Advanced Macroeconomics,

Econ 416

Homework #3

Solving and Analyzing a Model with Two Lucas Trees

This reproduces the calculations studied in Cochrane, Longstaff and Santa-Clara ('Two Trees', The Review of Financial Studies, vol. 21 no. 1, 2008) and Ian Martin, 'The Lucas Orchard,' Econometrica, January 2013 (see especially Figure 3).

Consider an economy with two trees, tree number 1 and tree number 2. The quantity of fruit that falls from tree i in period t is denoted $D_{i,t}$, $i = 1, 2$. The time $t+1$ growth rate of the amount of fruit falling off tree number i is denoted $\varepsilon_{i,t+1}$, $i = 1, 2$:

$$\frac{D_{1,t+1}}{D_{1,t}} = \varepsilon_{1,t+1},\ \frac{D_{2,t+1}}{D_{2,t}} = \varepsilon_{2,t+1},$$

where ε_{it} is serially uncorrelated and ε_{1t} and ε_{2t} are uncorrelated with each other.

The economy is populated by many identical households. The first date is $t = 0$ when the quantity of trees of type i owned by the representative household, $z_{i,0}$, is given, $i = 1, 2$. The household chooses $\{z_{1,t}, z_{2,t}\}_{t=1}^{\infty}$ to maximize discounted utility,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t),\ u(C) = \frac{C^{1-\gamma}}{1-\gamma},$$

subject to

$$C_t + p_{1,t}(z_{1,t+1} - z_{1,t}) + p_{2,t}(z_{2,t+1} - z_{2,t}) \leq z_{1,t}D_{1,t} + z_{2,t}D_{2,t}.$$

Here, $p_{i,t}$ denotes the price of tree i, $i = 1, 2$. You can imagine that the household chooses state contingent sequences, $\{z_{1,t}, z_{2,t}\}_{t=1}^{\infty}$, at date 0. Alternatively, you can imagine that the household chooses $z_{i,t+1}$ in each period t given $z_{i,t}$ and the period t shocks. The results are the same.

In equilibrium, the quantity of trees purchased must be equal to the outstanding stock. The outstanding stock of trees of type 1 is α and the outstanding stock of trees of type 2 is $1 - \alpha$. This stock never changes. Thus, market clearing requires

$$z_{1,t} = \alpha,\ z_{2,t} = 1 - \alpha,\ \text{for all } t.$$

This is so, even though the individual atomistic household believes they can choose to buy as much or as little trees as it wishes, subject to its budget constraint.

Goods market clearing requires that total consumption, C_t, is equal to the total quantity of fruit dropped from both trees:

$$C_t = \alpha D_{1,t} + (1-\alpha) D_{2,t}.$$

The type of fruit falling from each tree is identical in the sense that one unit of fruit from one tree contributes to utility in the same way as one unit of fruit from the other tree. The optimal choice of $z_{1,t+1}$ by the representative household leads to the following condition:

$$p_{1,t} = \beta E \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \left[D_{1,t+1} + p_{1,t+1} \right].$$

It is convenient to rewrite the previous expression. Thus, consider the (weighted) price consumption ratio:

$$\begin{aligned} P_{1,t} &\equiv \frac{\alpha p_{1,t}}{C_t}, \\ P_{2,t} &\equiv \frac{(1-\alpha) p_{2,t}}{C_t}. \end{aligned}$$

Then,

$$\begin{aligned} P_{1,t} &= \beta E \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \left[\frac{\alpha D_{1,t+1}}{C_{t+1}} + P_{1,t+1} \right] \frac{C_{t+1}}{C_t} \\ &= \beta E \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left[x_{t+1} + P_{1,t+1} \right], \end{aligned}$$

where x_t denotes the share in consumption of earnings from tree 1 :

$$x_t \equiv \frac{\alpha D_{1,t}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}}.$$

Also,

$$\begin{aligned} P_{2,t} &= \beta E \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \left[\frac{(1-\alpha) D_{2,t+1}}{C_{t+1}} + P_{2,t+1} \right] \frac{C_{t+1}}{C_t} \\ &= \beta E \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left[1 - x_{t+1} + P_{2,t+1} \right], \end{aligned}$$

1. Show that

$$\begin{aligned}\frac{C_{t+1}}{C_t} &= x_t\varepsilon_{1,t+1} + (1-x_t)\,\varepsilon_{2,t+1} \\ x_{t+1} &= \frac{\varepsilon_{1,t+1}x_t}{x_t\varepsilon_{1,t+1} + (1-x_t)\,\varepsilon_{2,t+1}}.\end{aligned}$$

Note also that:

$$\begin{aligned}\frac{p_{1,t}}{D_{1,t}} &= \frac{\alpha p_{1,t}}{C_t}\frac{C_t}{\alpha D_{1,t}} = \frac{P_{1,t}}{x_t} \\ \frac{p_{2,t}}{D_{2,t}} &= \frac{(1-\alpha)\,p_{2,t}}{C_t}\frac{C_t}{(1-\alpha)\,D_{2,t}} = \frac{P_{2,t}}{1-x_t}.\end{aligned}$$

Suppose the net growth rates of the two trees takes on three possible values as follows:

$$\varepsilon_1 - 1, \varepsilon_2 - 1 \in (\mu - \sigma, \mu, \mu + \sigma)\,.$$

Let the probability of the three states be 1/4, 2/4 and 1/4, respectively, for each tree. Also,

$$\sigma^2 = 0.06,\ \ \beta = 0.96,\ \ \mu = 0.02.$$

Let the $N = 9$ states be given by the 9 by 1 vector, s :

$$s = \begin{pmatrix} l,l \\ l,m \\ l,h \\ m,l \\ m,m \\ m,h \\ h,l \\ h,m \\ h,h \end{pmatrix}$$

Here, the variable before the ‘,’ corresponds to a realization of $\varepsilon_{1,t}$ and the variable after the ‘,’ corresponds to a realization of $\varepsilon_{2,t}$. The notation, l, m, h, denotes ‘low’, ‘medium’ and ‘high’, respectively, i.e., $\mu - \sigma$, μ, $\mu + \sigma$. Let the 9×9 matrix, π, denote the Markov transition matrix for the state.

2. What is the structure of π? Explain why the iid assumption across time for each shock

implies that each row of π is the same. What is the variance of $\varepsilon_{i,t}$, $i = 1, 2$?

3. Show that $P_2(x) = P_1(1-x)$. With this symmetry property there is no need compute both price functions, so we can work on $P_1(x)$.

The P_1 function satisfies the following expression:

$$P_1(x) - \beta \sum_{j=1}^{N} \pi_{ij} \left[x\varepsilon_1(j) + (1-x)\varepsilon_2(j) \right]^{1-\gamma} \left[x'(j) + P_1(x'(j)) \right] = 0$$

for all $0 \leq x \leq 1$, where

$$x'(j) = \frac{\varepsilon_1'(j)\, x}{x\varepsilon_1'(j) + (1-x)\varepsilon_2'(j)}. \tag{1}$$

Note in the above sum that the value of i is irrelevant, since all rows of π are identical.

One way to approximate the function, P_1, uses Chebyshev polynomials. The domain of P_1 is $[0,1]$, but the domain of the Chebyshev polynomial is $[-1,1]$. Thus, you require an invertible mapping,

$$\varphi : [0,1] \to [-1,1].$$

Let

$$T(x) = [T_0(\varphi(x)), T_1(\varphi(x)), ..., T_{M-1}(\varphi(x))]',$$

denote the set of Chebyshev polynomials of order 0, 1, ..., $M-1$. Let a denote the $M \times 1$ vectors of parameters. Then:

$$\hat{P}_1(x; a) = a'T(x).$$

The M zeros of the M^{th} order Chebyshev polynomial, T_M, are

$$r_j = \cos\left(\frac{\pi(j-0.5)}{M}\right), \; j = 1, \ldots, M,$$

and let

$$x_j = \varphi^{-1}(r_j) = \frac{r_j + 1}{2}, \; j = 1, ..., M. \tag{2}$$

4. Identify system of equations that is linear in the elements of a whose solution corresponds to the collocation method. Solve the equations. Graph the error function over a grid much finer than the grid of x's used in the collocation method, to check the accuracy of your solution. The relevant error function is:

$$E_1(x; a) = \hat{P}_1(x; a) - \beta \sum_{j=1}^{N} \pi_{ij} \left[x\varepsilon_1(j) + (1-x)\varepsilon_2(j) \right]^{1-\gamma} \left[x'(j) + \hat{P}_1(x'(j); a) \right].$$

How big does M have to be to generate an accurate solution for x over the range, $(0,1)$?

5. Redo part 4 using piecewise linear polynomials. Graph the error function. Which approach, Chebyshev or piecewise linear, is easier in the sense of requiring a lower level of M to achieve a given level of accuracy?

6. We now consider the return on investment in trees 1 and 2. Given the approximate solutions for the pricing functions, show that the return on investment in tree 1 is approximated by the following function:

$$\hat{R}_1(x,j;a) = \frac{\left[x'(j) + \hat{P}_1(x'(j);a)\right]}{\hat{P}_1(x;a)}\left[x\varepsilon_1' + (1-x)\varepsilon_2'\right],$$

where $x'(j)$ is given in (1). Derive the analogous approximation for the return on investment in tree 2.

Define the mean returns (conditional on the state, x) as follows:

$$\begin{aligned} M_1(x;a) &= \sum_{j=1}^{N} \pi_{ij}\hat{R}_1(x,j;a) \\ M_2(x;b) &= \sum_{j=1}^{N} \pi_{ij}\hat{R}_2(x,j;b). \end{aligned}$$

Finally,

$$\begin{aligned} Cov(x) &= \sum_{j=1}^{N} \pi_{i,j}\left[\hat{R}_1(x,j;a) - M_1(x;a)\right]\left[\hat{R}_2(x,j;a) - M_2(x;a)\right] \\ V_1(x) &= \sum_{j=1}^{N} \pi_{i,j}\left[\hat{R}_1(x,j;a) - M_1(x;a)\right]^2 \\ V_2(x) &= \sum_{j=1}^{N} \pi_{i,j}\left[\hat{R}_2(x,j;b) - M_2(x;b)\right]^2 \\ \rho(x) &= \frac{Cov(x)}{\sqrt{V_1(x)V_2(x)}}. \end{aligned}$$

Graph these objects as a function of x, using your best solutions using Chebyshev and piecewise linear approximations. Are your results consistent with the findings reported in the Martin and Cochrane-Longstaff-Santa Clara papers?