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Advanced Macro
416, Fall, 2013
Homework 2.

1. Consider the one-sector stochastic neoclassical growth model discussed in class. Consider the following model parameterization:

   (a) Compute the coefficients in the linear, second order perturbation on the policy rule, for several parameterizations of the model. Do the approximation in terms of the capital stock itself, rather than in terms of the log of capital, as in the handout. In all cases,

   $$\alpha = 0.36,\ \beta = 0.99,\ \delta = 0.02,\ \rho = 0.95.$$

   Solve the model for the following four cases: $\gamma = 2$, 20 and for $Var\left(\varepsilon_t\right) = 1$ and 0.01^2. Report the coefficients of the approximate policy rules and store them for analysis below.

   (b) One way to quantify the difference between first and second order perturbations, is to investigate their implications for the speed of adjustment. Although shocks potentially play an interesting role in this, let us abstract from the shocks at this point. Thus, the policy rule is:

   $$k_{t+1} - k = g_k \times (k_t - k) + \frac{1}{2} g_{kk} \times (k_t - k)^2,$$

   where k is the value of the capital stock in non-stochastic steady state (it is not the log of the capital stock, as in lecture). In percent terms:

   $$\hat{k}_t = \frac{k_t - k^s}{k^s}.$$

   Dividing the second order policy rule by k^s :

   $$\hat{k}_{t+1} = g_k \hat{k}_t + \frac{1}{2} g_{kk} \times k \times \left(\hat{k}_t\right)^2. \quad (1)$$

   Suppose that at $t = 1$, $k_1 = \lambda \times k$, so that $\hat{k}_1 = \lambda - 1$. Consider first the standard, first order perturbation, in which g_{kk} is set to zero. Then, $\hat{k}_{t+1} = g_k^t (\lambda - 1)$, $t = 0, 1, ...$. The time required to close 95 percent of the gap between the capital stock and steady state is the value of t such that

$$\hat{k}_{t+1} = (\lambda - 1) \times 0.05,$$

or,

$$\begin{aligned} g_k^t &= 0.05 \\ t &= \frac{\log(0.05)}{\log g_k}. \end{aligned}$$

Notice that t is independent of the value of λ. Compute the value of t for our model economy.

Now, consider the implications of the second-order expansion for speed of adjustment. Set $\hat{k}_1 = \lambda - 1$, and simulate (1) for $t = 1, ..., 100$. Set λ to a very small number, $\lambda = 0.0001$ and put in the large value of γ. Graph the two trajectories, one for the first order approximation and the other for the second order approximation. Are they very similar?

2. (Simulating a rule obtained by second-order perturbation.) Consider the following expression, which we can think of as the second order approximation of some unspecified law of motion:

$$y_t = f(y_{t-1}, \varepsilon_t) = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t, \ 0 < \rho < 1,$$

where ε_t is a mean zero, *iid* process, uncorrelated with past y_t. Note that there are two solutions, $y = f(y, 0) : y = 0, y = (1-\rho)/\alpha$. The first steady state is locally stable, since the root in its first order expansion is ρ. The second is not. So, if $y_t > (1-\rho)/\alpha$, then y_t is likely to explode. Verify this by simulating 100 observations of y_t with $\varepsilon_t \tilde{} N(0, \sigma_\varepsilon)$ and $\sigma_\varepsilon = 0.01$, 0.10. The problem illustrated here is something that can happen with policy rules that are quadratic expansions: "...they will have extra steady states not present in the original model, and some of these steady states are likely to mark transitions to unstable behavior".[1]

[1] Taken from page 17, Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher Sims, February 3, 2005, 'Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models,' manuscript.

Kim, Kim, Schaumburg and Sims (2005, section 7) argue that the system should be simulated by 'pruning', as follows. Let the first order system be $y_t^{(1)}$:

$$y_t^{(1)} = \rho y_{t-1}^{(1)} + \varepsilon_t.$$

The pruned simulated output is $y_t^{(2)}$, where $y_t^{(2)}$ solves the original system with the exception that wherever y_{t-1} appears in quadratic terms, it is replaced by $y_t^{(1)}$:

$$y_t^{(2)} = \rho y_{t-1}^{(2)} + \alpha \left[y_{t-1}^{(1)} \right]^2 + \varepsilon_t.$$

Construct two graphs for the case, $\sigma_\varepsilon = .01$. Display y_t and $y_t^{(2)}$ in one graph. Display $y_t^{(1)}$ and $y_t^{(2)}$ in the other. Note how pruning eliminates the explosive behavior. See Kim, Kim, Schaumburg and Sims for additional discussion.

3. Return to the policy rule in question 1a with $\gamma = 20$ and $Var(\varepsilon_t) = 1$. Simulate 10,000 observations using the linear approximate rule, the quadratic rule (e.g., 'naive simulation'), and the quadratic rule based on pruning (all three should be simulated in response to the same shocks). Display the three sequences of capital stocks in a graph (truncate super large values if you have to). How do mean value of k_t and its variance compare for the three rules? How does the mean value of k_t compare with its non-stochastic steady state value? Repeat this exercise using the more reasonable estimate of the variance, $V_\varepsilon = 0.01^2$. Finally, consider the same experiment with the more 'normal' risk aversion: $\gamma = 2$ (set $V_\varepsilon = 0.01^2$). Note that in the latter case, there is very little difference between the three solutions.

4. (Orthogonality property of Chebyshev polynomials). Consider the Chebyshev polynomial:

$$\begin{aligned} T_0(r) &= 1 \\ T_1(r) &= r \\ T_n(r) &= 2rT_{n-1}(r) - T_{n-2}(r), \end{aligned}$$

for $n = 2, 3, 3, ...$ Let $r_1, ..., r_N$ denote the zeros of the N^{th} order Chebyshev polynomial, $T_N(r)$:

$$r_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{N}\right), \ k = 1, ..., N.$$

The Chebyshev polynomial has the following discrete orthogonality property:

$$\sum_{k=1}^{N} T_i(r_k) T_j(r_k) = \begin{cases} 0 & i \neq j \\ N & i = j = 0 \\ \frac{N}{2} & i = j \neq 0 \end{cases}.$$

The way you did $N-$point Chebyshev interpolation in the previous homework is that you first computed the zeros of the $N-$dimensional Chebyshev polynomial, $r = [r_1, ..., r_N]'$. These map into the $N \times 1$ set of points, x, belonging to the domain of the Runge function via the transformation:

$$x = \frac{(b-a)(r+1)}{2} + a,$$

where $a = -5$, $b = 5$. Then, you constructed a matrix

$$X = \left[\begin{array}{ccc} T_0(r) & \cdots & T_{N-1}(r) \end{array}\right],$$

where $T_j(r)$ is the $N-$dimensional column vector formed by evaluating the j^{th} order Chebyshev polynomial at each of the elements of the column vector, r. Finally, you form the N dimensional column vector, $f(x)$, where f denotes the Runge function evaluated at each of the elements in x. Then the interpolation equation to be solved is given by:

$$X\gamma = f(x).$$

Premultiply by X' :

$$X'X\gamma = X'f(x)$$

Note from the discrete orthogonality property, that $X'X$ is a diagonal matrix with $\frac{N}{2}$ in all but one entry on the diagonal. In the 1,1 element of $X'X$ there appears N. To find γ simply invert the $X'X$ matrix:

$$\gamma = (X'X)^{-1} X'f(x),$$

or

$$\gamma_k = \begin{cases} \frac{T_0(r)'f(x)}{N} & k=0 \\ 2\frac{T_k(r)'f(x)}{N} & k=1,...,N-1 \end{cases}. \quad (2)$$

Notice the relatively trivial nature of this formula for the γ's. For example, it is no problem to make N *very* large. This stands in sharp contrast with the fixed interval interpolation that was done (and which works badly!) in the previous homework. In that case, for N large you have to invert a matrix, X, with columns of numbers that are of very different orders of magnitude, with negative implications for numerical accuracy.

We conclude that the $N-1^{th}$ order Chebyshev interpolating function for $f(x)$ is

$$f(x) \simeq \sum_{k=0}^{N-1} \gamma_k T_k(\varphi(x)), \quad (3)$$

where φ was defined in the previous homework. Verify that the γ's you computed in the previous homework are also the solution to the above formula.

5. (Using the Chebyshev interpolation theorem approximate an integral). The Chebyshev interpolation theorem tells us that for large enough N, (3) provides an excellent (in the sup norm sense) approximation to virtually *any* function, f (some restrictions, such as continuity, are required for f). And, as noted above, the Chebyshev orthogonality property implies that making N as large as you want creates no numerical problems. Thus, we have

$$\int_a^b f(x)\,dx \simeq \sum_{k=0}^{N-1} \gamma_k \int_{-1}^{1} T_k(r)\,dr.$$

Since $T_k(r)$ is a polynomial, it is easy to integrate. The formula for this is:

$$\int_{-1}^{1} T_k(r)\,dr = \left[\frac{kT_{k+1}(r)}{k^2-1} - \frac{rT_k(r)}{k-1}\right]_{r=-1}^{r=1}$$

This method for approximating an integral is called Chebyshev Quadrature integration (the Chebyshev quadrature integration formula is usually expressed in a simplified form). Consider the integral,

$$\int_0^1 e^{3x} dx.$$

The exact value of this integral is easy to work out analytically. Approximate it using Chebyshev quadrature, programming everything yourself (there is software all over the web for doing this, but it is most instructive to do it yourself, at least one time). Try it for a small value of N (say, $N = 2$), and determine what value of N you need to go to, to get a decent approximation.