Consider the model economy in homework #1, in the case, $\gamma = 0$.

1. (a) In homework #1, you solved the model by first deriving the first order conditions for consumption and hours worked, and then substituting out for consumption in the two equations from the resource constraint. Then, you log linearized the two equations and substituted out for hours worked in the linearized intertemporal equation using the linearized intra temporal equation. Write out the numerical values of the elements in the solution:

$$\hat{k}_{t+1} = \lambda \hat{k}_t$$
$$\hat{n}_t = h \hat{k}_t.$$  

That is, write out $\lambda$ and $h$.

(b) Now, use the QZ decomposition to do the same calculations, but without substituting the intratemporal equation into the intertemporal equation (always do the linearization of the equations using MATLAB’s symbolic algebra). In particular, express the system of two linearized equilibrium conditions as follows:

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0, z_t = \left( \begin{array}{c} \hat{k}_{t+1} \\ \hat{n}_t \end{array} \right),$$

where $z_{-1}$ is given at time 0. Define

$$Y_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix},$$

where $n = 2$ and the first order representation of $Y_t$ is

$$aY_{t+1} + bY_t = 0,$$

where

$$a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.$$
(i) Display the numerical value $\alpha_0$ and note that the rank of $\alpha_0$ is less than $n$, so that $\alpha_0$ is not invertible.

(ii) Compute the QZ decomposition of $a$ and $b$, i.e., two orthonormal matrices, $Q$ and $Z$, such that

$$QaZ = H_0, \quad QbZ = H_1,$$

where $H_0$ and $H_1$ are upper triangular matrices. (That $Q$ and $Z$ are orthonormal implies $QQ' = I$, $ZZ' = I$.) Follow the code distributed in class by choosing the decomposition so that all the zeros on the diagonal of $H_0$ appear in the bottom right part of the diagonal.\(^1\) How many zeros are there on the diagonal (denote this number by $l$)? Is the value of $l$ what you would have predicted given the rank of $\alpha_0$? Explain. Verify that the $l$ terms on the bottom right diagonal of $H_1$ are non-zero.

Consider the following partitioning of $H_0$ and $H_1$:

$$H_0 = \begin{bmatrix} G_0^{(2n-l)\times(2n-l)} & H_{12}^0 \\ 0 & H_{22}^0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} G_1^{(2n-l)\times(2n-l)} & H_{12}^1 \\ 0 & H_{22}^1 \end{bmatrix},$$

where $G_0$ and $H_{22}^0$ are upper triangular, the diagonal elements of $G_0$ are non-zero, while the diagonal elements of $H_{22}^0$ are all zero. The object, $G_1$, is upper triangular, and you may assume that the diagonal elements of the upper triangular matrix, $H_{22}^1$, are nonzero. Define

$$\gamma_t = \begin{bmatrix} \gamma_1^t \\ \gamma_2^t \end{bmatrix}^{(2n-l)\times1} = Z'Y_t = \begin{bmatrix} L_1^{(2n-l)\times2n} \\ L_2^{l\times2n} \end{bmatrix}Y_t,$$

where $Z' = \begin{bmatrix} L_1^{(2n-l)\times2n} \\ L_2^{l\times2n} \end{bmatrix}$.

(iv) Compute all the possible MSV solutions to the system, i.e., $n \times 2n$ matrices, $D$, such that $DY_0 = 0$. Suppose there are $J$ such matrices, $D^1, ..., D^J$. Partition each of these matrices into two $n \times n$ parts:

$$D^j = \begin{bmatrix} D_1^j & : & D_2^j \end{bmatrix}.$$

\(^1\)That is, implement the MATLAB routine, qzdiv right after qz, as in the code (solvea.m) distributed for class.
How many of these matrices have the property that $D_i^j$ is invertible? Real? In each invertible case, compute

$$A_j = -\left(D_i^j\right)^{-1}D_i^j,$$

and verify that

$$\alpha_0 (A_j)^2 + \alpha_1 A_j + \alpha_2 = 0,$$

that is, that you have found the zero of a particular matrix polynomial.

(v) How many matrices, $A_j$, have the property that their eigenvalues are less than unity in absolute value? There should be exactly one. Compare this solution with the one reported in part (a) of this question. They should be the same.