Dynamic Contracts in the Absence of Commitment

1 Introduction

We describe a simple model of financial frictions. There is an entrepreneur who has access to a project which uses capital, k_t, to produce output. However, the entrepreneur has no resources of her own and must borrow the funds needed to acquire capital. In period $t = 0$, the entrepreneur agrees to a contract with a lender, with terms that specify an amount of consumption, capital, debt, and interest payments in each of periods $t =$1, 2, 3, ... The problem is that the borrower cannot commit to her future actions. At the start of each period t, the borrower has two options: to continue to abide by the terms of the contract, or renege on the contract and run away ('abscond') with a fraction, θ, of the capital that she has at that time.[1] The remaining fraction of capital, $1-\theta$, is destroyed. The fraction, θ, is an exogenously fixed parameter of the model. The consequence for the borrower of absconding is that in addition to destroying a quantity, $(1 - \theta)k_t$, of capital, she forever loses the ability to exploit her project. Still, in principle there are contract terms that she is willing to agree to ex ante, but would in fact renege on ex post. Of course, the lender wishes to ensure that these circumstances do not arise. As a result, the menu of lending contracts available to entrepreneurs only includes those contracts with terms that are 'self enforcing'. That is, the contracts have the property that ex post, the borrower has an incentive to abide by the terms and not run away. Throughout, we assume that the lender is fully able to commit to future actions.

We study the properties of the optimal, self enforcing contract. Under certain model parameter values, enforceability is not a problem and in this case, the option to run away ex post is not an issue. To make the analysis interesting, we only consider model parameter values in which there are enforceability problems. Under these circumstances, in the initial periods of the contract the lender advances fewer funds to the borrower than he would if the borrower had the ability to commit to her future actions. We assume there is curvature in the investment project. So, the fact that relatively little is invested in the early phases of the project implies that the project has a high return. This high return on the project helps to reduce the entrepreneur's temptation to run away, for to do so means losing the ability to exploit a valuable project.

In the early phase of the project, all revenues received by the entrepreneur that exceed current loan servicing costs are used to purchase more capital. The entrepreneur has an incentive to plow all extra funds back into the project because the project has high returns in the early phases. If there were no risk of the entrepreneur absconding, then outside lenders would extend enough funds to the entrepreneur in the initial period so that the return on the project would not be higher than those available generally in financial markets. But, outsiders are afraid to lend so much to the entrepreneur out of a concern that the entrepreneur would renege on her loan commitment and run away with the funds.

An important principle emerges in the model analysis. In the initial phase of the contract, when the entrepreneur has no resources of her own, she is nevertheless able to borrow. This stands in contrast to models where borrowing is limited by the quantity of tangible capital owned by the entrepreneur. In those models, the tangible capital is something that the lenders can seize in

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[1]For an important early paper on these ideas, see Jonathan Thomas and Tim Worrall, 1994, 'Foreign Direct Investment and the Risk of Expropriation,' *Review of Economic Studies*, vol. 61, pp. 81-108. For related work, see Rui Albuquerque and Hugo A. Hopenhayn, 2004, 'Optimal Lending Contracts and Firm Dynamics,' *Review of Economic Studies*, Vol. 71, No. 2, April, pp. 285-315.

the event that the borrower reneges on the contract terms.[2] In the model below, lending occurs even though the lenders do not have the option to seize the assets of the borrower. Lenders are nevertheless willing to extend loans to the borrower because lenders understand that borrowers suffer a loss in the event that they renege.[3] In the early phases of a project, the capital operated by the borrower is primarily financed by the lender. As a result, a large share of the current earnings of capital are earmarked for interest payments to the lender. It follows that the destruction of physical capital which occurs when a borrower reneges provides little incentive in the early phase of a loan contract for the borrower to abide by the terms. But, the borrower in the early phase of a loan contract has something else to lose in the case of default: contact with a project that has the potential to generate valuable returns in the future. In later phases of a loan project, the borrower is in a position to borrow even more because at that point she has more to lose by reneging. As time evolves, a larger portion of the capital operated by entrepreneurs has been financed by their own savings. As a result the destruction of capital that occurs in the later phases of a project provides another reason for the entrepreneur to abide by contract terms.[4]

The assumptions of the model are designed in part to convey some wisdom about financial markets and in part with an eye on tractability. To ensure tractability, assumptions have been made that are in some cases hard to recognize in the data. For example, there are many ways a real-world borrower can in effect renege on a loan contract. She can, for example, be chronically late in making the required interest payments. Or she can attempt to renegotiate terms with the lender ex post. All these options and others are summarized here in the simple, blunt assumption that the borrower can 'run away'. Also, the model leaves out many real world factors. For example, lenders are assumed to not be able to seize the borrower's capital.[5] Although the model is unrealistic in many of its details, there is nevertheless much that is familiar in it. For example, when a real-world bank extends a mortgage contract, it takes into consideration the borrower's income. This information is useful to the bank because it wants to understand the incentives of a household to actually stick to the terms of the loan contract ex post. A given set of contract terms extended to a borrower with low income implies low consumption. But, contract terms that imply low consumption for the borrower raise concerns to a banker that the household has an incentive to renege later on.

2 Model

Consider an entrepreneur who has access to a technology for accumulating capital, k_{t+1}:

$$k_{t+1} \leq I_t + (1-\delta) k_t, \ 0 < \delta < 1 \tag{1}$$

where I_t denotes investment and δ denotes the rate of depreciation. The entrepreneur also has access to a technology whereby capital can be used to produced output as follows:

$$AF(k_t), \ F(k) = k^\alpha, \ 0 < \alpha < 1, \ A > 0. \tag{2}$$

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[2] See, e.g., Kiyotaki and Moore, 1997, 'Credit Cycles' *Journal of Political Economy*, 105, 211-248

[3] Think of passengers in an airplane. They are comfortable with the idea that the pilot will pay close attention to the task of flying the plane because they understand the pilot has a lot to lose by not paying attention. In particular, the passengers' peace of mind has nothing to do with any perception that the pilot will make some sort of transfer to them in the event of a crash.

[4] The ideas in this paragraph are an important theme of the paper by Albuquerque and Hopenhayn that is cited in a previous footnote.

[5] This assumption is thought to make the model particularly relevant for thinking about international capital markets, where there are often severe restrictions against seizing tangible assets.

We assume that the problem starts in period 0 when the entrepreneur has no capital (i.e., $k_0 = 0$) and no other resources. The entrepreneur has access to a financial firm which extends a sequence of one-period loans. If the entrepreneur borrows B_{t+1} in period t, then she must repay RB_{t+1} in $t+1$, where $R > 1$ denotes the gross rate of interest and $t = 0, 1, 2, ...$. For example, if the entrepreneur wishes to operate capital, k_1, in period 1, then (1) implies that she must build it in period 0 using investment goods acquired with a loan, B_1:

$$k_1 \leq B_1. \tag{3}$$

The entrepreneur has linear preferences:

$$\sum_{t=1}^{\infty} \beta^t c_t, \ 0 < \beta < 1. \tag{4}$$

The entrepreneur seeks to maximize (4) subject to $k_t, c_t \geq 0$ for $t \geq 1$ and to her budget constraints (no consumption occurs in period 0). The budget constraints are composed of (1), (3), a debt limit defined below and a sequence of flow budget constraints:

$$k_{t+1} - (1 - \delta) k_t + c_t \leq AF(k_t) + B_{t+1} - RB_t, \tag{5}$$

for $t = 1, 2, ...$. Here, I_t has been substituted out using (1) and R denotes the gross, one period rate of interest. Also, B_{t+1} corresponds to borrowing in period t. In principle, B_{t+1} could be positive or negative. Throughout, assume[6]

$$R = \frac{1}{\beta}.$$

We assume that the entrepreneur is required to respect the following borrowing limit:

$$\lim_{j \to \infty} \beta^j B_{j+1} \leq 0. \tag{6}$$

3 Properties of the intertemporal budget constraint

In studying the nature of the loan contract, it is convenient to consider an alternative representation of the budget constraint. The alternative, in which intertemporal decisions are restricted by a single equation, is identical to (3), (5) and (6). The present value representation of the budget constraint is:

$$k_1 \leq \sum_{t=1}^{\infty} \beta^t \left[AF(k_t) + (1 - \delta) k_t - c_t - k_{t+1} \right]. \tag{7}$$

Notice that this equation has no debt in it. The object on the left of (7) is the initial stock of capital, which must be fully financed by borrowing, B_1. That is, (7) says that the amount borrowed in the first period must be no greater than the present discounted value of the future surpluses generated by the entrepreneur. Expression (7) is equivalent to (3), (5) and (6). That is to say, any sequence of c's, k's and B's that satisfy (3), (5) and (6) satisfy (7). Moreover, for any sequence of c's and k's that satisfy (7), a sequence of B's may be found such that the c's, k's and B's satisfy (3), (5) and (6). To establish these results it is required that various infinite sums like

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[6]We do not explicitly describe the general macroeconomic framework in which the borrowers and lenders that we study are situated. Still, with the assumption of linear utility in consumption, it is very likely that the interest rate would be $1/\beta$ in a general equilibrium. If the interest rate were greater than $1/\beta$, then the suppliers of funds to the lender would supply an infinite amount. If the interest rate were less than $1/\beta$ they would supply nothing.

(7) are well defined. All these things are established rigorously in the Appendix. The discussion in the appendix clarifies the role played by (6). Without that restriction, the 'budget constraint', (3) and (5), is no constraint at all because it does not rule out setting B unboundedly large.

The implication of the equivalence of (7) and (3), (5), (6) is that the original problem of the entrepreneur can be recast as one of choosing $\{k_t, c_t\}_{t=1}^{\infty}$ to maximize utility, (4), subject to (7), without regard to debt. After the problem with the present discounted value budget constraint has been solved, the underlying sequence of debts can be backed out using (3) and (5).

4   Model properties in absence of financial frictions

Suppose the entrepreneur selects sequences, $\{k_t, c_t\}_{t=1}^{\infty}$, to maximize utility subject her budget constraint, (7), and non-negativity of consumption and capital. We:

- show that there is a unique sequence of capital stocks, $k_1, k_2, ...$ that solves the entrepreneur's problem.

- derive a simple function relating the optimal sequence of k's to the model parameters.

- show that although the sequence of k's that solve the problem is unique, the sequence of c's is not unique.

- show that the non-negativity constraints on consumption are never binding.

As discussed in the previous section, we can replace the flow budget constraint with the single intertemporal budget constraint, which involves only c's and k's. The Lagrangian representation of the resulting problem is:

$$
\max_{\{c_t, k_{t+1}\}} \sum_{t=1}^{\infty} \beta^t \left(1 + \gamma_t\right) c_t + \omega \left(\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - \overbrace{\sum_{t=1}^{\infty} \beta^t c_t}^{\equiv C} - k_1 \right),
$$

where $\omega \geq 0$ denotes the multiplier on the intertemporal budget constraint and C denotes the present discounted value of consumption. Also, $\gamma_t \geq 0$ is the multiplier on the period t non-negativity constraint on consumption. Collect terms in c_t :

$$
\max_{\{c_t, k_{t+1}\}} \sum_{t=1}^{\infty} \beta^t \left(1 - \omega + \gamma_t\right) c_t + \omega \left(\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - k_1 \right)
$$

The first order conditions are:

$$
c_t : 1 + \gamma_t = \omega, \ \overbrace{\gamma_t c_t = 0, \ \gamma_t \geq 0, c_t \geq 0}^{\text{complementary slackness}}, \ t > 0 \tag{8}
$$

$$
k_{t+1} : 1 = \beta \left[AF'(k_{t+1}) + (1-\delta) \right], \ t \geq 0 \tag{9}
$$

$$
\overbrace{\omega}^{\geq 0} \times \left(\overbrace{\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - C - k_1}^{\geq 0} \right) = 0 \tag{10}
$$

From (9), we see that the optimal sequence of k's is a sequence of constants, $k^*, k^*, ...$, where

$$
k^* = \left[\frac{\frac{1}{\beta} - (1-\delta)}{\alpha A} \right]^{\frac{1}{\alpha-1}}. \tag{11}
$$

From (8), we see that γ is constant. We conjecture, and subsequently verify, that $\gamma = 0$. As a result, (8) implies $\omega = 1$. The Lagrangian representation of the problem simplifies to:

$$\max_{\{k_{t+1}\}} \sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1 - \delta) k_t) \right] - k_1, \tag{12}$$

and the complementary slackness condition, (10), on the intertemporal budget constraint (given $\omega > 0$):

$$C = \sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1 - \delta) k_t) \right] - k_1.$$

The solution to (12) is given by (11), so that the unique solution for C, denoted C^*, is:

$$C^* = \frac{\beta}{1 - \beta} \left[AF(k^*) - (k^* - (1 - \delta) k^*) \right] - k^*,$$

where k^* is defined in (11).

We now verify that for each $A > 0$ and $\delta \in (0, 1)$, $C^* > 0$. Since the optimal k_t, $t = 1, 2, 3, ...$ satisfies the restriction that the k's are all constant, we can - without restricting the optimum - impose this restriction on the entrepreneur's problem. That is,

$$\begin{aligned} k^* &= \arg\max_k \left\{ \frac{\beta}{1 - \beta} \left[AF(k) - (k - (1 - \delta) k) \right] - k \right\} \tag{13} \\ &= \arg\max_k \left\{ \frac{\beta}{1 - \beta} \left[AF(k) + (1 - \delta) k \right] - \frac{\beta}{1 - \beta} k - k \right\} \\ &= \frac{1}{1 - \beta} \arg\max_k \left\{ \beta \left[AF(k) + (1 - \delta) k \right] - k \right\}. \end{aligned}$$

It is feasible to make the object in braces positive because the slope of the object in square brackets goes to ∞ as $k \to 0$. This establishes that $C^* > 0$. This in turn means that we can always find a sequence, $c_t \geq 0$, that supports the optimum. This verifies our assumption that the non-negativity constraint on c_t is non-binding for each t, i.e., $\gamma = 0$.

In sum, the problem may be solved by first computing the (unique) optimal k's and C, and then finding a sequence of c_t's that is compatible with C. Obviously, there is a very large set of c_t's that solves the problem, even though the optimal C is unique. Corresponding to each sequence of c_t's that is compatible with C^*, a sequence of B_t's that satisfy the flow budget constraints and debt limit can be found.

5   Model with financial frictions: General Observations

Now suppose that the entrepreneur lacks the ability to commit to an arbitrary set of contract terms that are consistent with the budget constraint. In particular, in period t the entrepreneur can choose to abscond with a fraction, $\theta \in (0, 1)$, of the capital, k_t, that is then in her possession. After running away, the agent forever loses access to the capital accumulation and production technologies. She retains full access to credit markets at the interest rate, $R = 1/\beta$. Thus, she can sell θk_t and - possibly by using credit markets - use the proceeds to finance a sequence of consumptions, $\{c_{t+j}^d\}_{j=0}^{\infty}$ subject to:

$$\theta k_t = \sum_{j=0}^{\infty} \beta^j c_{t+j}^d.$$

Here, the superscript, 'd', indicates consumption after deviating from a loan agreement.

In this section we discuss the lending arrangement that emerges under lack of commitment. The subsequent subsections contain the following results.

1. In the first subsection below, we describe the additional constraints that lenders impose on the loan contract, beyond the budget constraint. The constraints ('Incentive Constraints', IC) limit the entrepreneur to contracts whose terms will not give her the incentive to abscond at some point in the future. That is, the terms guarantee that the contract is self enforcing.

2. if the IC is binding in period $t + 1$, then the optimally chosen c_t is equal to zero.

3. if the IC is non-binding for period t, then it is also non-binding for periods $t, t+1, t+2, \ldots$.

4. if the IC ceases to bind in period t, then the non-negativity constraint on consumption ceases to bind in period $t - 1, t, t + 1, \ldots$.

5. if the non-negativity constraint on c_t is strictly binding, then the IC constraint is less binding in $t + 1$ than it is in t. A related observation is that the capital stock rises smoothly up to its unconstrained value of k^* in (11).

5.1   The Incentive Constraints (IC)

A lender will not extend just any budget-feasible contract to a borrower. The lender will only consider a contract that has the property that the borrower will not ex post find it optimal to abscond. Denote the utility enjoyed by the lender who absconds in period t by $V^a(k_t)$:

$$V^a(k_t) = \theta k_t.$$

That this is the correct representation of the utility value of absconding reflects two considerations. First, we assume that an absconder continues to have access to financial markets at interest rate, $R = 1/\beta$. Through the use of financial markets, the period t absconder can obtain any sequence of consumptions, c^d_{t+j}, $j \geq 0$, that satisfy

$$\theta k_t = \sum_{j=0}^{\infty} \beta^j c^d_{t+j}. \tag{14}$$

The second consideration is our assumption that the object on the right of (14) is the utility of the sequence, c^d_t. The utility value of not absconding in period t and abiding by the terms of the contract is:

$$c_t + \beta c_{t+1} + \beta^2 c_{t+2} + \cdots .$$

Here, c_t denotes the period t level of consumption implied by honoring the lending contract. Lenders will only extend loan contracts to borrowers which provide no incentive to abscond in any period. That is, the contract must satisfy:

$$c_t + \beta c_{t+1} + \beta^2 c_{t+2} + \cdots - \theta k_t \geq 0, \tag{15}$$

for $t = 1, 2, \ldots$.

5.2 If the incentive constraint is binding in $t+1$ then internal funds are used to the maximum possible extent in t

Let $\beta^t \Lambda_t \geq 0$ denote the multiplier on the period t incentive constraint, (15). The Lagrangian representation of the entrepreneur's problem is:

$$\sum_{t=1}^{\infty} \beta^t \{ c_t + \gamma_t c_t + \Lambda_t \left[c_t + \beta c_{t+1} + \beta^2 c_{t+2} + \cdots - \theta k_t \right] \} \tag{16}$$

$$+ \omega \left(\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - \sum_{t=1}^{\infty} \beta^t c_t - k_1 \right).$$

To collect terms on c_t, note that these terms come in two parts: one involving the incentive constraint and the other, not. The second part is:

$$\beta^t \left(1 + \gamma_t - \omega \right).$$

Now consider the contribution of the incentive constraints to the coefficient on c_t in (16). Note that c_t appears in the date t incentive constraint with coefficient $\beta^t \Lambda_t$; in the date $t-1$ constraint with coefficient $\beta^{t-1} \Lambda_{t-1} \beta$; in the date $t-2$ constraint with coefficient $\beta^{t-2} \Lambda_{t-2} \beta^2$; and so on, until the period 1 incentive constraint, where it appears with coefficient $\beta \Lambda_1 \beta^{t-1}$. That is, c_t enters (16) due to the incentive constraint with coefficient

$$\beta^t \Lambda_t + \beta^{t-1} \Lambda_{t-1} \beta + \beta^{t-2} \Lambda_{t-2} \beta^2 + \dots + \beta \Lambda_1 \beta^{t-1},$$

or,

$$\beta^t \left[\Lambda_t + \Lambda_{t-1} + \Lambda_{t-2} + \dots + \Lambda_1 \right].$$

Collecting terms in c_t in the entrepreneur's Lagrangian problem,

$$\sum_{t=1}^{\infty} \beta^t \{ \left[1 + \gamma_t + \Lambda_t + \Lambda_{t-1} + \Lambda_{t-2} + \dots + \Lambda_1 - \omega \right] c_t - \Lambda_t \theta k_t \}$$

$$+ \omega \left(\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - k_1 \right).$$

If the coefficient on c_t for some t in the Lagrangian were positive, the entrepreneur would set $c_t = \infty$, which violates the intertemporal budget constraint. If the coefficient were negative for some t the entrepreneur would set $c_t = -\infty$, which violates non-negativity. So, for the solution to the Lagrangian problem to be the solution to the constrained problem of the entrepreneur, it must be that the multipliers have the property that the coefficient on c_t is zero. That is,

$$c_t : \ 1 + \gamma_t + \Lambda_t + \Lambda_{t-1} + \Lambda_{t-2} + \dots + \Lambda_1 = \omega.$$

In this way, the Lagrangian representation of the entrepreneur's problem reduces to:

$$\max_{\{k_t\}} \left[-\sum_{t=1}^{\infty} \beta^t \Lambda_t \theta k_t + \omega \left(\sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - k_1 \right) \right]. \tag{17}$$

There are two interesting features of the Lagrangian representation of the entrepreneur's problem. First, c_t does not appear explicitly, only k_t does. Thus, to solve for c_t using this representation of the problem, first solve for k_t. Then find a sequence of c_t's that are consistent with the solved-for

k_t's and the budget and incentive constraints. The second interesting feature of (17) is that the effect of the incentive constraint is to attach an additional (shadow) cost to capital, so that we can expect there to be less capital than when the incentive constraint is ignored. As a result, we can also expect the ex post return on capital to be high in periods when the incentive constraint is strictly binding. To see this, note that the first order condition for k_{t+1} is:

$$k_{t+1} : \omega + \beta \Lambda_{t+1} \theta = \beta \omega \left[AF'(k_{t+1}) + (1 - \delta) \right],$$

for $t = 0, 1, \dots$.

Collecting first order conditions:

$$c_t \quad : \quad 1 + \gamma_t + \Lambda_t + \Lambda_{t-1} + \Lambda_{t-2} + \dots + \Lambda_1 = \omega, \quad \overset{\geq 0}{\overbrace{c_t}} \times \overset{\geq 0}{\overbrace{\gamma_t}} = 0 \tag{18}$$

$$k_{t+1} \quad : \quad \omega + \beta \Lambda_{t+1} \theta = \beta \omega \left[AF'(k_{t+1}) + (1 - \delta) \right] \tag{19}$$

According to (18), an ever growing sum of non-negative objects is equal to a constant. Evidently, it must be that γ_t is falling. To see this, write the expression at time $t + 1$:

$$1 + \gamma_{t+1} + \Lambda_{t+1} + \Lambda_t + \Lambda_{t-1} + \Lambda_{t-2} + \dots + \Lambda_1 = \omega.$$

Now, subtract, to obtain:

$$\gamma_t = \gamma_{t+1} + \Lambda_{t+1}. \tag{20}$$

Since $\Lambda_{t+1} \geq 0$, it follows that

$$0 \leq \gamma_{t+1} \leq \gamma_t \text{ for all } t.$$

Thus, the non-negativity constraint on consumption is non-increasing over time.

Suppose that there is some t, say t^*, in which the incentive constraint binds, i.e., $\Lambda_{t^*} > 0$. This implies, via (20), that $\gamma_{t^*} < \gamma_{t^*-1}$. Since $\gamma_t \geq 0$ for all t, it follows that $\gamma_{t^*-1} > 0$. By the complementary slackness condition, (18),

$$c_{t^*-1} = 0. \tag{21}$$

There is a simple intuition for (21). Dividing both sides of (19) by $\omega\beta$ and making use of $R = 1/\beta$:

$$R + \Lambda_{t+1} \frac{\theta}{\omega} = AF'(k_{t+1}) + (1 - \delta). \tag{22}$$

That is, if $\Lambda_{t+1} > 0$ then the return on capital built in period t is greater than R. But, for the entrepreneur to be compensated for a reduction in c_t by an increase in c_{t+1} the return need only be R. With the return via capital investment greater than R, utility is maximized by reducing c_t to its limit of zero and investing the proceeds in k_{t+1}. Put differently, 'internal funds' (i.e., consumption) are used to the maximum extent possible when the external source of funds is constrained.

5.3 If the IC fails to bind in a period then it fails to bind in all subsequent periods

The result, (20), implies:

$$\gamma_t = 0 \rightarrow \gamma_{t+1} = \Lambda_{t+1} = 0. \tag{23}$$

Since γ_{t+1} and Λ_{t+1} are both non-negative, the only way their sum can be zero is if each term is individually zero. In words, if the non-negativity constraint on c_t is non-binding, then it must be

that the return on investing internal funds in capital is no greater than the intertemporal rate of return on consumption, i.e., $\Lambda_{t+1} = 0$ (see (22)).

We now show:
$$\text{if } \Lambda_t = 0 \text{ then } \Lambda_{t+s} = 0 \text{ for all } s > 0. \tag{24}$$

Thus, suppose $\Lambda_t = 0$. By (20), we have

$$\gamma_{t-1} = \gamma_t + \Lambda_t = \gamma_t.$$

So, $\gamma_{t-1} = \gamma_t$. Suppose $\gamma_{t-1} = 0$. Then, by (23) we know that $\Lambda_{t+s} = 0$ for $s > 0$. Now suppose $\gamma_{t-1} > 0$ and $\Lambda_{t+1} > 0$. We show that this implies a contradiction. Since $\gamma_t > 0$ it must be that $c_t = 0$ by (18). The supposition, $\Lambda_t = 0$, implies:

$$\beta c_{t+1} + \beta^2 c_{t+2} + \cdots \geq \theta k_t,$$

using $c_t = 0$. Then,
$$\beta \left[c_{t+1} + \beta c_{t+2} + \cdots \right] \geq \theta k_t$$

Now, $\Lambda_{t+1} > 0$ implies (by complementary slackness):

$$c_{t+1} + \beta c_{t+2} + \cdots = \theta k_{t+1}.$$

Combining the last two results:
$$\beta \theta k_{t+1} \geq \theta k_t,$$

or, cancelling θ,
$$\beta k_{t+1} \geq k_t.$$

Since $\beta < 1$,
$$k_{t+1} > k_t. \tag{25}$$

Consider (19) for t and $t + 1$:

$$
\begin{aligned}
\omega &= \beta \omega \left[AF'(k_t) + (1 - \delta) \right] \\
\omega + \beta \Lambda_{t+1} \theta &= \beta \omega \left[AF'(k_{t+1}) + (1 - \delta) \right],
\end{aligned}
$$

where $\Lambda_t = 0$ and $\Lambda_{t+1} > 0$ has been used. The above expressions imply:

$$k_{t+1} < k_t,$$

which contradicts (25). Thus, the supposition, $\gamma_{t-1} > 0$ and $\Lambda_{t+1} > 0$, implies a contradiction. That supposition is therefore false. This leaves two possibilities: $\gamma_{t-1} = 0$, in which case (24) follows by the argument given above; or $\gamma_{t-1} > 0$ and $\Lambda_{t+1} = 0$. In the latter case, we repeat the argument to obtain $\Lambda_{t+2} = 0$, and so on. This establishes (24).

5.4 Internal funds cease to be exhausted starting in the period before the IC ceases to bind

We now argue that
$$\text{if } \Lambda_t = 0, \text{ then } \gamma_{t-1+s} = 0 \text{ for } s \geq 0.$$

That is, the non-negativity constraint on consumption ceases to bind starting in the period before the date when the incentive constraint ceases to bind. Thus, suppose $\Lambda_t = 0$. We know from the preceding subsection that $\Lambda_{t+s} = 0$ for $s > 0$. We know that the γ's are declining over time and

9

that initially, they are positive (the latter is by assumption, for otherwise the financial frictions would be uninteresting). Suppose that t^* is the first date when $\Lambda_t = 0$, so that $\Lambda_{t^*-1} > 0$. Using

$$\gamma_{t-1} = \gamma_t + \Lambda_t,$$

this means that $\gamma_t > 0$ for $t = t^* - 2$, $t^* - 3$, *etc.* That is, consumption is zero in all those dates. From the same expression, we obtain $\gamma_{t^*-1} = \gamma_{t^*} = \gamma_{t^*+1} = ...$. The multipliers are all constant from the date just before Λ_t hits zero. Suppose they were all positive. This would mean that consumption is always zero, from the start to date infinity. But, we assume that consumption is feasible at some date (otherwise, the problem would not be interesting). It follows that all the multipliers are constant at zero. This means that consumption is weakly positive at all those dates, beginning with the date before t^*. So, we conclude

$$\gamma_{t^*-1} = \gamma_{t^*} = \gamma_{t^*+1} = ... = 0.$$

But, $\Lambda_t = 0$ for each $t = t^* + s$, $s \geq 0$. Thus, $\Lambda_t = 0$ implies

$$\gamma_{t-1} = \gamma_t = \gamma_{t+1} = ... = 0.$$

5.5   The IC constraint is less binding over time

We now show that $\Lambda_t \geq \Lambda_{t+1}$, i.e., the IC constraint is less binding over time. Suppose $\gamma_t = 0$. In that case, we showed above that $\Lambda_{t+1} = 0$ and the result holds because $\Lambda_t \geq 0$. Suppose $\gamma_t > 0$, so that $c_t = 0$ and the incentive constraint for period t is:

$$\beta \left[c_{t+1} + \beta c_{t+2} + ... \right] \geq \theta k_t.$$

If $\Lambda_{t+1} = 0$ the result follows trivially. Suppose $\Lambda_{t+1} > 0$. In this case, by complementary slackness,

$$c_{t+1} + \beta c_{t+2} + ... = \theta k_{t+1},$$

and substituting,

$$\beta \theta k_{t+1} \geq \theta k_t,$$

or,

$$k_{t+1} > k_t.$$

Equation (19) implies $\Lambda_t > \Lambda_{t+1}$. Thus, the incentive constraint gets less and less binding over time.

6   Solving the Model with Financial Frictions

We use the results of the previous sections to guide the computation of an optimal contract in the presence of financial frictions. For the computations to be interesting, those frictions must obviously be binding. That is, with $k_1 = k^*$ it must be that the period 1 incentive constraint,

$$\frac{1}{1-\beta} \left(\beta \left[AF(k^*) + (1-\delta) k^* \right] - k^* \right) \geq \beta \theta k^*, \tag{26}$$

is violated. In (26), the object on the left of the inequality is the present discounted value of consumption as of date 0. The discussion that follows presumes that (26) does not hold, so that $T \geq 1$.

Expression (26) shows that for larger θ the likelihood that the incentive constraint violated for the first best allocations increases. It is useful to have a formula for the largest value of θ compatible with the incentive constraint:

$$\frac{(\beta \left[AF(k^*) + (1-\delta)k^*\right] - k^*)}{(1-\beta)\beta k^*}.$$

We can represent this expression directly in terms of parameters alone by substituting out the value of k^* in terms of model parameters:

$$\frac{(\beta \left[AF(k^*) + (1-\delta)k^*\right] - k^*)}{(1-\beta)\beta k^*} = \frac{\beta(1-\alpha)A(k^*)^{\alpha-1}}{(1-\beta)\beta} = \frac{(1-\alpha)}{\alpha(1-\beta)}\left[\frac{1}{\beta} - (1-\delta)\right]. \qquad (27)$$

From this expression we see that the incentive constraint is more likely to be violated at the first best allocations when α is larger and/or δ is smaller.

6.1 Computing the Sequence of Capital Stocks

We describe a simple algorithm that can be used to compute the sequence of capital stocks that solve the entrepreneur problem. We assume that the constraints are binding for T periods, after which the capital stock is at k^*. This assumption can be verified at the end. In what follows we first derive a result about the path that the capital stock must follow, according to the optimal contract. We use this result in the second subsection to motivate our simple computational algorithm.

6.1.1 The Slope of the Capital Path While the Constraint is Binding

According to the discussion in section 5.5, the incentive constraint is weakly less binding over time and once it hits zero it stays there. This suggests looking for a solution to the contract problem in which the incentive constraint is binding for a finite number of periods, $t = 1, 2, ..., T$, and is non-binding in $T + 1$ and all subsequent periods. Thus, we proceed under the assumption that the solution is characterized by

$$\begin{aligned}\Lambda_t &> 0, \ t = 1, ..., T, \\ \Lambda_t &= 0, \ t = T+1, T+2, ... \ ,\end{aligned}$$

for some $\infty > T > 0$. The fact that the incentive constraint binds in $t = 1$ implies, by the IC constraint, (15):

$$c_1 + \beta c_2 + ... + \beta^{T-1}c_T + \beta^T c_{T+1} + \beta^{T+1}c_{T+2}... = \theta k_1. \qquad (28)$$

In words, k_1 is determined in period 0 and the entrepreneur decides whether or not to abscond at the start of period 1, at which point lifetime utility is evaluated according to the object on the left of the equality in (28).

The discussion in section 5.4 shows that consumption is zero in periods 1 to $T - 1$, so that (28) simplifies as follows:

$$\beta^{T-1}c_T + \beta^T c_{T+1} + \beta^{T+1}c_{T+2} + ... = \theta k_1. \qquad (29)$$

The entrepreneur also has the opportunity to abscond at the start of period $t = 2$, at which point the utility of continuing under the contract is the object on the left of the equality in the following expression:

$$c_2 + \beta c_3 + ... + \beta^{T-2}c_T + \beta^{T-1}c_{T+1} + \beta^T c_{T+2}... = \theta k_2.$$

Imposing
$$c_t = 0 \text{ for } t = 2, ..., T - 1, \tag{30}$$

we have
$$\beta^{T-2} c_T + \beta^{T-1} c_{T+1} + \beta^T c_{T+2}... = \theta k_2,$$

or, after multiplying by β :
$$\beta^{T-1} c_T + \beta^T c_{T+1} + \beta^{T+1} c_{T+2}... = \beta \theta k_2. \tag{31}$$

Similarly,
$$\beta^{T-1} c_T + \beta^T c_{T+1} + \beta^{T+1} c_{T+2}... = \beta^2 \theta k_3 \tag{32}$$
$$\beta^{T-1} c_T + \beta^T c_{T+1} + \beta^{T+1} c_{T+2}... = \beta^3 \theta k_4 \tag{33}$$
$$...$$
$$\beta^{T-1} c_T + \beta^T c_{T+1} + \beta^{T+1} c_{T+2}... = \beta^{T-1} \theta k_T \tag{34}$$

That is,
$$\beta^{T-1} c_T + \beta^T c_{T+1} + \beta^{T+1} c_{T+2}... = \beta^{t-1} \theta k_t, \tag{35}$$

for $t = 1, 2, ..., T$. Note that the object on the left of (35) is the same for all t, so that
$$\beta^{t-2} \theta k_{t-1} = \beta^{t-1} \theta k_t,$$

or,
$$k_t = \left(\frac{1}{\beta}\right)^{t-1} k_1 < k^*, \tag{36}$$

for $t = 1, 2, ..., T$. Evidently, to compute the optimal k_t's in the contract, it remains only to find k_1 and T. We do so in the following subsection.

6.1.2   An Algorithm for Computing the Optimal Capital Path

To understand how we proceed, recall that the basic problem of the entrepreneur is so identify a sequence of c's and k's that satisfy

  (i)   the budget constraint, (7);

  (ii)   the sequence of IC constraints at $t = 1, 2, ...,$ (15);

  (iii)   the non-negativity constraint on c_t.

We impose two additional constraints:

  (iv)   the entrepreneur must choose a particular value of T and must choose $c_1 = c_2 = ... = c_{T-1} = 0$;

  (v)   the sequence of k's that is chosen must satisfy (36).

Our algorithm chooses the contract that maximizes entrepreneur utility by choice of k_1 and T, subject to (i)-(v). We know from the results in the previous section (and the conjecture that the solution is characterized by finite T) that the best contract which satisfies (i)-(iii) also satisfies

(iv) and (v). So, if we find the best contract that satisfies (i)-(v), that contract is then also the best one that satisfies (i)-(iii).[7]

We now derive a simple characterization of conditions (ii) and (iii), conditional on (iv), (v) and a specific value for T. First, we show that the period 1 IC constraint can be used to solve for k_1 as a function of T. As before, let C denote the present discounted value of consumption:

$$C \equiv \sum_{t=T}^{\infty} \beta^t c_t.$$

Optimality implies that the entrepreneur is on the boundary of her budget constraint, (7). Thus,

$$C = \sum_{t=1}^{\infty} \beta^t \left[AF(k_t) - (k_{t+1} - (1-\delta) k_t) \right] - k_1. \tag{37}$$

Rearranging the terms in (37) and imposing $k_t = k^*$ for $t > T$:[8]

$$
\begin{aligned}
C &= \sum_{t=1}^{T} \beta^{t-1} \left(\beta \left[AF(k_t) + (1-\delta) k_t \right] - k_t \right) \\
&\quad + \frac{\beta^T}{1-\beta} \left(\beta \left[AF(k^*) + (1-\delta) k^* \right] - k^* \right).
\end{aligned} \tag{38}
$$

The period 1 incentive constraint evaluated at equality is:[9]

$$C = \beta \theta k_1. \tag{39}$$

Equating (38) and (39):

$$
\begin{aligned}
&\sum_{t=1}^{T} \beta^{t-1} \left(\beta \left[AF(\left(\frac{1}{\beta}\right)^{t-1} k_1) + (1-\delta) \left(\frac{1}{\beta}\right)^{t-1} k_1 \right] - \left(\frac{1}{\beta}\right)^{t-1} k_1 \right) \\
&\quad + \frac{\beta^T}{1-\beta} \left(\beta \left[AF(k^*) + (1-\delta) k^* \right] - k^* \right) = \beta \theta k_1
\end{aligned} \tag{40}
$$

This expression defines k_1 as a function of T, and we denote this relationship by $k_1(T)$. To see that $k_1(T)$ is single-valued note first that for $k_1 = 0$ the term on the left of the equality in (40) is positive and has infinite slope (recall, $0 < \alpha < 1$ according to (2)). Consequently, for small k_1 the term on the left of the equality is greater than the term on the right of the equality. However, the slope of the term on the left is monotonically decreasing and as $k_1 \to \infty$ it becomes negative (here, we use the facts, $\alpha, \delta \in (0, 1)$)[10]. Thus, the term on the left must, for sufficiently large k_1

---

[7] We have not included a formal demonstration that the solution necessarily involves a finite value of T, We suspect that such a demonstration is possible.

[8] A key step uses the fact that, by rearranging terms, (37) can be written as:

$$
\begin{aligned}
C &= \lim_{N \to \infty} \sum_{t=1}^{N} \beta^{t-1} \left(\beta \left[AF(k_t) + (1-\delta) k_t \right] - k_t \right) \\
&\quad - \lim_{N \to \infty} \beta^N k_N.
\end{aligned}
$$

Then, note that the last term is zero given that $k_N = k^*$ for $N > T$.

[9] The fact that β multiplies the object on the right of the weak inequality reflects that $C = \beta c_1 + \beta^2 c_2 + ...$, while the incentive constraint involves $c_1 + \beta c_2 + ...$.

[10] Note that for $k_1 > k^*$ every term in the summand in (40) is decreasing.

be smaller than the term on the right. Continuity of the mappings on the left and the right of the equality implies that there must be a crossing. Because the derivative of the function on the left is monotone decreasing in k_1, there can at most be only one crossing. We conclude that there is a unique value of k_1 that solves (40) for every T. That is, $k_1(T)$ is single-valued.

It is also easy to verify that $k_1(T)$ is decreasing in T. To do so, consider the infinite sum from $t = 1$ to infinity on the left of the equality in (40). Each summand for $t > T$ is at its maximized value because it is evaluated at k^* (recall (13). The summands for $t \leq T$ are evaluated at capital stocks that are less than k^* and so they are smaller than the summands for $t > T$. It follows that for each k_1, the curve to the left of the equality in (40) shifts down as T increases. From this, we conclude that $k_1(T)$ is decreasing in T.

We know that $k_t < k^*$ for $t = 1, ..., T$, so that, using (36),

$$k_1(T) < k^* \beta^{T-1}. \tag{41}$$

We now show that when k_t evolves according to (36), the fact that the IC holds with equality at $t = 1$ implies that it also holds with equality in $t = 2, ..., T$. To see this, note that the value of remaining in the contract for $1 \leq t \leq T$ is:

$$\left(\frac{1}{\beta}\right)^t C,$$

by (30). The value of deviating from the contract in period t is (using (36)):

$$\theta \left(\frac{1}{\beta}\right)^{t-1} k_1.$$

It follows that the IC constraint evaluated at equality is:

$$\left(\frac{1}{\beta}\right)^t C - \left(\frac{1}{\beta}\right)^{t-1} \theta k_1 = 0,$$

or,

$$\left(\frac{1}{\beta}\right)^t [C - \beta \theta k_1] = 0.$$

This is zero if, and only if $C = \beta \theta k_1$. This is indeed satisfied by the construction of k_1. Thus, the IC constraint is satisfied as a strict equality in periods $t = 1, ..., T$.

Finally, we turn to the IC constraints after T, that is,

$$c_{T+l} + \beta c_{T+l+1} + ... \geq \theta k^*,$$

for $l = 1, ...$. The IC constraints at T and $T + 1$ are:

$$c_T + \beta c_{T+1} + ... = \theta k_T$$
$$c_{T+1} + \beta c_{T+2} + ... \geq \theta k^*.$$

Multiply the second equation by β and subtract:

$$c_T \leq \theta (k_T - \beta k^*). \tag{42}$$

Thus, a constraint on T is that

$$k_T \geq \beta k^*. \tag{43}$$

14

It is easy to see that if (43) is satisfied, then the IC constraints in $T+2, T+3, ...$ are also satisfied. Thus, consider the IC constraints at T and $T+l$:

$$c_T + \beta c_{T+1} + ... = \theta k_T$$
$$c_{T+l} + \beta c_{T+l+1} + ... \geq \theta k^*.$$

Multiply the second term by β^l and subtract:

$$c_T + \beta c_{T+1} + ... + \beta^{l-1} c_{T+l-1} \leq \theta \left(k_T - \beta^l k^* \right). \tag{44}$$

From (44) we can see that non-negativity of consumption requires

$$k_T \geq \beta^l k^*,$$

for $l = 1, 2, ...$. To guarantee that the latter condition is satisfied, it is sufficient to verity (43).

We conclude that for the IC constraints to be satisfied for $t \geq 1$ and for given T, it must be that $k_1 = k_1(T)$ and

$$\beta k^* \leq \frac{k_1(T)}{\beta^{T-1}} < k^*. \tag{45}$$

An equivalent representation of (45) is, by (36), that

$$\beta k^* \leq k_T < k^*.$$

That is, the optimal contract has the property that in the last period in which the IC constraint is binding, the capital stock is 'close' to k^*.

The entrepreneur's problem is to maximize discounted utility, C, subject to the constraints, (i)-(v). The constraints are satisfied as long as we set $k_1 = k_1(T)$ and are consistent with (45). By construction of $k_1(T)$, we have $C(T) = \beta \theta k_1(T)$. Thus, the entrepreneur's problem reduces to that of solving

$$\max_T k_1(T), \tag{46}$$

subject to (45). Given that $k_1(T)$ is decreasing in T, we have a simple numerical algorithm for finding the optimal contract. First, verify that $k_1 = k^*$ is not incentive compatible (i.e., does not satisfy (26)). Then, consider $T = 1$. If (45) is violated then increase the value of T by unity and try againt. Continue in this way until (45) is satisfied.

6.2 Computing a Sequence of Consumptions and Debts

Once the preceding computations have been executed, we can look for a sequence, $c_T, c_{T+1}, ...,$ that is compatible with the budget constraint (i.e., C) and with incentive compatibility in periods $T+1, T+2, ...$. In fact, there is more than one consumption sequence that satisfies these two requirements.

A very natural consumption sequence, one in which $c_t = c$ for $t \geq T$, does *not* satisfy the incentive constraints. In this case,

$$c = \frac{1-\beta}{\beta^T} C = (1-\beta) \frac{\theta k_1}{\beta^{T-1}} = (1-\beta) \theta k_T = \theta (k_T - \beta k_T) > \theta (k_T - \beta k^*). \tag{47}$$

Here, the first equality ensures that c satisfies the budget constraint; the second equality uses the fact, $C = \beta \theta k_1$ (see (39)); and the third equality uses (36). The strict inequality in (55) uses the fact, $k_T < k^*$ (see (45)) and so a necessary condition for an optimal contract, (44) with

15

$l = 1$, is contradicted. Although this consumption sequence is budget feasible, it is not incentive compatible. With the constant level of consumption, c, the entrepreneur would abscond in period $T + 1$.

Consider a different consumption stream in which $c_T = 0$ and $c_t = c$ for $t \geq T + 1$. In this case,

$$c = \frac{1 - \beta}{\beta^{T+1}} C = \frac{1 - \beta}{\beta} \theta k_T. \tag{48}$$

Here, the first equality assures the budget feasibility of c and the second equality corresponds to (36) and (39). It is trivial to use (48) to verify that the period T participation constraint, (34), is satisfied as a strict equality. Now consider the incentive constraints for periods $t \geq T + 1$. Similarly, (48) can be used to verify that the periods $t \geq T+1$ participation constraints, (35), are satisfied as weak inequalities because of the first inequality in (45). If it is the case that the first inequality in (45) is strict, then the periods $t \geq T + 1$ participation constraints are also satisfied as strict inequalities. We refer to this sequence of consumptions as our 'baseline sequence'.

We now consider an alternative sequence of consumptions (the 'alternative sequence') that satisfy the budget constraint and participation constraints. Let c_t, $t = T, T + 1, ...$ denote the sequence of consumptions defined by (44) evaluated at equality for $l = 1, ...$:

$$c_t = \begin{cases} \theta \left(k_T - \beta k^* \right) & t = T \\ \theta \left(1 - \beta \right) k^* & t > T \end{cases}. \tag{49}$$

The alternative consumption sequence coincides with our baseline sequence in the special case where model parameter values imply that $k_T = \beta k^*$, in which case $c_T = 0$. Of course, this case has measure zero in the space of parameters, so that typically, $k_T > \beta k^*$ and the baseline and alternative consumption sequences differ.

We confirm that the sequence of c_t's defined by (49) enforce (44) as a strict equality for $l > 1$ by noting (i) that the c_t's which solve (44) with equality strict is unique and (ii) that (49) satisfies (44) as a strict equality for each $l > 1$. To see (ii), note

$$c_T + \beta c_{T+1} + ... + \beta^{l-1} c_{T+l-1}$$
$$= \theta \left(k_T - \beta k^* \right) + \beta \frac{1 - \beta^{l-1}}{1 - \beta} \theta \left(1 - \beta \right) k^*$$
$$= \theta \left(k_T - \beta^l k^* \right).$$

It remains to confirm that the c_t's defined by (49) satisfy the budget constraint. In particular,

$$\beta^T \left[c_T + \beta c_{T+1} + ... \right]$$
$$= \beta^T \left[\theta \left(k_T - \beta k^* \right) + \left(\beta + \beta^2 + ... \right) \theta \left(1 - \beta \right) k^* \right]$$
$$= \beta^T \theta k_T$$
$$= \theta \beta k_1,$$

by (36). The result then follows by (39).

Given a choice for the sequence of consumptions and capitals, we compute a sequence of debts as follows:

$$B_1 = k_1,$$

$$B_{t+1} = \overbrace{k_{t+1} - (1 - \delta) k_t + c_t + R B_t}^{\text{expenditures}} - \overbrace{A F(k_t)}^{\text{income}},$$

for $t \geq 1$.

6.3 A Numerical Example

We first consider the set of parameters values,

$$\alpha = 0.3, \ \beta = 0.97, \ \delta = 0.10,$$

and we set A so that $k^* = 1$. From (26) and (27),

$$\frac{(1-\alpha)}{(1-\beta)}\frac{\frac{1}{\beta} - (1-\delta)}{\alpha} = 10.18, \tag{50}$$

after rounding. Thus, θ would have to exceed 10.18 for the period 1 incentive constraint to be binding. Of course, $\theta > 1$ has no interpretation. So, we conclude that for these parameter values, the first best allocations are incentive compatible for any $0 \le \theta \le 1$. Equation (50) indicates that with a sufficiently high value of α, the first best allocations will not be incentive compatible for some $0 < \theta \le 1$.

With these considerations in mind, we set $\alpha = 0.90$ and $A = 0.1455$, so that $k^* = 1$ and the object on the left of the equality in (50) is 0.48. We set $\theta = 0.49$, so that the first best allocations are not incentive compatible. We found that the best incentive compatible allocations (i.e., the ones that solve (46)) have the property, $T = 1$. If the bank were to lend $B_1 = 1$ to the entrepreneur in period 0, then at the start of period 1 the entrepreneur would take $\theta k^* = 0.49$ and renege on her debt to the bank. In the best incentive compatible contract, the entrepreneur borrows $B_1 = 0.9896$ and acquires $k_1 = 0.9896$ units of capital. In periods $2, 3, \dots$, $k_t = 1$, so that output is simply equal to A in each of these periods. As discussed in the previous subsection, consumption in periods $t = T, T+1, \dots$ is not determined, though it is restricted by incentive compatibility and by the budget constraint. We display results only for the baseline consumption and debt plan. The details of the solution appear in the following table:

Solution to Dynamic Debt Contract Problem for Particular Parameter Values				
	Period, t			
	0	1	2	3
interest payments on debt, RB_t	0	1.0202	1.0159	1.0159
beginning of period t debt, B_t	0	0.9896	0.9855	0.9855
beginning of period t net worth, $k_t - B_t$	0	0	0.0145	0.0145
beginning of period t capital, k_t	0	0.9896	1	1
$I_t = k_{t+1} - (1-\delta)k_t$	0.9896	0.1093	0.10	0.10
c_t	0	0	0.0150	0.0150
C	0.4704			
output, Ak_t^α	0	0.1441	0.1455	0.1455
utility of deviating (running away), θk_t	NA	0.4849	0.490	0.490
utility of abiding by the contract	NA	$0.4849 \ (=\frac{\beta}{1-\beta}c)$	$0.4999 \ (=\frac{1}{1-\beta}c)$	0.4999
Note: parameter values - $\theta = 0.49$, $\alpha = 0.90$, $\beta = 0.97$, $\delta = 0.10$, $A = 0.1455$; NA - 'not applicable'				

Exercise 1. *Compute the alternative consumption and debt plan for the above example. Consider a version of the above example with one change, $\theta = 0.51$. Construct a table like the one above for this model. Report the baseline and alternative debt and consumption paths.*

We considered a version of the above model with $\theta = 0.7$. The results are reported in Figures 1-4. Each figure displays the properties of the solution for the baseline specification of the model, as well as for a perturbation. Consider Figure 1. The starred line displays the properties of the baseline model, in which $T = 13$. Note how the borrowing, B, of the entrepreneur rises over time,

and that this occurs while the entrepreneur's net worth, $k - B$, is also increasing. Another way to see this is that leverage, $k/(k - B)$, falls monotonically as the capital stock increases. Figure 1 also shows what happens to the solution when α is increased from 0.90 to 0.95. The value of k^* is increased substantially, and the value of T nearly doubles. Figure 2 displays the impact of reducing β from 0.97 in the baseline to 0.96. Not surprisingly, the reduction in β results in a fall in k^*. The amount of time required to grow away from the binding financial constraints, T, increases. Figure 3 shows the impact of raising θ. The value of k^* is not changed, but the amount of time needed to reach k^* increases because the higher value of θ makes the financial constraint more binding. Figure 4 compares the baseline and alternative paths for consumption. They are in fact quite similar. Consumption rises in period $T = 13$ in the alternative path, and as a result, it settles to a smaller number after period T. The difference in the debt is very small.

7   Appendix: Properties of Budget Constraints

In this appendix, we establish the results summarized in section 3. To establish the equivalence of the present discounted value budget constraint (7) and the flow budget constraints, (3), (5) and (6), we must first establish the boundedness of various objects in the model. We do this in three lemmas, before turning to our main proposition. The first lemma is the following:

Lemma 1. *Suppose* $\{c_t, k_t, B_t\}_{t=1}^{\infty}$ *satisfies (3), (5) and (6) and that* β, δ, F *satisfy (1), (2) and (4). Then* k_1, B_1 *are bounded above.*

Proof. Rearranging (5) for $t = 1$:

$$RB_1 - [AF(k_1) + (1 - \delta)k_1] + c_1 \leq B_2 - k_2,$$

or, because $c_2 \geq 0$,

$$RB_1 - [AF(k_1) + (1 - \delta)k_1] \leq B_2 - k_2. \tag{51}$$

For $t = 2$, after rearranging and using $\beta = R^{-1}$:

$$B_2 \leq \beta[AF(k_2) + (1 - \delta)k_2] + \beta B_3 - \beta k_3. \tag{52}$$

Using (52) to substitute out for B_2 in (51), we obtain:

$$RB_1 - [AF(k_1) + (1 - \delta)k_1] \leq \beta[AF(k_2) + (1 - \delta)k_2] - k_2 + \beta B_3 - \beta k_3. \tag{53}$$

Evaluating (52) for $t = 3$:

$$B_3 \leq \beta[AF(k_3) + (1 - \delta)k_3] + \beta B_4 - \beta k_4.$$

Using this to substitute out for B_3 in (53),

$$RB_1 - [AF(k_1) + (1 - \delta)k_1] \quad \leq \quad \beta[AF(k_2) + (1 - \delta)k_2] - k_2$$
$$+ \beta\left(\beta[AF(k_3) + (1 - \delta)k_3] - k_3\right) + \beta^2 B_4 - \beta^2 k_4$$

Continuing in this way,

$$RB_1 - [AF(k_1) + (1 - \delta)k_1] \quad \leq \quad \sum_{j=2}^{N} \beta^{j-2}\left(\beta[AF(k_j) + (1 - \delta)k_j] - k_j\right) + \beta^{N-1}B_{N+1} - \beta^{N-1}k_{N+1}$$

$$\leq \quad \sum_{j=2}^{N} \beta^{j-2}\left(\beta[AF(k_j) + (1 - \delta)k_j] - k_j\right) + \beta^{N-1}B_{N+1},$$

18

since $k_{N+1} \geq 0$. Under our assumptions on F and δ, M in

$$M = \max_k \{\beta [AF(k) + (1 - \delta) k] - k\} > 0, \tag{54}$$

and the value of k, k^*, that attains the maximum exists and is finite.[11] Thus,

$$
\begin{aligned}
RB_1 - [AF(k_1) + (1 - \delta) k_1] &\leq \sum_{j=2}^{N} \beta^{j-2} \max_{k_j} (\beta [AF(k_j) + (1 - \delta) k_j] - k_j) + \beta^{N-1} B_{N+1} \\
&= \frac{1 - \beta^{N-1}}{1 - \beta} M + \beta^{N-1} B_{N+1}
\end{aligned}
$$

Driving $N \to \infty$ and making use of (6):

$$RB_1 - [AF(k_1) + (1 - \delta) k_1] \leq \frac{1}{1 - \beta} M$$

From (3), $k_1 \leq B_1$. Also, the object in square brackets to the left of the inequality is strictly increasing in k_1. Thus,

$$RB_1 - [AF(B_1) + (1 - \delta) B_1] \leq \frac{1}{1 - \beta} M. \tag{55}$$

Note that the derivative of the left side with respect to B_1 is $R - AF'(B_1) - (1 - \delta)$. The object on the left of (55) is zero at $B_1 = 0$, slopes down for small B_1 and is eventually monotonically increasing with slope $R - (1 - \delta) > 0$. So, there is exactly one value of B_1 such that equality in (55) is attained. We denote this value of B_1, $\bar{B}_1$:

$$R\bar{B}_1 - [AF(\bar{B}_1) + (1 - \delta) \bar{B}_1] = \frac{1}{1 - \beta} M.$$

We conclude that (3), (5) and (6) imply:

$$k_1 \leq B_1 \leq \bar{B}_1. \tag{56}$$

$\square$

Our second lemma is as follows:

Lemma 2. *Under assumptions (3), (5) and (6),*

$$\sum_{t=1}^{N} \beta^t [k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) - B_{t+1} + RB_t]$$

converges to a finite number as $N \to \infty$.

Proof. Proof: let c_N denote the following partial sum:

$$c_N = \sum_{t=1}^{N} \beta^t [AF(k_t) + (1 - \delta) k_t - k_{t+1} - c_t + B_{t+1} - RB_t]. \tag{57}$$

---

[11] Note that in the graph of $\beta [AF(k) + (1 - \delta) k]$ versus k, the slope of the curve is $\beta AF'(k) + \beta (1 - \delta) \to \beta (1 - \delta) < 1$ as $k \to \infty$. The slope $\to \infty$ as $k \to 0$. Thus, the curve, $\beta [AF(k) + (1 - \delta) k]$, rises above the 45 degree line (i.e., the graph of k against k) for small k and eventually cuts the 45 degree line from above at a point, say $\bar{k}$. It is easy to see that $0 < k^* < \bar{k}$. The object, k^* solves $\beta AF'(k) + \beta (1 - \delta) = 1$;

Rearranging the terms in (57) and using $\beta R = 1$, we obtain:

$$
\begin{aligned}
c_N \;=\;& \beta^N B_{N+1} + \beta \left(AF(k_1) + (1-\delta)\, k_1 \right) - \beta c_1 \\
&+ \sum_{t=2}^{N} \beta^{t-1} \left[\beta \left(AF(k_t) + (1-\delta)\, k_t \right) - k_t - \beta c_t \right] - B_1 - \beta^N k_{t+N}.
\end{aligned}
$$

Taking into account the non-negativity of B_1 and k_{t+N},

$$
\begin{aligned}
c_N \;\leq\;& \beta^N B_{N+1} + \beta \left(AF(k_1) + (1-\delta)\, k_1 \right) + \sum_{t=2}^{N} \beta^{t-1} \left[\beta \left(AF(k_t) + (1-\delta)\, k_t \right) - k_t \right] \\
\leq\;& \beta^N B_{N+1} + \beta \left(AF(\bar{B}_1) + (1-\delta)\, \bar{B}_1 \right) + \frac{M}{1-\beta}.
\end{aligned}
$$

The second inequality makes use of (54) and (56).

Equation (6) implies that for each fixed $\varepsilon > 0$ there exists an $\bar{N}$ such that for all $N > \bar{N}$,

$$
\beta^N B_{N+1} \leq \varepsilon.
$$

Thus, for all $N > \bar{N}$,

$$
c_N \leq \varepsilon + \beta \left(AF(\bar{B}_1) + (1-\delta)\, \bar{B}_1 \right) + \frac{M}{1-\beta}.
$$

Because the object in square brackets in (57) is non-negative for each t (see equation (5)), we have that

$$
c_1 \leq c_2 \leq \;....\; \leq c_N.
$$

It follows that each element in the sequence, $\{c_N\}$, is bounded. Since $\{c_N\}$ is bounded and non-decreasing, we can infer that

$$
\lim_{N \to \infty} c_N
$$

exists, i.e., it is finite. $\qquad\square$

The third lemma is:

Lemma 3. *Suppose $\{c_t, k_t, B_t\}_{t=1}^{\infty}$ satisfies (3), (5) and (6) and that β, δ, F satisfy (1), (2) and (4). Then,*

$$
\sum_{t=1}^{N} \beta^t \left[AF(k_t) + (1-\delta)\, k_t - c_t - k_{t+1} \right]
$$

converges to a finite number as $N \to \infty$.

Proof. Let a_N be defined as follows:

$$
a_N \equiv \sum_{t=1}^{N} \beta^t \left[AF(k_t) + (1-\delta)\, k_t - c_t - k_{t+1} \right]. \tag{58}
$$

Then,

$$
a_N = b_N + c_N,
$$

where

$$
b_N = -\sum_{t=1}^{N} \beta^t \left(B_{t+1} - R B_t \right) = R B_1 - \beta^N B_{N+1},
$$

where we have used $R\beta = 1$ and c_N is defined in (57). The previous lemma established the convergence of a_N. The convergence of b_N is immediate because we assume in (6) that $\beta^N B_{N+1}$ converges to a specific number. $\qquad\square$

Our central proposition is:

Proposition 4. *The entrepreneur's intertemporal consumption and capital accumulation opportunities implied by the flow budget constraints, (3), (5), and the borrowing limit, (6), are equivalent to the consumption and capital opportunities implied by the following single budget constraint:*

$$k_1 \leq \sum_{t=1}^{\infty} \beta^t \left[AF(k_t) + (1 - \delta) k_t - c_t - k_{t+1} \right].$$ (59)

The proposition has a straightforward interpretation. The budget constraint, (7), prevents the entrepreneur from borrowing more in period 0 to buy k_1 than can be repaid by the discounted future surpluses of production over expenditures, $c_t + k_{t+1}$.

Proof. We establish the above proposition in two steps. We first show that if there is a sequence of k's and c's and B's that satisfy the flow budget constraints and debt limit, then those k's and c's satisfy the single budget constraint, (59). Consider the weighted sum of the flow budget constraints:

$$\begin{aligned}
0 \geq\ & k_1 - B_1 + \lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) - B_{t+1} + RB_t \right] \\
=\ & k_1 - B_1 + \beta \left[k_2 - (1 - \delta) k_1 + c_1 - AF(k_1) - B_2 + RB_1 \right] \\
& + \beta^2 \left[k_3 - (1 - \delta) k_2 + c_2 - AF(k_2) - B_3 + RB_2 \right] \\
& + \beta^3 \left[k_4 - (1 - \delta) k_3 + c_3 - AF(k_3) - B_4 + RB_3 \right] + ... \\
=\ & k_1 - B_1 + \beta RB_1 + \lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} + c_t - (AF(k_t) + (1 - \delta) k_t) \right] - \lim_{N \to \infty} \beta^N B_{N+1} \\
\geq\ & k_1 + \lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} + c_t - (AF(k_t) + (1 - \delta) k_t) \right].
\end{aligned}$$ (60)

The limits in (60) are well defined by the preceding lemmas. The first inequality (60) reflects the sign of each item in the sum. The second equality reflects the cancellation of B_t with βRB_t when $\beta R = 1$, for $t = 2, 3, ..., N - 1$. The last inequality reflects $RB = 1$ and the sign restriction on $\lim_{N \to \infty} \beta^N B_{N+1}$. We conclude that if (5) and (6) are satisfied, then (59) is satisfied.

Now we consider the reverse result. In particular, suppose we have a sequence of k's and c's that satisfy (59). Then, a sequence of B's can be found that satisfy the debt limit, (6), and that has the property that together with the given k's and c's, the flow budget constraints, (5), are satisfied. Our strategy is to select a particular sequence of B's that satisfy the flow budget constraints and then verify that that sequence satisfies the debt limit. Set B_1 as follows:

$$B_1 = \sum_{t=1}^{\infty} \beta^t \left[(1 - \delta) k_t + AF(k_t) - k_{t+1} - c_t \right],$$

an object that is well defined by an earlier lemma. By (59), the period 0 flow budget constraint, (3), is satisfied. Next, set B_{t+1} so that the flow budget constraint is satisfied for each $t \geq 1$ as a strict equality. That is,

$$k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) + RB_t - B_{t+1} = 0,$$

for $t = 1, 2, ...$. Multiply the above expression by β^t and add over all t :

$$\lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) + RB_t - B_{t+1} \right] = 0,$$

because each term in the sum is, by construction, equal to zero. Rearranging terms, we obtain, for fixed N :

$$\sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) + RB_t - B_{t+1} \right]$$

$$= \sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) \right] + B_1 - \beta^N B_{N+1},$$

where $R\beta = 1$ has been used. Then,

$$0 = \lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) + RB_t - B_{t+1} \right]$$

$$= \lim_{N \to \infty} \sum_{t=1}^{N} \beta^t \left[k_{t+1} - (1 - \delta) k_t + c_t - AF(k_t) \right] + B_1 - \lim_{N \to \infty} \beta^N B_{N+1}$$

$$= - \lim_{N \to \infty} \beta^N B_{N+1},$$

by construction of B_1. We conclude that the chosen sequence of B's satisfies the debt limit. Since we have already verified that $k_1 \leq B_1$, it follows that we have found a sequence of B's with the property that the sequence of budget constraints and the debt limit are satisfied. Our result has now been established. $\qquad \square$

Figure 1: Baseline (*) and Perturbation on Alpha
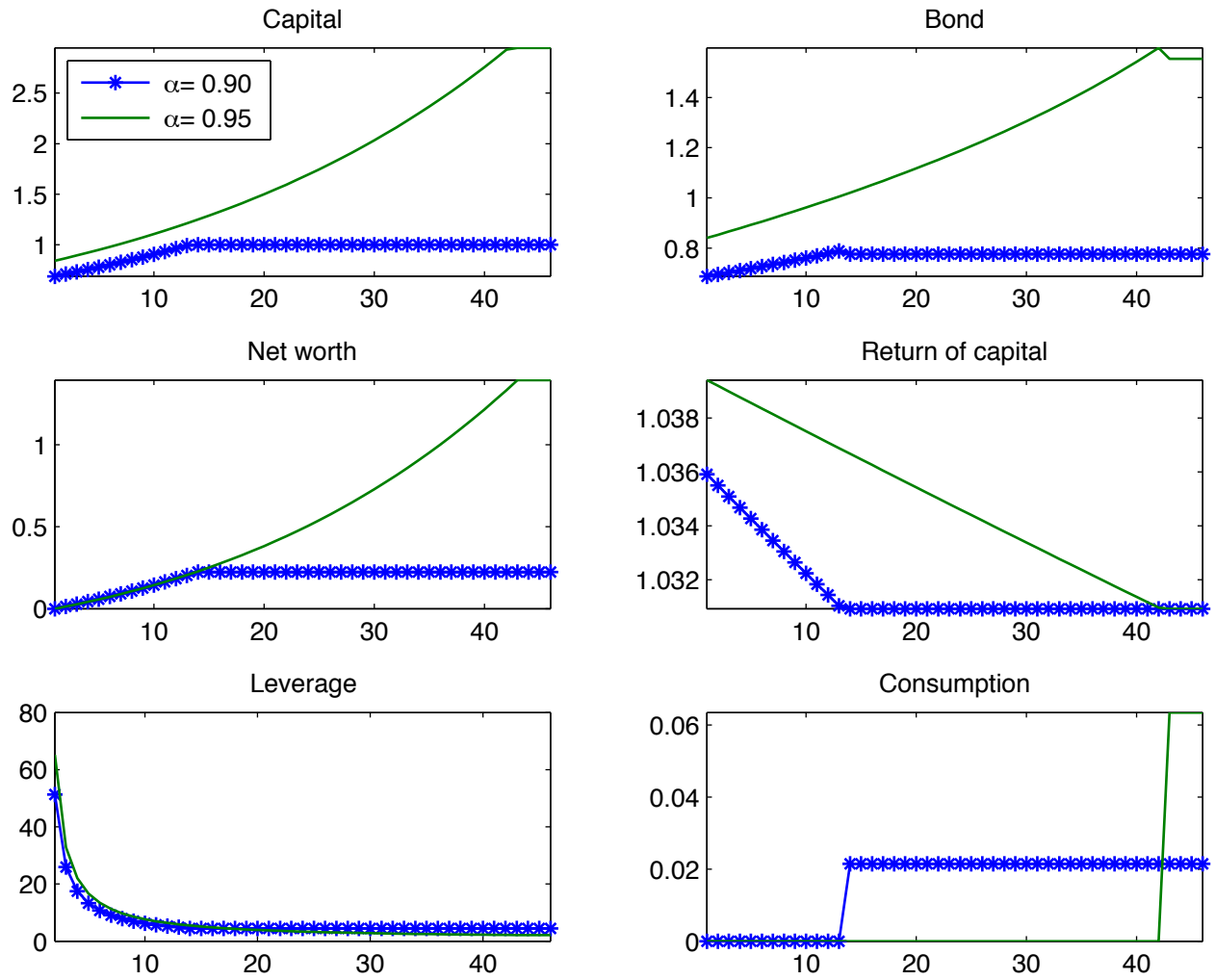
Figure 2: Baseline (*) and Perturbation on Beta
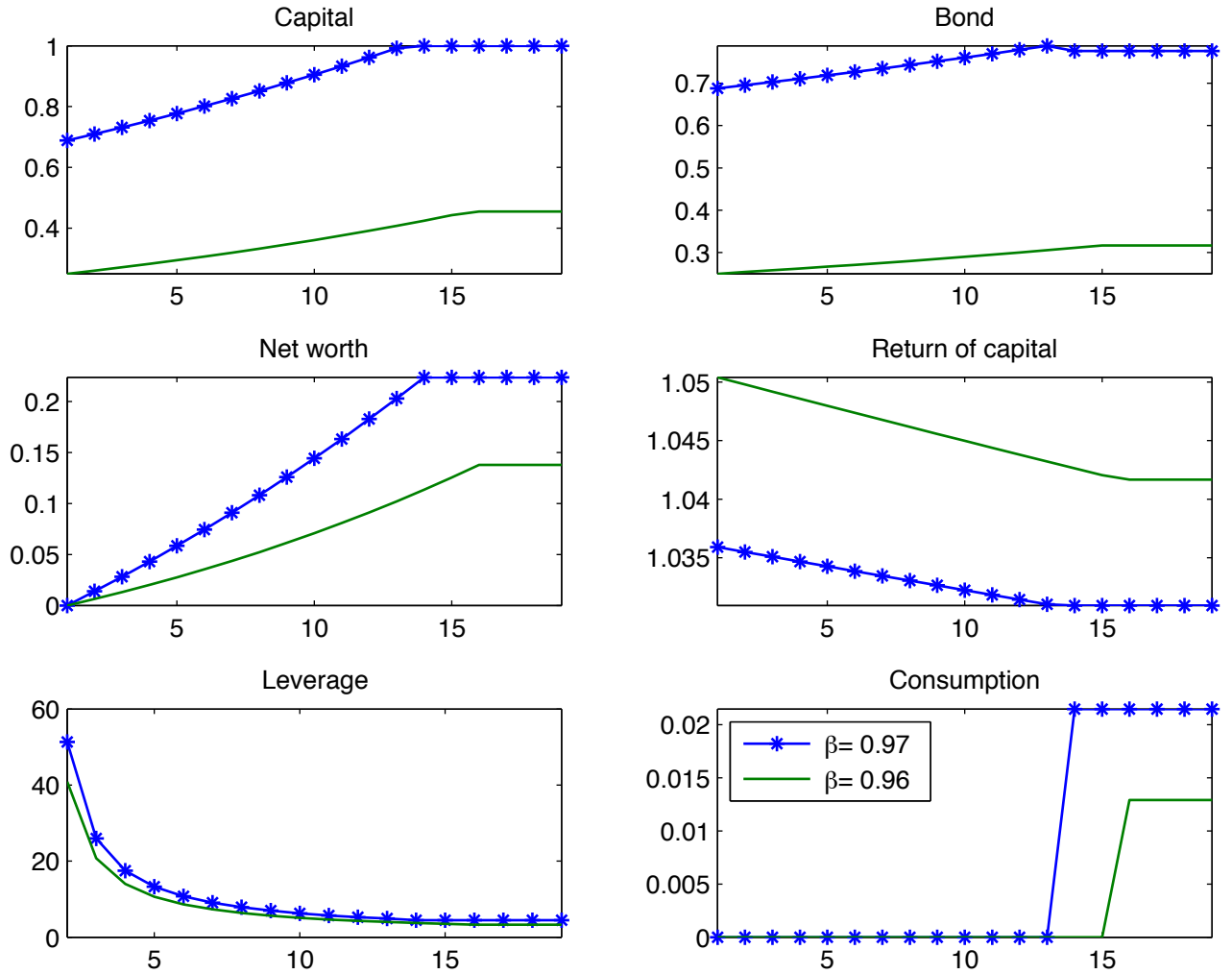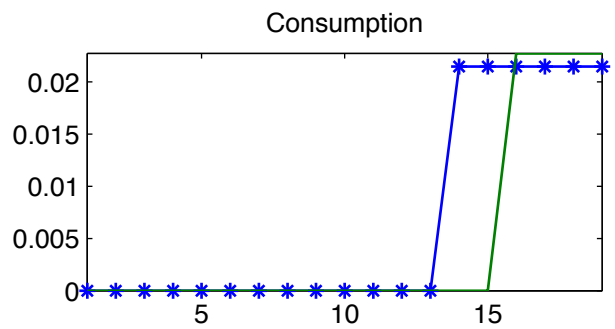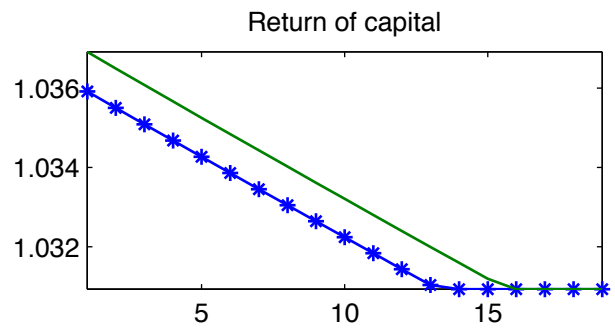
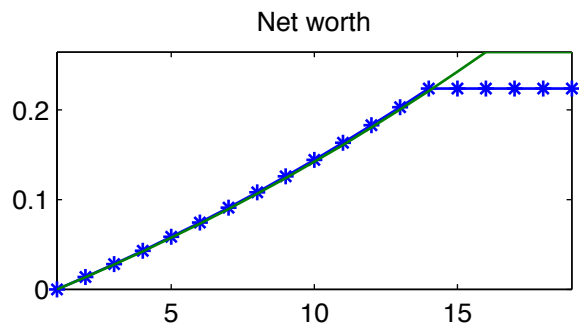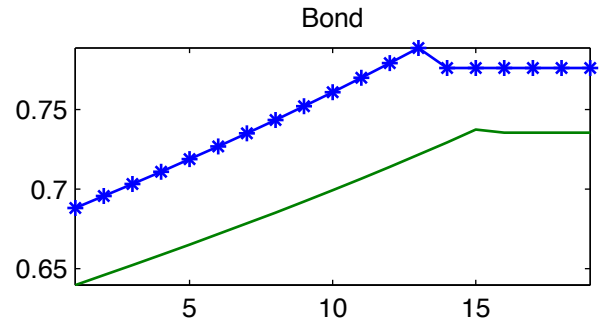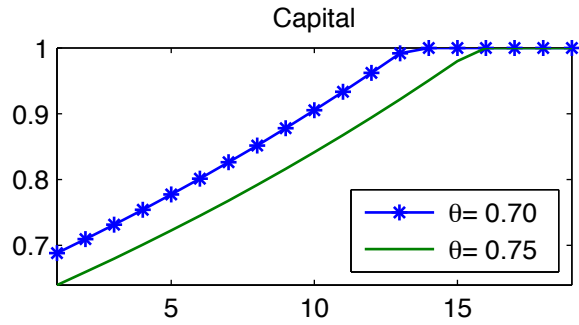Figure 3: Baseline (*) and Perturbation on Theta
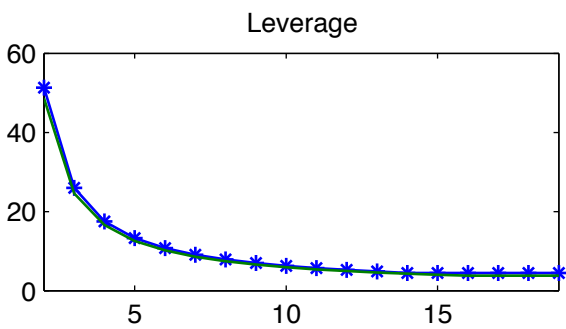
Figure 4: Baseline (*) and Alternative Consumption and Debt Paths, Baseline Model Parameter Values