow 0$ as $t \rightarrow \infty$. Let m denote the the number of explosive eigenvalues of π (i.e., the ones that exceed unity in absolute value).

1. Suppose $m = n$. Show that there is exactly one convergent solution and that that solution is an MSV. Write

$$D = \begin{bmatrix} D_1 : D_2 \end{bmatrix},$$

where D_1 is square and D is the unique matrix with the property, $DY_t = 0$. Then, define

$$A = -D_1^{-1}D_2.$$

(You may simply assume that D_1 is invertible.) Show that

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0,$$

and that the eigenvalues of A lie inside the unit circle.

2. Show that if $m < n$, then there are many convergent solutions. How many solutions are MSV solutions? Show that there are more solutions.
 3. Show that if $m > n$, then there is no convergent solution.
2. Suppose that the rank of a is less than $2n$, so that a is not invertible. The QZ decomposition of a and b is two orthonormal matrices, Q and Z , such that

$$QaZ = H_0, \quad QbZ = H_1,$$

where H_0 and H_1 are upper triangular matrices.¹ (That Q and Z are orthonormal implies $QQ' = I$, $ZZ' = I$.) Without loss of generality, you can assume that the l diagonal elements of H_0 that are zero all appear along the bottom right of H_0 .² Consider the following partitioning of H_0 and H_1 :

$$H_0 = \begin{bmatrix} G_0 & H_{12}^0 \\ (2n-l) \times (2n-l) & \\ 0 & H_{22}^0 \\ & l \times l \end{bmatrix}, \quad H_1 = \begin{bmatrix} G_1 & H_{12}^1 \\ (2n-l) \times (2n-l) & \\ 0 & H_{22}^1 \\ & l \times l \end{bmatrix},$$

where G_0 and H_{22}^0 are upper triangular, the diagonal elements of G_0 are non-zero, while the diagonal elements of H_{22}^0 are all zero. The object, G_1 ,

¹This approach is described in detail in the section, ‘the non-invertible a case’ in Christiano, Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients, published in a special issue of the journal, Computational Economics, 2001. You can find a copy at <http://faculty.wcas.northwestern.edu/~lchrist/research/Solve/papernew.pdf>.

²To compute the QZ decomposition as it is defined in the text, first run the MATLAB command, `[H0,H1, Q, Z] = qz(a,b)`. Then, to ensure that the diagonal elements of H_0 satisfy the property described in the text, you can use the MATLAB command `ordqz`. Alternatively, you can use a routine that Christopher Sims wrote. Set `stake1 = 1e+08`, and `[H0,H1,q,z] = qzdiv(stake1,H0,H1,Q,Z)`. To run `qzdiv`, you will also need `qzswitch.m`.

is upper triangular, and you may assume that the diagonal elements of the upper triangular matrix, H_{22}^1 , are nonzero. Define

$$\gamma_t = \begin{pmatrix} \gamma_{1t} \\ (2n-l) \times 1 \\ \gamma_{2t} \\ l \times 1 \end{pmatrix} = Z' Y_t = \begin{pmatrix} L_1 \\ (2n-l) \times 2n \\ L_2 \\ l \times 2n \end{pmatrix} Y_t, \text{ where } Z' = \begin{pmatrix} L_1 \\ (2n-l) \times 2n \\ L_2 \\ l \times 2n \end{pmatrix}.$$

1. Explain why $l < n$ and show that

$$H_0 \gamma_{t+1} + H_1 \gamma_t = 0, \quad t = 0, 1, 2, \dots$$

2. Prove that the previous expression, plus the structure on H_{22}^0 and H_{22}^1 , imply $\gamma_{2t} = 0$, $t = 0, 1, 2, \dots$. (Hint: consider the very last element of γ_{2t} first, then the next-to-last element, etc.) Thus, the system may be written as the following ‘smaller’ system:

$$\begin{aligned} G_0 \gamma_{1,t+1} + G_1 \gamma_{1,t} &= 0, \quad t = 0, 1, 2, \dots, \\ L_2 Y_t &= 0 \end{aligned}$$

in which the matrix, G_0 , is invertible. We may write

$$\gamma_{1,t} = \pi^t \gamma_{1,0}, \quad \pi = -G_0^{-1} G_1.$$

3. How many solutions to the system are there? (Hint: recall that there are n free elements in Y_0 and that l of these are ‘used up’ by the requirement, $L_2 Y_t = 0$.)
4. How many MSV solutions are there?
5. In the previous question, eigenvalue conditions were discussed that guarantee that there a unique convergent solution, more than one convergent solution and no convergent solution. Derive the relevant eigenvalue conditions in for the current situation.
6. Explain how to construct the matrix, A , such that

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0.$$

3. Consider the economy in homework 1.

Solve the model again, but this time do so *without* substituting out for labor in the dynamic Euler equation. Set the externality parameter, γ to zero. Set up the system:

$$aY_{t+1} + bY_t = 0, \quad t = 0, 1, 2, \dots \quad (0.1)$$

Note that a is not invertible. One way to proceed is to do the substitutions you did in homework 1, identifying $z_t = \begin{bmatrix} \hat{k}_{t+1} & \hat{n}_t \end{bmatrix}'$. An alternative strategy uses the QZ decomposition, as in the previous question. It also removes the singularity in a system by reducing its size and producing a new system whose a matrix is invertible (the a matrix in the reduced system is denoted G_0).

1. In this example, H_0 , H_1 , Q , Z , a and b matrices are 4×4 .³ Display these matrices for the parameterization used in homework 1 (except, $\gamma = 0$). What is the value of l ? Verify that the diagonal terms in G_0 and H_{22}^1 are nonzero, and that the diagonal terms in H_{22}^0 are zero.
2. How many explosive eigenvalues of π do you require in order for the system to have a unique non-explosive solution? How many explosive eigenvalues does it have in fact?
3. Display the (hopefully!) unique 2×4 matrix, D , having the property

$$DY_t = 0,$$

for $t = 0, 1, 2, \dots$. Verify that when D is partitioned as follows

$$D = \begin{bmatrix} D_1 & \vdots & D_2 \end{bmatrix},$$

where D_1 is 2×2 , then D_1 is invertible.

4. Display:

$$A = -D_1^{-1}D_2.$$

Verify: the second column of A is zero; $\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0$; and this solution coincides with the solution you obtained in homework 1.

5. Consider a value of γ that produces multiplicity. Can you find more than one MSV? If so, display two values of A that have eigenvalues less than unity, which satisfy $\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0$.

³There is a way to ammend the discussion in question 2, so that the matrices are 3×3 .

4. Consider the nonstochastic version of the Clarida-Gali-Gertler model:

$$\begin{aligned}\delta\pi_{t+1} + \lambda y_t - \pi_t &= 0 \\ y_{t+1} - y_t - \frac{1}{\sigma}(r_t - \pi_{t+1}) &= 0 \\ \rho r_{t-1} + (1 - \rho)\beta\pi_{t+1} + (1 - \rho)\gamma y_t - r_t &= 0.\end{aligned}$$

Write this system in the form:

$$AY_{t+1} + BY_t = 0,$$

for $t = 1, \dots$. Here, $Y_t = (\pi_t, y_t, r_{t-1})$. All variables are in deviations from their steady state. Use the following parameter values, $\delta = 0.99$, $\sigma = 1$, $\lambda = 0.3$, $\gamma = 0.15$, $\rho = 0.5$, $\beta = 1.5$. Let the initial interest rate, r_0 , have its steady state value of 0.

1. Compute T and Λ in the eigenvector, eigenvalue decomposition of $-A^{-1}B$.
2. Compute the set of MSV solutions (i.e., the set of 2×3 matrices with the property $DY_1 = 0 \implies DY_t = 0$ for all t).
3. Suppose the first period is $t = 1$, and time evolves for $t = 1, 2, 3, \dots, 20$. Construct the following 3 graphs, one corresponding to each MSV solution. In each graph, display two lines of length 20 each, with $t = 1, \dots, 20$ on the horizontal axis. One line, a line of zeros, corresponds to the nonstochastic, steady state equilibrium of the model. The other is a convergent solution to the Euler equations in which π_1 is a little greater than zero. Note how the high inflation equilibrium, while eventually returning to steady state, displays a period of high inflation and high output.
4. Repeat c, replacing the first equation with

$$\delta\pi_{t+1} + \lambda y_t + r_t - \pi_t = 0.$$

The presence of r_t is intended to capture a ‘financial friction’ in the supply side of the economy.

5. Repeat c, with the original version of the first model equation, and with $\beta = 0.5$.