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416, Spring, 2009
Homework 7, Due May 28.

1. Consider the one-sector stochastic neoclassical growth model discussed in the class and in the handout, 'Notes on the Perturbation Method'. Consider the following model parameterization:

   (a) Compute the coefficients in the linear, second order perturbation on the policy rule, for several parameterizations of the model. It is convenient to do the approximation in terms of the capital itself, rather than in terms of the log of capital, as in the handout. In all cases,

   $$\alpha = 0.3,\ \beta = 0.99,\ \delta = 0.025,\ \rho = 0.8.$$

   Solve the model for the following four cases: $\gamma = 1.1$, 9 and for $Var(\varepsilon_t) = 1$ and 0.012. Store the coefficients of the approximate policy rules, for analysis below.

   (b) One way to quantify the difference between first and second order perturbations, is to investigate their implications for the speed of adjustment. Although shocks potentially play an interesting role in this, let us abstract from the shocks at this point. Thus, the policy rule is:

   $$k_{t+1} - k = g_k \times (k_t - k) + \frac{1}{2} g_{kk} \times (k_t - k)^2,$$

   where k is the value of the capital stock in non-stochastic steady state. In percent terms:

   $$\hat{k}_t = \frac{k_t - k^s}{k^s}.$$

   Dividing the second order policy rule by k^s :

   $$\hat{k}_{t+1} = g_k \hat{k}_t + \frac{1}{2} g_{kk} \times k \times \left(\hat{k}_t\right)^2. \tag{1}$$

   Suppose that at $t = 1$, $k_1 = \lambda \times k$, so that $\hat{k}_1 = \lambda - 1$. Consider first the standard, first order perturbation, in which g_{kk} is set to zero. Then, $\hat{k}_{t+1} = g_k^t (\lambda - 1)$, $t = 0, 1, \ldots$. The time required to close 95 percent of the gap between the capital stock and steady state is the value of t such that

$$\hat{k}_{t+1} = (\lambda - 1) \times 0.05,$$

or,

$$\begin{aligned} g_k^t &= 0.05 \\ t &= \frac{\log(0.05)}{\log g_k}. \end{aligned}$$

Notice that this is independent of the value of λ. Compute the value of t for our model economy.

Now, consider the implications of the second-order expansion for speed of adjustment. Set $\hat{k}_1 = \lambda - 1$, and simulate (1) for $t = 1, \ldots, 100$. Set λ to a very small number, $\lambda = 0.0001$ and put in the large value of γ. Graph the two trajectories, one for the first order approximation and the other for the second order approximation. Are they very similar?

2. (Simulating a rule obtained by second-order perturbation.) Consider the following expression, which we can think of as the second order approximation of some unspecified law of motion:

$$y_t = f(y_{t-1}, \varepsilon_t) = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t, \;\; 0 < \rho < 1,$$

where ε_t is a mean zero, *iid* process, uncorrelated with past y_t. Note that there are two solutions, $y = f(y, 0) : y = 0, y = (1-\rho)/\alpha$. The first steady state is locally stable, since the root in its first order expansion is ρ. The second is not. So, if $y_t > (1-\rho)/\alpha$, then y_t is likely to explode. Verify this by simulating 100 observations of y_t with $\varepsilon_t \tilde{} N(0, \sigma_\varepsilon)$ and $\sigma_\varepsilon = 0.01$, 0.10. The problem illustrated here is something that can happen with policy rules that are quadratic expansions: "...they will have extra steady states not present in the original model, and some of these steady states are likely to mark transitions to unstable behavior".[1]

[1] Taken from page 17, Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher Sims, February 3, 2005, 'Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models,' manuscript.

Kim, Kim, Schaumburg and Sims (2005, section 7) argue that the system should be simulated by 'pruning', as follows. Let the first order system be $y_t^{(1)}$:

$$y_t^{(1)} = \rho y_{t-1}^{(1)} + \varepsilon_t.$$

The pruned simulated output is $y_t^{(2)}$, where $y_t^{(2)}$ solves the original system with the exception that wherever y_{t-1} appears in quadratic terms, it is replaced by $y_t^{(1)}$:

$$y_t^{(2)} = \rho y_{t-1}^{(2)} + \alpha \left[y_{t-1}^{(1)} \right]^2 + \varepsilon_t.$$

Construct two graphs for the case, $\sigma_\varepsilon = .01$. Display y_t and $y_t^{(2)}$ in one graph. Display $y_t^{(1)}$ and $y_t^{(2)}$ in the other. Note how pruning eliminates the explosive behavior. See Kim, Kim, Schaumburg and Sims for additional discussion.

3. Return to the policy rule in question 2. Simulate 10,000 observations using the linear approximate rule, the quadratic rule, and the quadratic rule based on pruning (all three should be simulated in response to the same shocks). Display the three sequences of capital stocks in a graph. How do Ek_t, $Var(k_t)$ compare for the three rules? How do Ek_t compare with the magnitude of k? Repeat this exercise using a more reasonable estimate of the variance, $V_\varepsilon = 0.012$. Finally, consider the same experiment with a more 'normal' risk aversion: $\gamma = 1.1$ (seep $V_\varepsilon = 0.012$). Note that in the latter case, there is very little difference between the three solutions.

4. Consider the function,

$$f(x) = \frac{1}{1+x^2}, \quad f : [5, 5] \rightarrow R.$$

Approximate this function by an $n-$dimensional polynomial, for $n = 2, 8, 16, 40$.

   (a) Compute a sequence of n^{th} order approximating polynomials, $\hat{g}^n(x)$, by choosing the $n+1$ parameters so that $\hat{g}^n$ and f coincide at $n+1$ equally spaced points on the domain of f.

(b) Do the same as above, but this time choose the points where $\hat{g}^n$ and f match to be the ones associated with the zeros of the $n+1$ dimensional Chebyshev polynomial. Note how one polynomial approximation strategy converges and the other one does not.