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Solving Dynamic General Equilibrium Models Using Log Linear Approximation

Log-linearization strategy

- Example #1: A Simple RBC Model.
  - Define a Model 'Solution'
  - Motivate the Need to Somehow Approximate Model Solutions
  - Describe Basic Idea Behind Log Linear Approximations
  - Some Strange Examples to be Prepared For
    'Blanchard-Kahn conditions not satisfied'
- Example #2: Putting the Stochastic RBC Model into General Canonical Form
- Example #3: Stochastic RBC Model with Hours Worked (Matrix Generalization of Previous Results)
- Example #4: Example #3 with 'Exotic' Information Sets.
- Summary so Far.
- Example #5: Sticky price model with no capital - log linearizing about a particular benchmark.
  - Will exploit the example to derive the nonlinear equilibrium conditions of a New Keynesian model (will be used later in discussions of optimal policy).

Example #1: Nonstochastic RBC Model

$$\text{Maximize}_{\{c_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to:

$$C_t + K_{t+1} - (1-\delta)K_t \ = \ K_t^{\alpha}, \ K_0 \text{ given}$$

First order condition:

$$C_t^{-\sigma} - \beta C_{t+1}^{-\sigma}\left[\alpha K_{t+1}^{\alpha-1} + (1-\delta)\right],$$

or, after substituting out resource constraint:

$$v(K_t, K_{t+1}, K_{t+2}) \ = \ 0, \ t = 0, 1, ..., \ \text{with } K_0 \text{ given.}$$

Example #1: Nonstochastic RBC Model ...

- 'Solution': a function, $K_{t+1} = g(K_t)$, such that

$$v(K_t, g(K_t), g[g(K_t)]) \ = \ 0, \text{ for all } K_t.$$

- Problem:

This is an Infinite Number of Equations
(one for each possible K_t)
in an Infinite Number of Unknowns
(a value for g for each possible K_t)

- With Only a Few Rare Exceptions this is Very Hard to Solve Exactly
  - Easy cases:
    - If $\sigma = 1, \delta = 1 \Rightarrow g(K_t) = \alpha\beta K_t^{\alpha}$.
    - If v is linear in K_t, K_{t+1}, K_{t+1}.
  - Standard Approach: Approximate v by a Log Linear Function.

Approximation Method Based on Linearization

- Three Steps
  - Compute the Steady State
  - Do a Log Linear Expansion About Steady State
  - Solve the Resulting Log Linearized System
- Step 1: Compute Steady State -
  - Steady State Value of K, K^* -

$$\begin{aligned} & C^{-\sigma} - \beta C^{-\sigma}\left[\alpha K^{\alpha-1} + (1-\delta)\right] = 0, \\ \Rightarrow\ & \alpha K^{\alpha-1} + (1-\delta) = \frac{1}{\beta} \\ \Rightarrow\ & K^* = \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

  - K^* satisfies:

$$v(K^*, K^*, K^*) = 0.$$

- Step 2:
  - Replace v by First Order Taylor Series Expansion About Steady State:

$$v_1(K_t - K^*) + v_2(K_{t+1} - K^*) + v_3(K_{t+2} - K^*) \ = \ 0,$$

  - Here,

$$v_1 \ = \ \frac{dv_u(K_t, K_{t+1}, K_{t+2})}{dK_t}, \text{ at } K_t = K_{t+1} = K_{t+2} = K^*.$$

  - Conventionally, do *Log-Linear Approximation*:

$$(v_1 K)\,\hat{K}_t + (v_2 K)\,\hat{K}_{t+1} + (v_3 K)\,\hat{K}_{t+2} \ = \ 0,$$

$$\hat{K}_t \equiv \frac{K_t - K^*}{K^*}.$$

  - Write this as:

$$\alpha_2 \hat{K}_t + \alpha_1 \hat{K}_{t+1} + \alpha_0 \hat{K}_{t+2} \ = \ 0,$$

$$\alpha_2 = v_1 K, \ \alpha_1 = v_2 K, \ \alpha_0 = v_3 K$$

Approximation Method Based on Linearization ...

- Step 3: Solve
  - Posit the Following Policy Rule:

    $$\hat{K}_{t+1} = A\hat{K}_t,$$

    Where A is to be Determined.
  - Compute A :

    $$\alpha_2\hat{K}_t + \alpha_1 A\hat{K}_t + \alpha_0 A^2\hat{K}_t = 0,$$

    or

    $$\alpha_2 + \alpha_1 A + \alpha_0 A^2 = 0.$$
  - A is the Eigenvalue of Polynomial
- In General: Two Eigenvalues.
  - Can Show: In RBC Example, One Eigenvalue is Explosive. The Other Not.
  - There Exist Theorems (see Stokey-Lucas, chap. 6) That Say You Should Ignore the Explosive A.

Some Strange Examples to be Prepared For

- Other Examples Are Possible:
  - Both Eigenvalues Explosive
  - Both Eigenvalues Non-Explosive
  - What Do These Things Mean?

Some Strange Examples to be Prepared For ...

- Example With Two Explosive Eigenvalues
- Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^\gamma}{\gamma}, \; \gamma < 1.$$

- Technology:
  - Production of Consumption Goods

$$C_t = k_t^\alpha n_t^{1-\alpha}$$

  - Production of Capital Goods

$$k_{t+1} = 1 - n_t.$$

Some Strange Examples to be Prepared For ...

- Planning Problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left[k_t^{\alpha}\left(1-k_{t+1}\right)^{1-\alpha}\right]^{\gamma}}{\gamma}$$

- Euler Equation:

$$\begin{aligned} v(k_t, k_{t+1}, k_{t+2}) &= -(1-\alpha)k_t^{\alpha\gamma}(1-k_{t+1})^{[(1-\alpha)\gamma-1]} + \beta\alpha k_{t+1}^{(\alpha\gamma-1)}\left(1-k_{t+2}\right)^{(1-\alpha)\gamma} \\ &= 0, \end{aligned}$$

$t = 0, 1, ...$

- Steady State:

$$k = \frac{\alpha\beta}{1-\alpha+\alpha\beta}.$$

Some Strange Examples to be Prepared For ...

- Log-linearize Euler Equation:

$$\alpha_0\hat{k}_{t+2} + \alpha_1\hat{k}_{t+1} + \alpha_2\hat{k}_t \;=\; 0$$

- With $\beta = 0.58$, $\gamma = 0.99$, $\alpha = 0.6$, *Both* Roots of Euler Equation are explosive:

$$-1.6734, \;\; -1.0303$$

- Other Properties:
  - Steady State:

$$0.4652$$

  - Two-Period Cycle:

$$0.8882, \; 0.0870$$

Some Strange Examples to be Prepared For ...

- Meaning of Stokey-Lucas Example
  - Illustrates the Possibility of All Explosive Roots
  - Economics:
    - If Somehow You Start At Single Steady State, Stay There
    - If You are Away from Single Steady State, Go Somewhere Else
  - If Log Linearized Euler Equation Around Particular Steady State Has Only Explosive Roots
    - All Possible Equilibria Involve Leaving that Steady State
    - Log Linear Approximation Not Useful, Since it Ceases to be Valid Outside a Neighborhood of Steady State
  - Could Log Linearize About Two-Period Cycle (That's Another Story...)
  - The Example Suggests That *Maybe* All Explosive Root Case is Unlikely

  - 'Blanchard-Kahn conditions not satisfied, too many explosive roots'

Some Strange Examples to be Prepared For ...

- Another Possibility:
  - Both Roots Stable
  - Many Paths Converge Into Steady State: Multiple Equilibria
  - Can Happen For Many Reasons
    - Strategic Complementarities Among Different Agents In Private Economy
    - Certain Types of Government Policy
  - This is a More Likely Possibility
  - Avoid Being Surprised by It By Always Thinking Through Economics of Model.

Some Strange Examples to be Prepared For ...

- Strategic Complementarities Between Agent A and Agent B
  - Payoff to agent A is higher if agent B is working harder
  - In following setup, strategic complementarities give rise to two equilibria:

Me	Everyone else	
	work hard	take it easy
work hard	3	0
take it easy	1	1

  - Example closer to home: every firm in the economy has a 'pet investment project' which only seems profitable if the economy is booming
    - Equilibrium #1: each firm conjectures all other firms will invest, this implies a booming economy, so it makes sense for each firm to invest.
    - Equilibrium #2: each firm conjectures all other firms will not invest, so economy will stagnate and it makes sense for each firm not to invest.

Some Strange Examples to be Prepared For ...

- Example even closer to home:
  - $*$ firm production function -

$$y_t = A_t K_t^{\alpha} h_t^{1-\alpha},$$

$$A_t = Y_t^{\gamma},\ Y_t \text{ ˜ economy-wide average output}$$

  - $*$ resource constraint -

$$C_t + K_{t+1} - (1-\delta) K_t = Y_t$$

  - $*$ equilibrium condition -

$Y_t = y_t$ ‘economy-wide average output is average of individual firms’ production’

  - $*$ household preferences -

$$\sum_{t=0}^{\infty} \beta^t u(C_t, h_t)$$

  - $*$ γ large enough leads to two stable eigenvalues, multiple equilibria.

Some Strange Examples to be Prepared For ...

- Lack of Commitment in Government Policy Can Lead to Multiple Equilibria

  – Simple economy: households solve

$$\max u(c,h) = c - \frac{1}{2}l^2$$

$$c \le (1-\tau)wh,$$

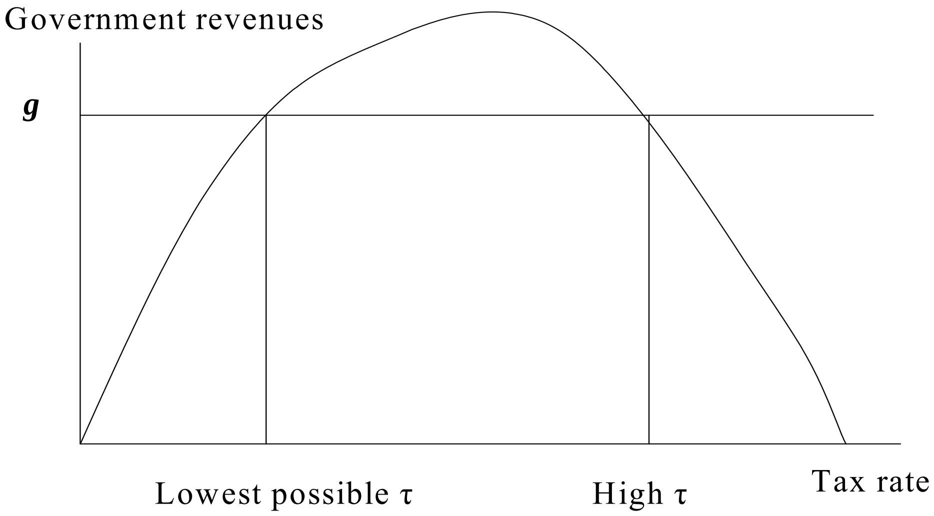
  w is technologically determined marginal product of labor.

  – Government chooses τ to satisfy its budget constraint

$$g \le \tau wl$$

Some Strange Examples to be Prepared For ...

- Laffer curve



- Two scenarios depending on ‘order of moves’
  - $*$ commitment: (i) government sets τ (ii) private economy acts
    - $\cdot$ lowest possible τ only possible outcome
  - $*$ no commitment: (i) private economy determines h (ii) government chooses τ
    - $\cdot$ at least two possible equilibria - lowest possible τ or high τ

- For an environment like this that leads to too many stable eigenvalues, see Schmitt-Grohe and Uribe paper on balanced budget, *JPE*.

Example #2: RBC Model With Uncertainty

- Model

  $$\text{Maximize } E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

  subject to

  $$C_t + K_{t+1} - (1-\delta)K_t \; = \; K_t^{\alpha}\varepsilon_t,$$

  where ε_t is a stochastic process with $E\varepsilon_t = \varepsilon$, say. Let

  $$\hat{\varepsilon}_t \; = \; \frac{\varepsilon_t - \varepsilon}{\varepsilon},$$

  and suppose

  $$\hat{\varepsilon}_t \; = \; \rho\hat{\varepsilon}_{t-1} + e_t, \; e_t \tilde{} N(0, \sigma_e^2).$$

- First Order Condition:

  $$E_t \left\{ C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[\alpha K_{t+1}^{\alpha-1} \varepsilon_{t+1} + 1 - \delta \right] \right\} = 0.$$

Example #2: RBC Model With Uncertainty ...

- First Order Condition:

$$E_t v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0,$$

where

$$\begin{aligned} & v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \; & \left(K_t^{\alpha}\varepsilon_t + (1-\delta)K_t - K_{t+1}\right)^{-\sigma} \\ & -\beta \left(K_{t+1}^{\alpha}\varepsilon_{t+1} + (1-\delta)K_{t+1} - K_{t+2}\right)^{-\sigma} \\ & \times \left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta\right]. \end{aligned}$$

- Solution: a $g(K_t, \varepsilon_t)$, Such That

$$E_t v\left(g(g(K_t, \varepsilon_t), \varepsilon_{t+1}), g(K_t, \varepsilon_t), K_t, \varepsilon_{t+1}, \varepsilon_t\right) = 0,$$

For All K_t, ε_t.

- Hard to Find g, Except in Special Cases
  - One Special Case: v is Log Linear.

Example #2: RBC Model With Uncertainty ...

- Log Linearization Strategy:
  - Step 1: Compute Steady State of K_t when ε_t is Replaced by $E\varepsilon_t$
  - Step2: Replace v By its Taylor Series Expansion About Steady State.
  - Step 3: Solve Resulting Log Linearized System.
- Logic: If Actual Stochastic System Remains in a Neighborhood of Steady State, Log Linear Approximation Good

- Caveat: Strategy not accurate in all conceivable situations.
  - Example: suppose that where I live -

$$\varepsilon \equiv \text{temperature} = \begin{cases} \text{140 Fahrenheit, 50 percent of time} \\ \text{0 degrees Fahrenheit the other half} \end{cases}.$$

  - On average, temperature quire nice: $E\varepsilon = 70$ (like parts of California)
  - Let $K =$ capital invested in heating and airconditioning
    - EK *very*, *very* large!
    - Economist who predicts investment based on replacing ε by $E\varepsilon$ would predict $K = 0$ (as in many parts of California)
  - In standard model this is not a big problem, because shocks are not so big....steady state value of K (i.e., the value that results eventually when ε is replaced by $E\varepsilon$) is approximately $E\varepsilon$ (i.e., the average value of K when ε is stochastic).

Example #2: RBC Model With Uncertainty ...

- Step 1: Steady State:

$$K^* = \left[\frac{\alpha\varepsilon}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.$$

- Step 2: Log Linearize -

$$\begin{aligned}
& v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
\simeq\ & v_1\left(K_{t+2} - K^*\right) + v_2\left(K_{t+1} - K^*\right) + v_3\left(K_t - K^*\right) \\
& + v_3\left(\varepsilon_{t+1} - \varepsilon\right) + v_4\left(\varepsilon_t - \varepsilon\right) \\
=\ & v_1 K^*\left(\frac{K_{t+2} - K^*}{K^*}\right) + v_2 K^*\left(\frac{K_{t+1} - K^*}{K^*}\right) + v_3 K^*\left(\frac{K_t - K^*}{K^*}\right) \\
& + v_3\varepsilon\left(\frac{\varepsilon_{t+1} - \varepsilon}{\varepsilon}\right) + v_4\varepsilon\left(\frac{\varepsilon_t - \varepsilon}{\varepsilon}\right) \\
=\ & \alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t.
\end{aligned}$$

- Step 3: Solve Log Linearized System
  - Posit:
$$\hat{K}_{t+1} = A\hat{K}_t + B\hat{\varepsilon}_t.$$
  - Pin Down A and B By Condition that log-linearized Euler Equation Must Be Satisfied.
    - Note:
$$\begin{aligned}\hat{K}_{t+2} &= A\hat{K}_{t+1} + B\hat{\varepsilon}_{t+1} \\ &= A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}.\end{aligned}$$
    - Substitute Posited Policy Rule into Log Linearized Euler Equation:
$$E_t\left[\alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t\right] = 0,$$

      so must have:
$$\begin{aligned}&E_t\{\alpha_0\left[A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}\right] \\ &+\alpha_1\left[A\hat{K}_t + B\hat{\varepsilon}_t\right] + \alpha_2\hat{K}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1\hat{\varepsilon}_t\} = 0\end{aligned}$$

Example #2: RBC Model With Uncertainty ...

$*$ Then,

$$\begin{aligned}
&E_t\left[\alpha_0\hat{K}_{t+2}+\alpha_1\hat{K}_{t+1}+\alpha_2\hat{K}_t+\beta_0\hat{\varepsilon}_{t+1}+\beta_1\hat{\varepsilon}_t\right]\\
&=E_t\{\alpha_0\left[A^2\hat{K}_t+AB\hat{\varepsilon}_t+B\rho\hat{\varepsilon}_t+Be_{t+1}\right]\\
&+\alpha_1\left[A\hat{K}_t+B\hat{\varepsilon}_t\right]+\alpha_2\hat{K}_t+\beta_0\rho\hat{\varepsilon}_t+\beta_0e_{t+1}+\beta_1\hat{\varepsilon}_t\}\\
&=\alpha(A)\hat{K}_t+F\hat{\varepsilon}_t\\
&=0
\end{aligned}$$

where

$$\begin{aligned}
\alpha(A) &= \alpha_0A^2+\alpha_1A+\alpha_2,\\
F &= \alpha_0AB+\alpha_0B\rho+\alpha_1B+\beta_0\rho+\beta_1
\end{aligned}$$

$*$ Find A and B that Satisfy:

$$\alpha(A) = 0,\ F=0.$$

Example #3 RBC Model With Hours Worked and Uncertainty

- Maximize

$$E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \;=\; f(K_t, N_t, \varepsilon_t)$$

and

$$E\varepsilon_t = \varepsilon,$$

$$\hat{\varepsilon}_t \;=\; \rho\hat{\varepsilon}_{t-1} + e_t,\; e_t\,\tilde{}\,N(0, \sigma_e^2)$$

$$\hat{\varepsilon}_t \;=\; \frac{\varepsilon_t - \varepsilon}{\varepsilon}.$$

Example #3 RBC Model With Hours Worked and Uncertainty ...

- First Order Conditions:

$$E_t v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0$$

and

$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

where

$$\begin{aligned} & v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \ & U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & -\beta U_c\left(f(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, N_{t+1}\right) \\ & \times\left[f_K(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + 1 - \delta\right] \end{aligned}$$

and,

$$\begin{aligned} & v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\ = \ & U_N\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & +U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\ & \times f_N(K_t, N_t, \varepsilon_t). \end{aligned}$$

- Steady state K^* and N^* such that Equilibrium Conditions Hold with $\varepsilon_t \equiv \varepsilon$.

- Log-Linearize the Equilibrium Conditions:

$$
\begin{aligned}
& v_K(K_{t+2}, N_{t+1,}K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
= \; & v_{K,1}K^*\hat{K}_{t+2} + v_{K,2}N^*\hat{N}_{t+1} + v_{K,3}K^*\hat{K}_{t+1} + v_{K,4}N^*\hat{N}_t + v_{K,5}K^*\hat{K}_t \\
& + v_{K,6}\varepsilon\hat{\varepsilon}_{t+1} + v_{K,7}\varepsilon\hat{\varepsilon}_t
\end{aligned}
$$

$v_{K,j}$ ~ Derivative of v_K with respect to j^{th} argument, evaluated in steady state.

$$
\begin{aligned}
& v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\
= \; & v_{N,1}K^*\hat{K}_{t+1} + v_{N,2}N^*\hat{N}_t + v_{N,3}K^*\hat{K}_t + v_{N,4}\varepsilon\hat{\varepsilon}_{t+1}
\end{aligned}
$$

$v_{N,j}$ ~ Derivative of v_N with respect to j^{th} argument, evaluated in steady state.

Example #3 RBC Model With Hours Worked and Uncertainty ...

- Representation Log-linearized Equilibrium Conditions
  - Let
$$z_t = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{N}_t \end{pmatrix},\ s_t = \hat{\varepsilon}_t,\ \epsilon_t = e_t.$$
  - Then, the linearized Euler equation is:
$$\begin{aligned} E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] &= 0, \\ s_t &= P s_{t-1} + \epsilon_t,\ \epsilon_t \sim N(0, \sigma_e^2),\ P = \rho. \end{aligned}$$
  - Here,
$$\begin{aligned} \alpha_0 &= \begin{bmatrix} v_{K,1}K^* & v_{K,2}N^* \\ 0 & 0 \end{bmatrix},\ \alpha_1 = \begin{bmatrix} v_{K,3}K^* & v_{K,4}N^* \\ v_{N,1}K^* & v_{N,2}N^* \end{bmatrix}, \\ \alpha_2 &= \begin{bmatrix} v_{K,5}K^* & 0 \\ v_{N,3}K^* & 0 \end{bmatrix}, \end{aligned}$$
$$\beta_0 = \begin{pmatrix} v_{K,6}\varepsilon \\ 0 \end{pmatrix},\ \beta_1 = \begin{pmatrix} v_{K,7}\varepsilon \\ v_{N,4}\varepsilon \end{pmatrix}.$$
- Previous is a Canonical Representation That Essentially All Log Linearized Models Can be Fit Into (See Christiano (2002).)

- Again, Look for Solution

$$z_t \ = \ Az_{t-1} + Bs_t,$$

where A and B are pinned down by log-linearized Equilibrium Conditions.

- Now, A is *Matrix* Eigenvalue of *Matrix* Polynomial:

$$\alpha(A) \ = \ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0.$$

- Also, B Satisfies Same System of Log Linear Equations as Before:

$$F \ = \ (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0.$$

- Go for the 2 Free Elements of B Using 2 Equations Given by

$$F \ = \ \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example #3 RBC Model With Hours Worked and Uncertainty ...

- Finding the Matrix Eigenvalue of the Polynomial Equation,

$$\alpha(A) = 0,$$

  and Determining if A is Unique is a Solved Problem.

- See Anderson, Gary S. and George Moore, 1985, 'A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models,' *Economic Letters*, 17, 247-52 or Articles in Computational Economics, October, 2002. See also, the program, DYNARE.

Example #3 RBC Model With Hours Worked and Uncertainty ...

- Solving for B
  - Given A, Solve for B Using Following (Log Linear) System of Equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

  - To See this, Use

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2),$$

to Convert $F = 0$ Into

$$vec(F') = d + q\delta = 0, \; \delta = vec(B').$$

  - Find B By First Solving:

$$\delta = -q^{-1} d.$$

Example #4: Example #3 With ‘Exotic’ Information Set

- Suppose the Date t Investment Decision is Made Before the Current Realization of the Technology Shock, While the Hours Decision is Made Afterward.
- Now, Canonical Form Must Be Written Differently:

$$\mathcal{E}_t\left[\alpha_0 z_{t+1}+\alpha_1 z_t+\alpha_2 z_{t-1}+\beta_0 s_{t+1}+\beta_1 s_t\right] \;=\; 0,$$

where

$$\mathcal{E}_t X_t=\left[\begin{array}{c} E\left[X_{1t}|\hat{\varepsilon}_{t-1}\right] \\ E\left[X_{2t}|\hat{\varepsilon}_t\right] \end{array}\right].$$

- Convenient to Change s_t:

$$s_t \;=\; \left(\begin{array}{c} \hat{\varepsilon}_t \\ \hat{\varepsilon}_{t-1} \end{array}\right),\; P=\left[\begin{array}{cc} \rho & 0 \\ 1 & 0 \end{array}\right],\; \epsilon_t=\left(\begin{array}{c} e_t \\ 0 \end{array}\right).$$

- Adjust β_i’s:

$$\beta_0=\left(\begin{array}{cc} v_{K,6}\varepsilon & 0 \\ 0 & 0 \end{array}\right),\; \beta_1=\left(\begin{array}{cc} v_{K,7}\varepsilon & 0 \\ v_{N,4}\varepsilon & 0 \end{array}\right),$$

Example #4: Example #3 With 'Exotic' Information Set ...

- Posit Following Solution:

$$z_t = Az_{t-1} + Bs_t.$$

- Substitute Into Canonical Form:

$$\begin{aligned} & \mathcal{E}_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] \\ = \ & \alpha(A)z_{t-1} + \mathcal{E}_t F s_t + \mathcal{E}_t \beta_0 \epsilon_{t+1} = \alpha(A)z_{t-1} + \mathcal{E}_t F s_t = 0, \end{aligned}$$

- Then,

$$\begin{aligned} \mathcal{E}_t F s_t \ &= \ \mathcal{E}_t \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} s_t = \mathcal{E}_t \begin{bmatrix} F_{11}\hat{\varepsilon}_t + F_{12}\hat{\varepsilon}_{t-1} \\ F_{21}\hat{\varepsilon}_t + F_{22}\hat{\varepsilon}_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & F_{12} + \rho F_{11} \\ F_{21} & F_{22} \end{bmatrix} s_t = \tilde{F} s_t. \end{aligned}$$

- Equations to be solved:

$$\alpha(A) = 0, \ \tilde{F} = 0.$$

- $\tilde{F}$ Only Has *Three* Equations How Can We Solve for the Four Elements of B?
- Answer: Only *Three* Unknowns in B Because B Must Also Obey Information Structure:

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Summary so Far

- Solving Models By Log Linear Approximation Involves Three Steps:
  a. Compute Steady State
  b. Log-Linearize Equilibrium Conditions
  c. Solve Log Linearized Equations.
- Step 3 Requires Finding A and B in:

$$z_t = Az_{t-1} + Bs_t,$$

to Satisfy Log-Linearized Equilibrium Conditions:

$$\mathcal{E}_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right]$$
$$s_t = Ps_{t-1} + \epsilon_t,\ \epsilon_t \sim \text{ iid}$$

- We are Led to Choose A and B so that:

$$\begin{aligned} \alpha(A) &= 0, \\ \text{(standard information set) } F &= 0, \\ \text{(exotic information set) } \tilde{F} &= 0 \end{aligned}$$

and Eigenvalues of A are Less Than Unity In Absolute Value.

Example #5: A Sticky Price Model (Clarida-Gali-Gertler)

- Technology grows forever: equilibrium of model has no constant steady state.
- Deviations of the equilibrium from a particular benchmark does have a steady state.
  - Benchmark: best equilibrium given preferences and production technology
    - ‘natural equilibrium’ (later, will investigate whether benchmark is achievable by some monetary/fiscal policy)
  - Natural equilibrium is trivial to compute because of absence of capital.
- Model is approximately log-linear around natural equilibrium allocations.

Example #5: A Sticky Price Model (Clarida-Gali-Gertler) ...

- Model:
  - Households choose consumption and labor.
  - Monopolistic firms produce and sell output using labor, subject to sticky prices
  - Monetary authority obeys a Taylor rule.

Household

- Preferences & budget constraint:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp\left(\tau_t\right) \frac{N_t^{1+\varphi}}{1+\varphi} - \upsilon \frac{\left(\frac{P_t C_t}{M_t^d}\right)^{1+\sigma_q}}{1+\sigma_q} \right),\ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}.$$

$$P_t C_t + M_t^d + B_{t+1} \leq M_{t-1}^d + B_t R_{t-1} + W_t N_t + Transfers\ and\ profits_t$$

- Household efficiency conditions (ignore money because υ is small):

$$\begin{aligned} C_t^{-1} &= \beta E_t C_{t+1}^{-1} R_t / \bar{\pi}_{t+1}, \\ \bar{\pi}_{t+1} &\equiv \frac{P_{t+1}}{P_t}, \\ MRS_t &= \exp\left(\tau_t\right) N_t^{\varphi} C_t = \frac{W_t}{P_t}. \end{aligned}$$

Firms

- Final Good Firms (simple!)
  - Buy $Y_{i,t}$, $i \in [0,1]$ at prices $P_{i,t}$ and sell Y_t at price P_t
  - Technology:

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \ \varepsilon \geq 1. \tag{1}$$

  - Demand for intermediate good (fonc for optimization of $Y_{i,t}$):

$$Y_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \tag{2}$$

  - Eqs (6) and (2) imply:

$$P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} \tag{3}$$

Firms ...

- Intermediate Good Firms (an astounding amount of algebra!)
  - Technology:

$$\begin{aligned} Y_{i,t} &= A_t N_{i,t}, \;\; a_t = \log A_t, \\ \Delta a_t &= \rho \Delta a_{t-1} + \varepsilon_t. \end{aligned}$$

  - Marginal cost of production for i^{th} firm (with subsidy, ν_t) :

$$s_t = \overbrace{(1-\nu_t)}^{\text{net of subsidy}} \overbrace{\frac{W_t}{A_t P_t}}^{\text{‘Normal’ Marginal Cost}} \overbrace{(1-\psi+\psi R_t)}^{\text{fraction, } \psi \text{, of labor costs requires bank finance}}$$

  - Calvo price-setting frictions:
    - A fraction, θ, of intermediate good firms cannot change price:

$$P_{i,t} = P_{i,t-1}$$

    - A fraction, $1-\theta$, set price optimally:

$$P_{i,t} = \tilde{P}_t$$

Firms ...

- Decision of intermediate good firm
  - Only choice problem: optimize price, $P_{i,t}$, whenever opportunity arises.
  - Otherwise, always produce the quantity dictated by demand.
- The firm's periodic optimization gives rise to equilibrium conditions needed to solve the model.
  - Ultimate equilibrium conditions simple.
  - Lot's of (simple) algebra to get them.

Firms ...

- Solving intermediate good firm optimization problem
  - Discounted profits:

$$E_t \sum_{j=0}^{\infty} \beta^j \overset{\text{Lagrange multiplier on household budget constraint}}{\overbrace{\upsilon_{t+j}}} \overbrace{\left[\overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} s_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}^{\text{period } t+j \text{ profits sent to household}}$$

  - Each of the $1-\theta$ firms that have opportunity to reoptimize price, $P_{i,t}$, select $\tilde{P}_t$ so maximize:

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\theta^j}^{\substack{\text{in selecting price, firm only cares about} \\ \text{future states in which it can't reoptimize}}} \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right].$$

Firms ...

- Substitute out for intermediate good firm output using demand curve:

$$
\begin{aligned}
& E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right] \\
= \; & E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon} \right].
\end{aligned}
$$

- Differentiate with respect to $\tilde{P}_t$:

$$
E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[(1-\varepsilon) \left(\tilde{P}_t \right)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1} \right] = 0,
$$

or,

$$
E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
$$

- When $\theta = 0$, get standard result - price is fixed markup over marginal cost.

- Substitute out the multiplier:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \overbrace{\frac{u'(C_{t+j})}{P_{t+j}}}^{\text{marginal utility of a dollar} = v_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

- Use utility functional form and goods market clearing condition, $C_{t+j} = Y_{t+j}$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

or,

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t},\ X_{t,j} = \left\{ \begin{array}{c} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}},\ j \geq 1 \\ 1,\ j = 0. \end{array} \right.,\ X_{t,j} = X_{t+1,j-1}\frac{1}{\bar{\pi}_{t+1}},\ j > 0$$

Firms ...

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \left(X_{t,j}\right)^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j}\right] = 0$$

- Solve for $\tilde{p}_t$:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \left(X_{t,j}\right)^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- We've almost finished solving the intermediate firm problem!

- But, still need expressions for K_t, F_t.

Firms ...

$$\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left(\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j}}^{\text{exactly } K_{t+1}!} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} K_{t+1}
\end{aligned}$$

- So,

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} K_{t+1}.$$

- Simplify marginal cost term:

$$\begin{aligned} \frac{\varepsilon}{\varepsilon - 1} s_t &= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu_t) \frac{W_t}{A_t P_t} (1 - \psi + \psi R_t) \\ &= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu_t) \overbrace{\exp(\tau_t) N_t^{\varphi} C_t}^{=\frac{W_t}{P_t} \text{ by household optimization}} \frac{1 - \psi + \psi R_t}{A_t}. \end{aligned}$$

- Conclude:
  - Optimal price:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

where

$$K_t = (1-\nu_t)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t) N_t^{\varphi} C_t}{A_t}(1-\psi+\psi R_t) + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} K_{t+1}.$$

Similarly,

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{1-\varepsilon} F_{t+1}$$

Firms ...

- We now have optimization conditions for households and firms.

- Need some aggregate conditions:
  - Relationship among prices
  - Relationship between aggregate inputs (e.g., technology, A_t, and labor, N_t) and aggregate output (e.g., Y_t).

Firms ...

- Aggregate Price Relationship

$$P_t = \left[\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

$$= \left[\int_{\text{firms that reoptimize price}} P_{i,t}^{(1-\varepsilon)} di + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

$$\overbrace{=}^{\text{all reoptimizers choose same price}} \left[(1-\theta)\,\tilde{P}_t^{(1-\varepsilon)} + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

Firms ...

- Rewrite integral of prices of intermediate good firms that do not reoptimize:

$$\int_{\text{firms that don't reoptimize price in } t} P_{i,t}^{(1-\varepsilon)} di$$

$$\overbrace{=}^{\text{add over prices, weighted by \# of firms posting that price}}$$

$$\int \left[\overbrace{f_{t-1,t}\left(\omega\right)}^{\text{`number' of firms that had price, } P(\omega)\text{, in } t-1 \text{ and were not able to reoptimize in } t} P\left(\omega\right)^{(1-\varepsilon)} \right] d\omega$$

In principle, HARD integral to evaluate!

Firms ...

- By Calvo randomization assumption:

$$f_{t-1,t}(\omega) = \theta \times \overbrace{f_{t-1}(\omega)}^{\text{total `number' of firms with price } P(\omega) \text{ in } t-1}, \text{ for all } \omega$$

- Substituting:

$$\begin{aligned} \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di &= \int f_{t-1,t}(\omega) P(\omega)^{(1-\varepsilon)} d\omega \\ &= \theta \int f_{t-1}(\omega) P(\omega)^{(1-\varepsilon)} d\omega \\ &= \theta P_{t-1}^{(1-\varepsilon)} \end{aligned}$$

- Trivial!

- Conclude that the following relationship holds between prices:

$$P_t = \left[(1-\theta)\,\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)}\right]^{\frac{1}{1-\varepsilon}}.$$

- Divide by P_t :

$$1 \;=\; \left[(1-\theta)\,\tilde{p}_t^{(1-\varepsilon)} + \theta\left(\frac{1}{\bar{\pi}_t}\right)^{(1-\varepsilon)}\right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t \;=\; \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}$$

Firms ...

- Aggregate inputs and outputs

  - Technically, there is no ‘aggregate production function’:

    i.e., simple relationship between output, Y_t, and aggregate inputs, N_t, A_t

  - Aggregate output, Y_t, is not only a function of total labor input, N_t, and A_t, but also of the *distribution* of labor input among intermediate goods.

  - Tak Yun (JME) developed a simple characterization of the connection between N, A, Y and the distribution of resources.

Firms ...

– Define Y_t^*:

$$Y_t^* = \int_0^1 Y_{i,t} di \left(= \int_0^1 A_t N_{i,t} di \overbrace{=}^{\text{labor market clearing}} A_t N_t \right)$$

$$\overbrace{=}^{\text{demand curve}} Y_t \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \int_0^1 \left(P_{i,t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \left(P_t^* \right)^{-\varepsilon}$$

where

$$P_t^* \equiv \left[\int_0^1 P_{i,t}^{-\varepsilon} di \right]^{\frac{-1}{\varepsilon}} = \left[(1-\theta) \tilde{P}_t^{-\varepsilon} + \theta \left(P_{t-1}^* \right)^{-\varepsilon} \right]^{\frac{-1}{\varepsilon}}$$

Firms ...

- Relationship between aggregate inputs and outputs:

$$\begin{aligned} Y_t &= \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} Y_t^* \\ &= p_t^* A_t N_t, \end{aligned}$$

  where

$$p_t^* \equiv \left(\frac{P_t^*}{P_t}\right)^{\varepsilon}.$$

- 'Efficiency distortion', p_t^*:

$$p_t^* : \begin{cases} \leq 1 & \\ = 1 & P_{i,t} = P_{j,t}, \text{ all } i, j \end{cases}$$

- When prices of different intermediate goods differ, then resources allocated inefficiently.

Firms ...

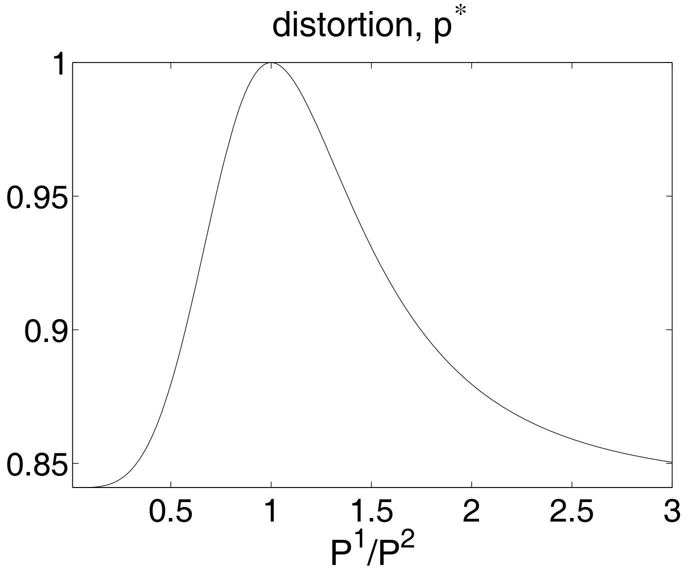
- Example:

$$P_{j,t} = \begin{cases} P^1 & 0 \leq j \leq \alpha \\ P^2 & \alpha \leq j \leq 1 \end{cases}.$$

- Then

$$p_t^* = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}\right)^{\varepsilon}$$

$$\alpha = 0.5, \varepsilon = 5$$



Firms ...

- Bottom line:
  - Combining efficiency condition of intermediate firms with household static efficiency:

$$K_t = (1-\nu_t)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t)\,N_t^{\varphi}C_t}{A_t}(1-\psi+\psi R_t)+\beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon}K_{t+1} \quad (1)$$

$$F_t = 1+\beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon-1}F_{t+1} \quad (2)$$

  - Intermediate good firm optimality and restriction across prices:

$$\frac{\overbrace{K_t}^{=\tilde{p}_t \text{ by firm optimality}}}{F_t} = \overbrace{\left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \quad (3)$$

Firms ...

– Law of motion for efficiency distortion:

$$p_t^* = \left[(1-\theta) \left(\frac{1-\theta \bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4)$$

– Household intertemporal condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \; (5)$$

– Aggregate inputs and outputs:

$$C_t = p_t^* A_t N_t \; (6)$$

– 8 unknowns - $\nu_t, C_t, p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$ - 6 equations.

– Need to find more equations to close the model!

Firms ...

- Closing the model: optimal and exogenous policy.

  – Choose optimal policy (I substituted out C_t):

$$
\begin{aligned}
&\max_{\nu_t, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \\
&+\lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{A_t \beta}{p_{t+1}^* A_{t+1} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
&+\lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta (\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \\
&+\lambda_{3t} \left[1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right] \\
&+\lambda_{4t} \left[(1-\nu_t) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* (1-\psi+\psi R_t) + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right] \\
&+\lambda_{5t} \left[F_t \left[\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} - K_t \right] \}
\end{aligned}
$$

Firms ...

- Optimal policy
  - Unknowns: 7 plus 5 multipliers = 12.
  - Equations: 5 plus 7 first order conditions = 12
  - Can solve this system using the linearization methods
  - Can ask: should we stabilize inflation, $\bar{\pi}_t$, the price level, P_t?
  - More on this later...

Firms ...

- Another way to close the model: add two exogenous equations (Taylor rule and specification of ν_t)

  - A standard formulation of Taylor rule policy focuses on 'natural' equilibrium (equilibrium in which relative price distortions have been eliminated and employment is not distorted by monopoly power) as benchmark.

  - Consider a policy that eliminates the 'labor wedge' in steady state and sets $\bar{\pi} = 1$.

- Steady state (delete time subscripts from the 5 equilibrium conditions and solve):

$$R = \frac{1}{\beta},\ p^* = 1,\ F = K = \frac{1}{1-\beta\theta},\ N = \exp\left(-\frac{\tau}{1+\varphi}\right) = 1 \text{ (since } \tau = 0).$$

- Natural equilibrium: absence of price distortions induces cross-industry efficiency:

  $$N_{i,t} = N_t \text{ all } i$$

  so that

  $$Y_t = A_t N_t,\ y_t = a_t + n_t \tag{4}$$

  - Labor market efficiency (in logs):

    $$\overbrace{c_t + \varphi n_t + \tau_t}^{\log\ MRS_t} = \overbrace{a_t}^{\log\ MP_{L,t}} \tag{5}$$

  - Combine (4) and (5):

    $$a_t = y_t + \varphi\left(y_t - a_t\right) + \tau_t$$

    so that natural level of output and employment is:

    $$y_t^* = a_t - \frac{1}{1+\varphi}\tau_t,\ n_t^* = y_t^* - a_t = -\frac{1}{1+\varphi}\tau_t$$

Firms ...

- Interest rate in the ‘natural’ equilibrium steers households to choose efficient levels of employment and consumption.
  - Household intertemporal Euler equation:

$$C_t^{-1} = \beta E_t C_{t+1}^{-1} R_t / \bar{\pi}_{t+1}.$$

  - In logs:

$$\begin{aligned} -c_t &= \log\beta + r_t + \log\left[E_t C_{t+1}^{-1}/\bar{\pi}_{t+1}\right] \\ &= \log\beta + r_t + \log\left[E_t \exp\left(-c_{t+1} - \pi_{t+1}\right)\right] \\ &\simeq \log\beta + r_t + \log\left[\exp\left(-E_t c_{t+1} - E_t \pi_{t+1}\right)\right] \\ &= r_t - rr - E_t c_{t+1} - E_t \pi_{t+1} \end{aligned}$$

$$rr \equiv -\log\beta$$

Firms ...

- Intertemporal Euler equation (repeated)

$$c_t = -\left[r_t - E_t\pi_{t+1} - rr\right] + E_t c_{t+1}$$

- To determine 'natural' real interest rate, rr_t^*, substitute 'natural' output, y_t^*, and inflation, $\pi_t = 0$, into household Euler equation:

$$\overbrace{a_t - \frac{1}{1+\varphi}\tau_t}^{y_t^*} = -\left[rr_t^* - rr\right] + E_t\overbrace{\left(a_{t+1} - \frac{1}{1+\varphi}\tau_{t+1}\right)}^{y_{t+1}^*}$$

or,

$$rr_t^* = rr + \rho\Delta a_t + \frac{1}{1+\varphi}\left(1-\lambda\right)\tau_t.$$

- Recall:

$$\tau_t = \lambda\tau_{t-1} + \varepsilon_t^\tau,\ \Delta a_t = \rho\Delta a_{t-1} + \varepsilon_t$$

Firms ...

- Natural rate:

$$rr_t^* = rr + \rho\Delta a_t + \frac{1}{1+\varphi}(1-\lambda)\tau_t.$$

  - Δa_t jumps
    - a_t will keep rising in future (if $\rho > 0$)
    - rise in c_t^* smaller than rise in c_{t+1}^*
    - people would like to use financial markets to smooth away from this
    - discourage this by having a high interest rate.

  - τ_t jumps
    - τ_t will be less high in the future (unless $\lambda > 1$)
    - c_t^* falls more than c_t^*
    - people want to smooth away
    - discourage this by having a high interest rate.

- Taylor Rule
  - Target interest rate, $\hat{r}_t$:

$$\hat{r}_t = rr + \phi_\pi \pi_t + \phi_x x_t,\ x_t \equiv y_t - y_t^*.$$

  - Actual interest rate, r_t :

$$\begin{aligned} r_t &= \alpha r_{t-1} + (1-\alpha)\,\hat{r}_t + u_t \\ u_t &= \delta u_{t-1} + \eta_t. \end{aligned}$$

  - Policy rule:

$$r_t - rr = \alpha\,(r_{t-1} - rr) + u_t + (1-\alpha)\phi_\pi \pi_t + (1-\alpha)\phi_x x_t$$

Firms ...

- Intertemporal equations:

Taylor rule equilibrium: $y_t = -\left[r_t - E_t\pi_{t+1} - rr\right] + E_t y_{t+1}$

Natural equilibrium: $y_t^* = -\left[rr_t^* - rr\right] + E_t y_{t+1}^*$

- Subtract, to obtain ‘New Keynesian IS equation’:

$$x_t = -\left[r_t - E_t\pi_{t+1} - rr_t^*\right] + E_t x_{t+1}$$

Firms ...

- With Taylor rule, cannot rule out fluctuations in inflation. So, in presence of shocks $N_{i,t}$ varies across i and:

$$y_t = \log p_t^* + n_t + a_t, \ \ \log p_t^* = \begin{cases} = 0 & \text{if } P_{i,t} = P_{j,t} \text{ for all } i, j \\ \leq 0 & \text{otherwise} \end{cases}.$$

- Along a nonstochastic steady state, zero inflation growth path, $\log p_t^* = 0$. Log-linear expansion of equilibrium law of motion for p_t^* yields:

$$\hat{p}_t^* \approx \theta \hat{p}_{t-1}^* + 0 \times \bar{\pi}_t$$

- We still need the equilibrium conditions associated with sticky prices.

Firms ...

- Log-linearizing each sticky price equilibrium condition (i.e., equilibrium conditions, 3-5) respectively:

$$\begin{aligned} \beta\theta E_t\left[(\varepsilon-1)\,\widehat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}\right] &= \hat{F}_t \text{ (3)} \\ d\tau_t+(1+\varphi)\,\hat{N}_t+\frac{R\psi}{1-\psi+\psi R}\hat{R}_t+\frac{\beta\theta}{1-\beta\theta}E_t\left[\varepsilon\widehat{\bar{\pi}}_{t+1}+\hat{K}_{t+1}\right] &= \frac{1}{1-\beta\theta}\hat{K}_t \text{ (4)} \\ \hat{F}_t+\frac{\theta}{1-\theta}\widehat{\bar{\pi}}_t &= \hat{K}_t \text{ (5)} \end{aligned}$$

- equations (3)-(5) reduce to the usual Phillips curve
  - substitute (5) into (4)

$$\begin{aligned} & d\tau_t+(1+\varphi)\,\hat{N}_t+\hat{p}_t^*+\frac{R\psi}{1-\psi+\psi R}\hat{R}_t+\frac{\beta\theta}{1-\beta\theta}E_t\left[\varepsilon\widehat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}+\frac{\theta}{1-\theta}\widehat{\bar{\pi}}_{t+1}\right] \\ = \ & \frac{1}{1-\beta\theta}\left[\hat{F}_t+\frac{\theta}{1-\theta}\widehat{\bar{\pi}}_t\right] \end{aligned}$$

- substitute (3) into the previous expression, and rearrange:

$$\widehat{\bar{\pi}}_t = \frac{(1-\beta\theta)(1-\theta)}{\theta} \overbrace{\left[d\tau_t + (1+\varphi)\hat{N}_t + \frac{R\psi}{1-\psi+\psi R}\hat{R}_t \right]}^{\text{percent deviation of real marginal cost from ss}} + \beta\widehat{\bar{\pi}}_{t+1},$$

- Note:

$$x_t = y_t - y_t^* = a_t + n_t - \left[a_t - \frac{1}{1+\varphi}\tau_t \right] = n_t + \frac{1}{1+\varphi}\tau_t,$$

so (recall, $\hat{N}_t = \log(N_t/N) = \log(N_t)$, $d\tau_t = \tau_t - \tau = \tau_t$)

$$\widehat{\bar{\pi}}_t = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\varphi)\left[x_t + \frac{R\psi}{(1+\varphi)(1-\psi+\psi R)}\hat{R}_t \right] + \beta\widehat{\bar{\pi}}_{t+1},$$

- We now have three equations (‘IS curve, Phillips curve and policy rule’) in three unknowns: π_t, r_t, x_t.

Equations of Taylor rule Equilibrium ($\psi = 0$)

$$\beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \text{ (Calvo pricing equation)}$$

$$-\left[r_t - E_t \pi_{t+1} - rr_t^*\right] + E_t x_{t+1} - x_t = 0 \text{ (intertemporal equation)}$$

$$\alpha r_{t-1} + u_t + (1-\alpha)\phi_\pi \pi_t + (1-\alpha)\phi_x x_t - r_t = 0 \text{ (policy rule)}$$

$$rr_t^* - \rho \Delta a_t - \frac{1}{1+\varphi}\left(1-\lambda\right)\tau_t = 0 \text{ (definition of natural rate)}$$

- r_t and rr_t^* expressed in deviations from steady state
- Preference and technology shocks enter system through rr_t^*
- Optimal equilibrium can be supported by setting nominal rate to natural rate:

$$r_t = rr_t^*.$$

- Practical issue: how to measure rr_t^*???

Solving the Sticky Price Model

- Exogenous shocks:

$$s_t = \begin{pmatrix} \Delta a_t \\ u_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ u_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \varepsilon_t^{\tau} \end{pmatrix}$$

$$s_t = P s_{t-1} + \epsilon_t$$

- Equilibrium conditions:

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ rr^*_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \kappa & 0 & 0 \\ 0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\ (1-\alpha)\phi_\pi & (1-\alpha)\phi_x & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr^*_t \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ x_{t-1} \\ r_{t-1} \\ rr^*_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s_{t+1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma\psi\rho & 0 & -\frac{1}{\sigma+\varphi}(1-\lambda) \end{pmatrix} s_t$$

$$E_t \left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t \right] = 0$$

Solving the Sticky Price Model ...

- Collecting:

$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] \;=\; 0$$

$$s_t - P s_{t-1} - \epsilon_t \;=\; 0.$$

- Solution:

$$z_t = A z_{t-1} + B s_t$$

- As before, want A such that

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0,$$

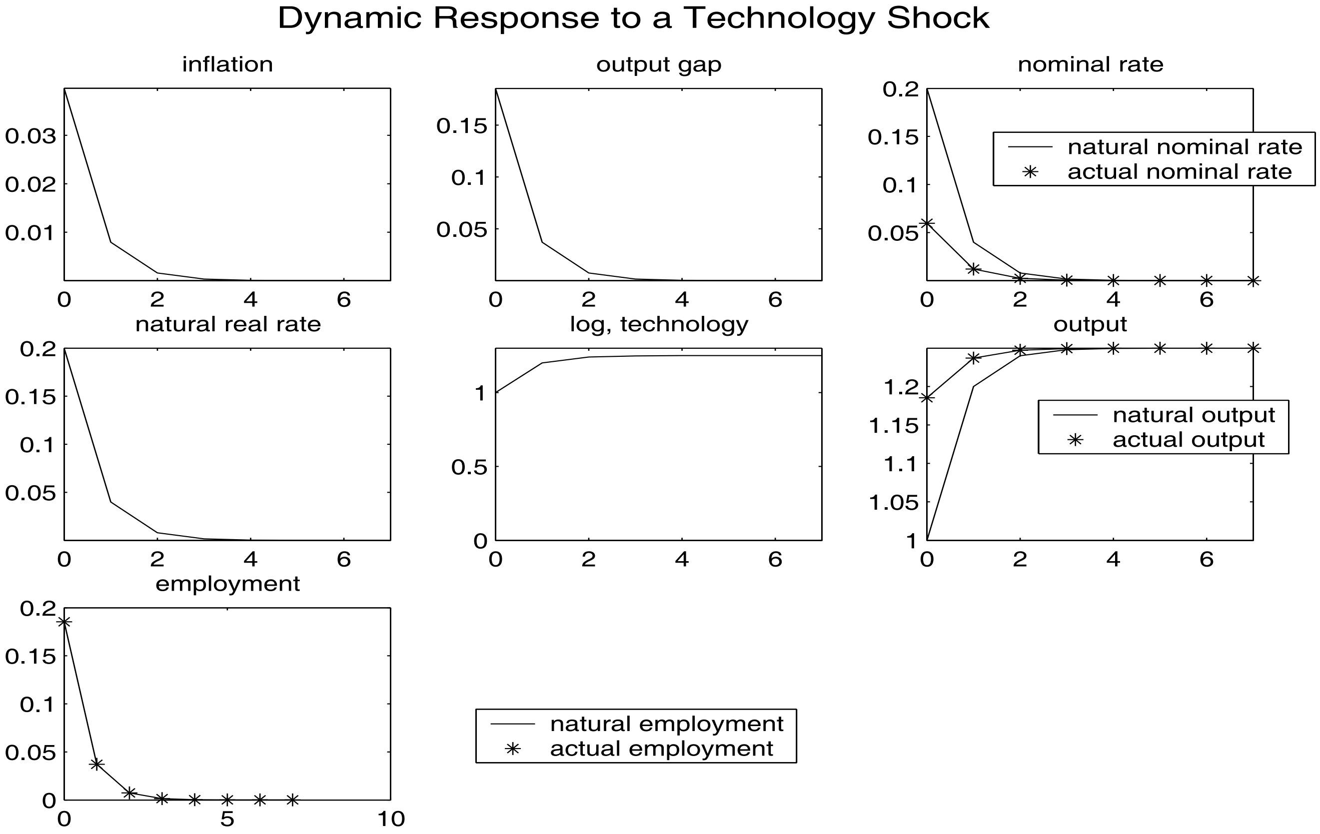
- Want B such that:

$$F \;=\; (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

- Note: if $\alpha = 0$, $A = 0$.

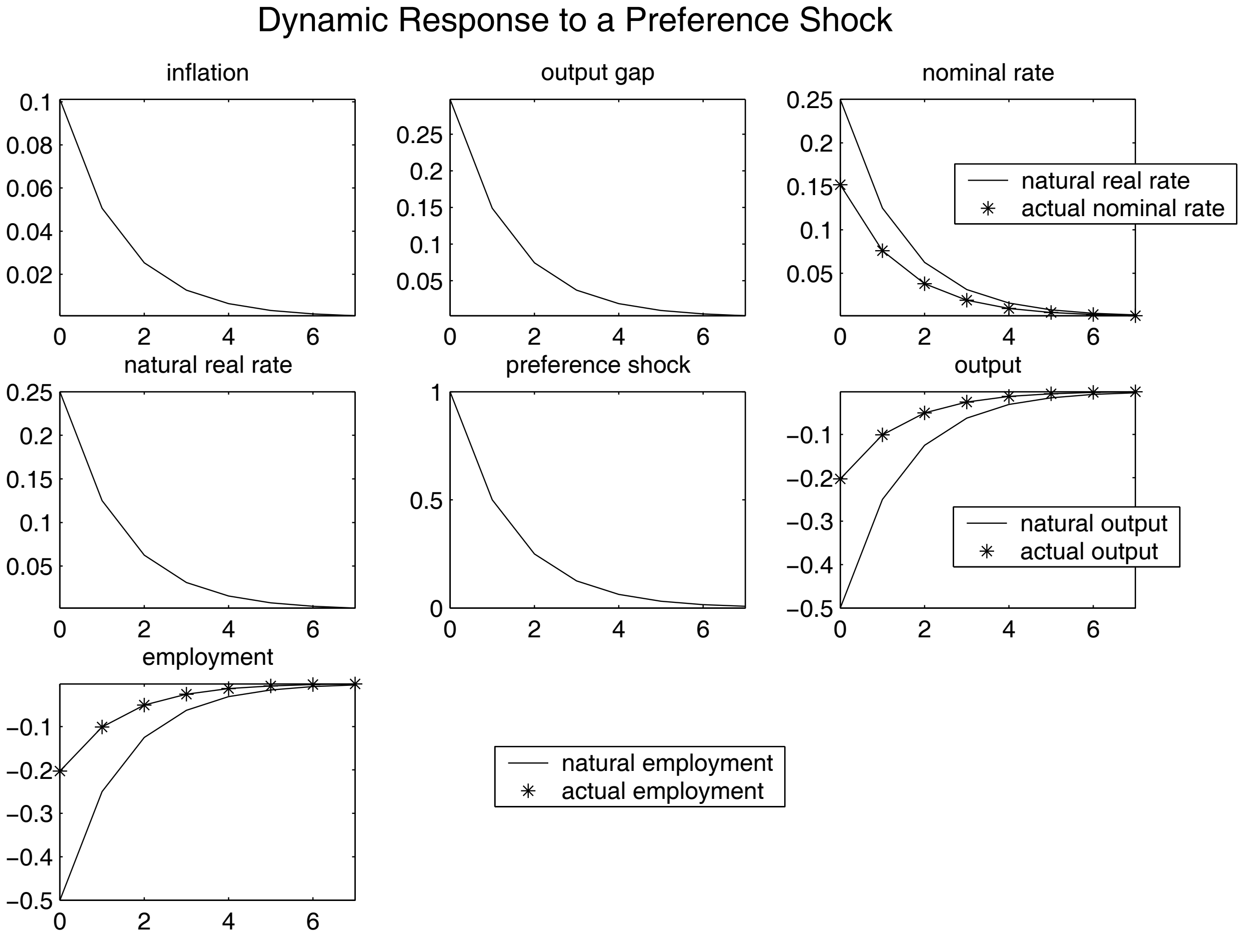
Examples with Sticky Price Model

$$\phi_x = 0,\ \phi_\pi = 1.5,\ \beta = 0.99,\ \varphi = 1,\ \rho = 0.2,\ \theta = 0.75,\ \alpha = 0,\ \delta = 0.2,\ \lambda = 0.5.$$



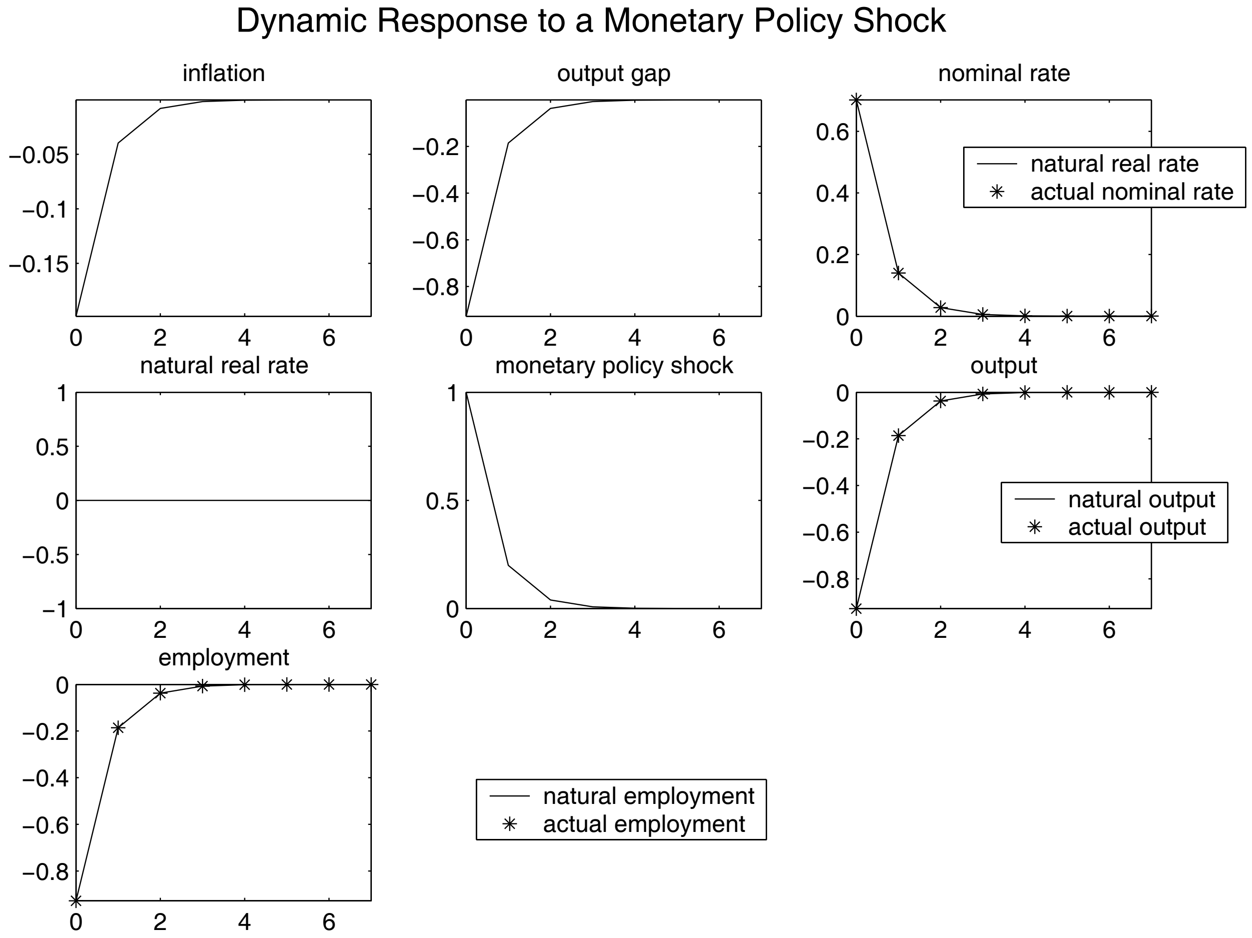
- Interest rate not increased enough, employment and inflation rise.

Examples with Sticky Price Model ...



- Under policy rule, interest rate not increased enough.
  - This encourages consumption above what is needed for the zero-inflation equilibrium.
  - The extra demand drives up output gap, inflation

Examples with Sticky Price Model ...



- Monetary policy shock drives up the interest rate
  - High interest rate discourages current consumption
  - Output, output gap and employment fall
  - Fall in costs causes inflation to drop.