Econ 416
Christiano
Fall, 2014

Solutions to a Class of Linear Expectational Difference Equations

1. Introduction

These notes provide an informal characterization of the solution to a class of expectational difference equations.[1] I first consider a very simple example in order to illustrate the main results. I then turn to the more general case without uncertainty. After that I consider the case with uncertainty. The previous two sections assumed that a certain matrix is singular, an assumption that is not typically satisfied in practice. The next section shows how to address this problem using the QZ decomposition. The last two sections work with an example to illustrate the various points in detail.

2. A Simple Example to Illustrate Some of the Basic Points

The intertemporal Euler equation associated with the neoclassical growth model, after that equation has been linearized about steady state, has the following representation:

$$k_{t+2} - \phi k_{t+1} + \frac{1}{\beta} k_t = 0, \tag{2.1}$$

for $t = 0, 1, 2, ...$ Here, k_{t+1} denotes the value of the capital stock selected at time t, expressed in deviation from steady state. Also, $\bar{k}_0$ represents the deviation of the initial stock of capital from steady state. We assume $\bar{k}_0 \neq 0$. Also, $0 < \beta < 1$ and

$$\phi > 1 + \frac{1}{\beta}. \tag{2.2}$$

We refer to a sequence, $\{k_t\}_{t=1}^{\infty}$, which satisfies the above sequence of difference equations as well as the initial condition, as a *solution*. (The notion of a solution is slightly different here than it was in Econ411). It is easy to see how many solutions there are. Consider the sequence of equations, (2.1):

$$\begin{aligned} k_2 - \phi k_1 + \frac{1}{\beta} k_0 &= 0 \\ k_3 - \phi k_2 + \frac{1}{\beta} k_1 &= 0 \\ &... \end{aligned}$$

If we arbitrarily set a value for k_1, the first equation can be solved for k_2. Then the second equation can be solved for k_3 and in this way we can find an entire sequence, $\{k_t\}_{t=1}^{\infty}$. Since

[1] Much of this material is taken from section 3 in Christiano, 2002, "Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients," *Computational Economics*, 20, pp. 21-55.

this sequence is completely determined by the selected value of k_1, we conclude that the set of solutions, $\{k_t\}_{t=1}^{\infty}$, has one dimension, indexed by the value of k_1.

Unfortunately, this way of characterizing the set of solutions to (2.1) is not very convenient. There is little else one can say about that set. An alternative characterization is more convenient. It represents the solutions in the space of roots of the polynomial equation associated with (2.1):

$$f(\lambda) = \lambda^2 - \phi\lambda + \frac{1}{\beta} \tag{2.3}$$

Let λ_1 and λ_2 satisfy $f(\lambda_i) = 0$ for $i = 1, 2$. Under the assumption that λ_1 and λ_2 are distinct (a condition established in Stokey and Lucas' textbook), the complete set of solutions to (2.1) can be represented as follows:

$$k_t = \left(\bar{k}_0 - a\right)\lambda_1^t + a\lambda_2^t, \tag{2.4}$$

where a is arbitrary. It is easy to verify that (2.4) satisfies (2.1) and the initial condition for every possible value of a.

From (2.4) we see, like before, that the set of solutions to (2.1) is one-dimensional and is indexed by the scalar, a. There are two other characteristics of the space of solutions that we can we see from (2.4). First, there exist two solutions in which k_t solves a first order difference equation. The two solutions are the ones associated with $a = \bar{k}_0$ and $a = 0$, respectively, that is:

$$k_t = \bar{k}_0\lambda_2^t, \; k_t = \bar{k}_0\lambda_1^t.$$

In each case, divide the expression by itself evaluated at $t-1$, to obtain:

$$k_t = \lambda_2 k_{t-1}, \; k_t = \lambda_1 k_{t-1}.$$

This verifies that in the case of these two solutions, the sequence of k_t's which satisfy the second order difference equation, (2.1), also satisfy a first order difference equation. It is natural to call these solutions minimal state variable (MSV) solutions, because in each case every capital stock in the sequence can be expressed as a function of only one previous value of the capital stock. This is a 'natural' solution if we think of k_t as being decided at time $t-1$. At that time, k_{t-1} is sufficient information to characterize the current situation and the value of capital in period $t-2$ or earlier seems superfluous. For example, to understand the nature of production opportunities at date $t-1$ one needs only know k_{t-1}. We can see from (2.4) that there are exactly two MSV solutions. All the other solutions make the capital stock a non-trivial function of two eigenvalues, and so the capital stock follows a second order difference equation. These are not MSV solutions because k_t at any point in the solution cannot be determined exclusively as a function of k_{t-1}.

We can also define a solution as being 'non-explosive' if

$$k_t \to 0$$

as $t \to \infty$. From (2.4) we can see three cases.

- Case 1: $|\lambda_1| > 1$ and $|\lambda_2| < 1$. In this case, there is precisely one solution that is non-explosive, the one associated with $a = \bar{k}_0$.

- Case 2: $|\lambda_1|, |\lambda_2| > 1$. In this case there is no solution that satisfies non-explosiveness.
- Case 3: $|\lambda_1|, |\lambda_2| < 1$. In this case, all solutions are non-explosive.

Note that in case 1, when there is exactly one non-explosive solution, that solution is a MSV.[2]

These properties of solutions are quite general. (i) The number of solutions correspond to the number of points in a finite-dimensional Euclidean space (in the example, it's R^1). (ii) There is a finite number of MSV solutions and that set is easy to characterize. (iii) The number of non-explosive solutions requires comparing the absolute value of eigenvalues with unity, and if there is only one solution that is non-explosive, then that solution is MSV.

3. Deterministic Case, Invertible a

We now develop a matrix version of the analysis in the previous section. Suppose that (2.1) is actually

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0,\ t = 0, 1, 2, ... \tag{3.1}$$

where z_t is the $n \times 1$ vector of time t endogenous variables, expressed in deviation from steady state. Also, α_i are known $n \times n$ matrices for $i = 0, 1, 2$. The '0' after the equality in (3.1) is an $n \times 1$ vector of zeros.[3] The initial conditions, z_{-1} are given. It is convenient to

[2] It is readily verified that the neoclassical model falls in case 1. Dividing f in (3.1) by λ, we find that the zero condition corresponds to

$$g(\lambda) = \phi,$$

where

$$g(\lambda) = \lambda + \frac{1}{\beta\lambda}.$$

The graph of g against λ has a 'U' shape, and reaches a minimum at $\lambda = \sqrt{1/\beta}$, when g takes on a value of $2\sqrt{\beta}$. Note

$$0 < \left(1 - \sqrt{1/\beta}\right)^2 = 1 + \frac{1}{\beta} - 2\sqrt{1/\beta},$$

so that $1 + \frac{1}{\beta} > 2\sqrt{1/\beta}$. It follows from (2.2) that the ϕ line cuts the g curve above the point where g reaches its minimum. As a result, roots of f are definitely distinct. In addition, it is easily verified that one root is less than unity and the other is greater than unity.

[3] In the case, $n = 1$, (3.1) could be the linearized intertemporal Euler equation for the neoclassical growth model. Thus, supppose the problem is to maximize $\sum_{t=0}^{\infty} \beta^t \log c_t$ subject to $c_t + k_{t+1} - (1-\delta) k_t \leq k_t^\alpha$, $\beta, \alpha, \delta \in (0, 1)$ and k_0 given. Then the first order condition is

$$\frac{1}{c_t} - \beta \frac{1}{c_{t+1}} \left[\alpha k_{t+1}^\alpha + 1 - \delta\right] = 0,$$

for $t \geq 0$. Substituting out for c_t and c_{t+1} in this expression from the resource constraint, we obtain

$$v(k_t, k_{t+1}, k_{t+2}) = 0,\ t = 0, 1, 2, ...$$

Compute k^* such that $v(k^*, k^*, k^*) = 0$ (i.e., the steady state). Then, (3.6) represents the above condition in which the v function has been replaced by its first order Taylor series expansion about steady state. In (3.6), k_t stands for $k_t - k^*$. I do not adopt a new piece of notation to express deviations from steady state in order to keep the notation simple and because this should not cause confusion in this setting.

express (3.1) as a first order difference equation system:

$$aY_{t+1} + bY_t = 0,\ t \geq 0, \tag{3.2}$$

where

$$Y_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix},\ a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix},\ b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.$$

The first set of n equations in (3.2) reproduce the n equations in (3.1), while the second set of n equations in (3.2) capture the fact that the second set of variables in Y_{t+1} coincide with the first set of variables in Y_t. A solution is a sequence of Y_t's such that (3.2) holds for each t.

Here we assume that a is invertible (i.e., we assume that α_0 is invertible). This assumption is rarely satisfied in practice, although it is satisfied in the simple example of the previous section when $n = 1$. It is convenient to separate the question of how to deal with the singularity in a from other aspects of model solution. For this reason we defer addressing the singular a case until a later section.

Premultiplying (3.2) by a^{-1}, we obtain:

$$Y_t = \Pi^t Y_0,\ \Pi = -a^{-1}b. \tag{3.3}$$

According to this equation, a solution is completely determined by the initial condition, Y_0. We assume that the eigenvalues of Π are distinct, which guarantees that Π has the following eigenvector-eigenvalue decomposition:

$$\Pi = P\Lambda P^{-1}. \tag{3.4}$$

Write

$$P = \begin{pmatrix} P_1 & \cdots & P_{2n} \end{pmatrix},\ P^{-1} = \begin{pmatrix} \tilde{P}_1 \\ \vdots \\ \tilde{P}_{2n} \end{pmatrix},\ \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{2n} \end{bmatrix},$$

where Λ is a diagonal matrix. The elements along the diagonal of Λ are the eigenvalues of the matrix, Π. The column vectors, P_i, are the right eigenvectors of Π :

$$\Pi P_i = \lambda_i P_i,\ i = 1, ..., 2n.$$

After premultiplying (3.4) by P^{-1}, we see that $\tilde{P}_i$ are the left eigenvectors of Π :

$$\tilde{P}_i \Pi = \lambda_i \tilde{P}_i,\ i = 1, ..., 2n.$$

The left eigenvectors of Π play a fundamental role in shaping the dynamics of the system.

From (3.3), we see that the space of solutions is $n-$dimensional. This is because Y_0 has n 'free parameters' in it, namely the n elements of z_0 (recall, z_{-1} is given). That is, the set of solutions corresponds to the n-dimensional Euclidean space, R^n. This is the analog of result (i) in the previous section. We now proceed to the analogs of results (ii) and (iii).

It is easy to verify that

$$\Pi^j = P\Lambda^j P^{-1}. \tag{3.5}$$

So, multiplying both sides of (3.3) by P^{-1} we obtain:

$$\tilde{Y}_t = \Lambda^t \tilde{Y}_0, \ \tilde{Y}_t \equiv P^{-1} Y_t.$$

The equations that determine the evolution of $\tilde{Y}_t$ are completely independent, and can be expressed as follows:

$$\tilde{Y}_{it} = \lambda_i^t \tilde{Y}_{i,0}, \text{ for } i = 1, 2, ..., 2n, \tag{3.6}$$

where,

$$\tilde{Y}_t = \begin{pmatrix} \tilde{Y}_{1,t} \\ \vdots \\ \tilde{Y}_{2n,t} \end{pmatrix}.$$

Equation (3.6) shows that if $\tilde{Y}_{i,0} = 0$, then $\tilde{Y}_{i,t} = 0$ for all t. Put differently,

$$\tilde{P}_i Y_0 = 0 \rightarrow \tilde{P}_i Y_t = 0, \text{ for } t = 1, 2, ... \tag{3.7}$$

That is, if Y_0 is orthogonal to the i^{th} left eigenvector, then Y_t will be orthogonal to that eigenvector too, for $t \geq 1$. This results will be useful in what comes next.

We define an MSV solution as a sequence of Y_t's that satisfy (3.2) and have the property that there exists an $n \times n$ real matrix, A which satisfies

$$\left[\begin{array}{ccc} I & \vdots & -A \end{array} \right] Y_t = 0 \text{ for } t = 0, 1, 2, ... \tag{3.8}$$

We find a *candidate* MSV solution by selecting n left eigenvectors of Π and using them to construct an $n \times 2n$ matrix D. Note that, by (3.7)

$$\text{for any solution with } DY_0 = 0, \text{ we have } DY_t = 0 \text{ for } t > 0. \tag{3.9}$$

We split the D matrix into components that are conformable with the two components of Y_t :

$$DY_t = \left[\begin{array}{ccc} D_1 & \vdots & D_2 \end{array} \right] \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = D_1 z_t + D_2 z_{t-1}.$$

If

$$D_1 \text{ invertible, } A \equiv -D_1^{-1} D_2 \text{ real,} \tag{3.10}$$

then the candidate MSV is an MSV (an *actual* MSV, for emphasis) in which Y_0 is uniquely determined by:

$$Y_0 = \begin{pmatrix} A \\ I \end{pmatrix} z_{-1}, \ I \text{ ~} n \times n \text{ identity matrix.}$$

Condition (3.8) holds because of (3.9).[4]

Note that there exist

$$\begin{pmatrix} 2n \\ n \end{pmatrix} \tag{3.11}$$

[4] Note that in effect we have described a strategy for computing a matrix zero, A, of the matrix polynomial, $\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0$.

candidate MSV's, since this is the number of ways of choosing n left eigenvectors from the set of $2n$ left eigenvectors. The set of actual MSV's is smaller than what is indicated in (3.11) if there are candidate MSV's which generate a complex A matrix and/or D_1 that is not invertible. (We discuss the complex case by way of an example below.) The key result is that there is a finite number of isolated MSV solutions. This is analogous to result (ii) in the previous section.

To determine the set of non-explosive solutions, note that $\tilde{Y}_t \to 0$ if, and only if, $Y_t \to 0$. Because of (3.6) it is convenient to consider the convergence properties of $\tilde{Y}_t$. According to (3.6), a non-explosive solution must have the property that the eigenvalues greater than unity in absolute value have been 'extinguished' from the system. If $|\lambda_i| > 1$ (i.e., λ_i is 'explosive') this eigenvalue is extinguished by choosing Y_0 so that $\tilde{P}_i Y_0 = 0$. The word, 'extinguished', is appropriate here because - according to (3.6) - λ_i has no impact on solution dynamics when $\tilde{P}_i Y_0 = 0$. Consider the same three cases delineated in the previous section. Let q denote the number of explosive eigenvalues.

- Case 1: $n = q$. In this case a candidate non-explosive solution is found by constructing a D matrix containing the n left eigenvectors associated with the explosive eigenvalues and determining the Y_0 such that $DY_0 = 0$. This candidate is an actual non-explosive solution if (3.10) is satisfied. The candidate non-explosive solution is unique.

- Case 2: $n < q$. In this case, every solution is explosive.

- Case 3: $n > q$. In this case, construct an $q \times 2n$ D_1 matrix by including the left eigenvectors of Π associated with the q explosive eigenvalues. The condition, $D_1 Y_0 = 0$ represents q restrictions on Y_0 which is not sufficient to determine Y_0 uniquely because Y_0 has $n > q$ free elements. One way to isolate a unique element in Y_0 is to construct an $(n - q) \times 2n$ matrix, D_2, using $n - q$ of the remaining $2n - q$ left eigenvectors of Π. Then,

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

is a $n \times 2n$ matrix that represents a candidate MSV solution. There are

$$\begin{pmatrix} 2n - q \\ n - q \end{pmatrix}$$

candidate MSV, non-explosive solutions.

Other non-explosive solutions can be found that are not MSV as follows. Partition D_1 and z_0 (i.e., the first n elements of Y_0) in the following way:

$$D_1 = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix},\ z_1 = \begin{bmatrix} z_{1,0} \\ z_{2,0} \end{bmatrix},\ y = \begin{bmatrix} z_{2,0} \\ z_{-1} \end{bmatrix}.$$

Here, D_{11} is $q \times q$ and D_{12} is $q \times 2n$. Also, $z_{1,0}$ is $q \times 1$, $z_{2,0}$ is $(n - q) \times 1$ and y is $(q + n) \times 1$. Then, consider:

$$D_1 Y_0 = D_{11} z_{1,0} + D_{12} y = 0.$$

As long as D_{11} is invertible and $-D_{11}^{-1}D_{12}$ is real we can write

$$z_{1,0} = -D_{11}^{-1}D_{12}y. \tag{3.12}$$

Note that y has n given initial conditions, as well the 'free' $n-q$ objects in $z_{2,0}$. For any specification of $z_{2,0}$ we have a particular z_0 and, hence, Y_0. In this way, we have identified an $n-1$ dimensional space of candidate solutions. I say 'candidate', because these solutions are actual solutions only if D_{11} is invertible and $D_{11}^{-1}D_{12}$ is real. Since the typical solution obtained in this way does not involve a Y_0 that is orthogonal to n left eigenvectors, almost all of these solutions are not MSV.

Although the equations we study have been linearized, when $q < n$ we know we have multiple equilibria in the underlying nonlinear system. This is because we know that when $z_{-1} = 0$, then one equilibrium is given by $z_t = 0$ for $t > 0$. When $q < n$ the procedure in the previous paragraph allows us to find other solutions arbitrality close to the steady state. Because these other solutions can be arbitrarily small deviations from the steady state equilibrium we know that the alternative solutions also satisfy the underlying non-linear equilibrium conditions.[5] This is because the first order Taylor series expansion is arbitraily accurate for paths sufficiently close to the steady state path. Formally, when $q < n$, then for any high-dimensional ball drawn around the steady state equilibrium we can find another equilibrium that lies within that ball. As a result, the steady state equilibrium is said to be *indeterminate*. It is worth noting that the MSV solutions that we studied above are less useful. This is because when $z_{-1} = 0$ then all MSV solutions have the property that $z_t = 0$ for $t > 0$, since each satisfies, $z_t = Az_{t-1}$ for $t \geq 0$ and some A. The non-MSV solutions described provide a constructive procedure to gaining insight into economic forces underlying the multiplicity of equilibria that exists when $q < n$.

4. Stochastic Case, Invertible a

In the stochastic case, (3.1) is written

$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0, \tag{4.1}$$

for each $z_t \in R^n$, $s_t \in R^{n_s}$. Here, s_t is the vector of exogenous shocks and is assumed to have the following time series representation:

$$s_t = Ps_{t-1} + \varepsilon_t,$$

where the eigenvalues of P are all less than unity in absolute value.[6] It is convenient to rewrite (4.1):

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t = \xi_{t+1}, \tag{4.2}$$

[5] This obviously also means any relevant transversality conditions since they are solutions that return to steady state.

[6] This example, in the case $n = 1$, could arise from a stochastic version of the neoclassical model in the previous footnote in which the production function is replaced by $k_t^{\alpha}\exp\left(\theta_t\right)^{1-\alpha}$. If $\theta_t - \theta = \rho\left(\theta_{t-1} - \theta\right) + u_t$, where u_t is *iid* and uncorrelated with past θ_t and $|\rho| < 1$, then $s_t = \theta_t$, $P = \rho$, $\varepsilon_t = u_t$. When $n_s > 1$ the

where ξ_{t+1} is a stochastic process satisfying

$$E_t\xi_{t+1} = 0, \tag{4.3}$$

for all t. Note that the set of ξ_{t+1} satisfying (4.3) is very large. For example,

$$\xi_{t+1}\tilde{}N(0, 1 - \cos(t)),$$

is a possibility.

As before, we proceed by expressing the system as a first order process. If

$$\underset{(2n+n_s)\times 1}{Y_t} = \begin{pmatrix} z_t \\ z_{t-1} \\ s_t \end{pmatrix}, \; a = \begin{bmatrix} \alpha_0 & 0 & \beta_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \; b = \begin{bmatrix} \alpha_1 & \alpha_2 & \beta_1 \\ -I & 0 & 0 \\ 0 & 0 & -P \end{bmatrix}, \; \omega_t = \begin{pmatrix} \xi_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix}, \tag{4.4}$$

then (4.2) and the law of motion of s_t are summarized as follows:

$$aY_{t+1} + bY_t = \omega_{t+1}. \tag{4.5}$$

As in the previous section, we assume α_0 is non-singular, so that a is invertible. Then, the analog of (3.3) is

$$Y_t = \Pi^t Y_0 + a^{-1}\omega_t + \Pi a^{-1}\omega_{t-1} + \cdots + \Pi^{t-1}a^{-1}\omega_1. \tag{4.6}$$

We see that now the $\{Y_t\}$ that solves the system is a stochastic process. We can still use the sort of approach of the previous section here in characterizing the set of solutions. The set of solutions is now indexed by Y_0 and a stochastic process for $\left\{\xi_{t+1}\right\}$ that satisfies (4.3). The other stochastic process, $\{\varepsilon_t\}$, is treated like z_{-1} in that it is given by the problem. Thus, the space of solutions is characterized by the choice of Y_0, $\left\{\xi_{t+1}\right\}$. This is a very large space. The vector, Y_0, has n free elements, the ones corresponding to z_0. The vector stochastic process, $\left\{\xi_{t+1}\right\}$, also has n free elements.

To further characterize the set of solutions, it is convenient to make use of (3.5) and premultiply (4.6) by P^{-1} :

$$\tilde{Y}_t = \Lambda^t\tilde{Y}_0 + P^{-1}a^{-1}\omega_t + \Lambda P^{-1}a^{-1}\omega_{t-1} + \cdots + \Lambda^{t-1}P^{-1}a^{-1}\omega_1, \tag{4.7}$$

or,

$$\tilde{Y}_{it} = \lambda_i^t\tilde{Y}_{i,0} + v_{i,t} + \lambda_i v_{i,t-1} + \cdots + \lambda_i^{t-1}v_{i,1}, \; v_{i,t} \equiv \tilde{P}_i a^{-1}\omega_t,$$

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notation accommodates more complicated θ_t processes and/or other shocks. For example, suppose

$$\theta_t - \theta = \rho_1(\theta_{t-1} - \theta) + \rho_2(\theta_{t-2} - \theta) + u_t + \gamma u_{t-1}$$

then, $s_t = \left[\begin{array}{ccc} \theta_t - \theta & \theta_{t-1} - \theta & u_t \end{array}\right]'$, $\varepsilon_t = \left[\begin{array}{ccc} u_t & 0 & u_t \end{array}\right]'$ and

$$F = \begin{bmatrix} \rho_1 & \rho_2 & \gamma \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that the roots of $\lambda^2 - \rho_1\lambda - \rho_2$ are less than unity in absolute if and only if the eigenvalues of F are less than unity in absolute value.

for $i=1,...,2n+n_s$. Note, selecting Y_0 to be orthogonal to the left eigenvector of λ_i, $\tilde{P}_i$, no longer ensures that Y_t remains orthogonal to that eigenvector forever after. For this to be the case, we must also set $v_{i,t}=0$ for each t. In fact we have the degrees of freedom to do this because of the n free stochastic processes in ξ_t. To see this, write

$$v_{i,t}=\left[\begin{array}{cc}\underset{1\times 1}{\delta_1^{(i)}} & \underset{1\times(n-1)}{\delta_2^{(i)}}\end{array}\right]\left(\begin{array}{c}\xi_t^1\\ \xi_t^2\end{array}\right)+\underset{1\times n_s}{\gamma^{(i)}}\varepsilon_t,\ \left[\begin{array}{cccc}\delta_1^{(i)} & \delta_2^{(i)} & \underset{1\times n}{0} & \gamma^{(i)}\end{array}\right]=\tilde{P}_i a^{-1},$$

where

$$\xi_t\equiv\left(\begin{array}{c}\underset{1\times 1}{\xi_t^1}\\ \underset{(n-1)\times 1}{\xi_t^2}\end{array}\right).$$

We suppose that $\delta_1^{(i)}\neq 0$. If ξ_t^2 is any given process that satisfies (4.3), then choose ξ_t^1 to satisfy

$$\xi_t^1=\frac{\delta_2^{(i)}\xi_t^2+\gamma^{(i)}\varepsilon_t}{\delta_1^{(i)}}.$$

Note that with ξ_t constructed in this way, (4.3) is satisfied. Thus, it is possible to find a solution (i.e., a Y_0 and a $\{\xi_t\}$ that satisfies (4.3)) with the property that $\tilde{P}_i Y_t=0$ for all t.

As before, we use this result to construct a set of MSV solutions. Here, we define a candidate MSV solution as a solution, $(Y_0,\{\xi_t\})$, having the property that there exists a $n\times(2n+n_s)$ matrix D such that

$$DY_t=0,\ t=0,1,2,....\ .$$

Note that if the analog of (3.10) holds (i.e., the left $n\times n$ block of D is invertible and the analog of A is real), an MSV solution has the property that z_t can be determined uniquely knowing only z_{t-1} and s_t. In this case, we say the candidate MSV solution is an actual MSV. As before, there are many candidate MSV solutions in the space of solutions. They can be found by constructing D matrices by selecting n left eigenvectors from the set of $2n+n_s$ eigenvectors of Π and by selecting the n free stochastic processes in ξ_t so that

$$Da^{-1}\omega_t=\underset{n\times 1}{0}, \tag{4.8}$$

for each $t=0,1,2,...$. Note that by selecting ξ_t in this way, we make ξ_t an exact function of ε_t. Such a solution is sometimes called by economists a 'fundamental' solution because it makes the stochastic processes driving the system, ξ_t,ε_t exclusively a function of the exogenous disturbances impacting on preferences and technology, namely the ε_t's.[7]

We now define a non-explosive solution as one having the property:

$$E_0Y_t\rightarrow 0,$$
$$Var_0\left(Y_t\right)\text{ bounded.}$$

[7] I say 'economists' here, because the word, 'fundamental' means something different in other areas, for example, time series analysis.

Note that the first condition is not enough, because it does not restrict the ξ_t's at all. Without restricting the ξ_t's, the first condition could be satisfied while the Y_t's have exploding variances...hardly 'non-explosive'.

As before, the non-explosive solutions are the ones in which the explosive eigenvalues have been suppressed. To suppress an explosive eigenvalue, λ_i, $a^{-1}\omega_t$ and Y_0 must be selected so that $\tilde{P}_i a^{-1}\omega_t = 0$ for all t and $\tilde{P}_i Y_0 = 0$. There are n degrees of freedom in setting ξ_t and in setting Y_0. As a result, we obtain the same three cases considered in the previous two sections. Let q denote the number of explosive eigenvalues.

- Case 1: if $q = n$, the number of candidate non-explosive solutions is unique and it is an MSV if the analog of (3.10) is satisfied.
- Case 2: if $q > n$, all solutions are explosive.
- Case 3: if $q < n$, there are many candidate non-explosive solutions. Some may be MSV's. In addition, non-explosive solutions that are not MSV's also exist. They can be found using the approach described at the end of the previous section.

It is worthwhile to elaborate a little on case 3. To identify a candidate non-explosive MSV solution, construct the first q rows of an $n \times (2n + n_s)$ D matrix using the q left eigenvectors of Π associated with the explosive eigenvalues. One could fill out the bottom $n - q$ rows of D with left eigenvectors of Π associated with the non-explosive eigenvalues. There are obviously several ways of doing this. For each D constructed in this way, choose Y_0 so that $DY_0 = 0$ and choose the n elements of ξ_t so that (4.8) is satisfied. Each resulting solution, $(Y_0, \{\xi_t\})$, is a candidate non-explosive MSV solution in which only fundamental shocks appear. There are also non-MSV solutions. Fill out the bottom $n - q$ rows of D with vectors other than left eigenvectors of Π. Choose them so that D has rank n. There is obviously a continuum of ways of doing this. Choose the ξ_t's any way you want, subject only to (4.3) and the requirement that $\tilde{P}_i a^{-1}\omega_t = 0$ for the explosive eigenvalues. There is obviously a continuum of ways of choosing ξ_t's to satisfy these conditions. To the extent that the ξ_t's are not a function of ε_t, these candidate non-MSV solutions are also non-fundamental. The elements of ξ_t that are not a function of ε_t are called 'sunspots'.

5. Non-Invertible a

Now consider the case when a is not invertible. This case arises when α_0 is singular. It occurs because in practice equilibrium conditions are different in terms of how many future variables they include. An intertemporal Euler equation includes variables stretching relatively far into the future while a resource constraint or an intratemporal Euler equation involves variables that extend less far in the future. A consequence of this is that rows of α_0 can be zero. A general procedure for handling this case is to substitute out variables in a way that makes the system smaller, and have the property that α_0 in that system is non-singular. A simple example is the neoclassical growth model with an hours worked decision. In that case, the system involves the intra- and inter- temporal Euler equations (I'm assuming the resource constraint has used to substitute out consumption). This system has a singular α_0. However, hours worked may be substituted out from the linearized intra-temporal Euler equation into the linearized inter temporal Euler equation. The resulting system has a non-singular α_0. In effect, we 'decouple' hours worked from the system and solve the smaller system. Once a solution for the smaller system is found, we can solve for the variable, hours worked, that had been substituted out.

Although manual substitution along the lines of the previous paragraph works in some examples, in general it is tedious. Fortunately, there is a simple matrix procedure based on the QZ decomposition that allows us to make the system smaller by decoupling some variables. To my knowledge, it was first suggested in Sims in a 1989 working paper, which was subsequently published in a 2001 issue of *Computational Economics.*

We consider the deterministic case in detail, and then explain the adjustments required to handle the stochastic case.

5.1. Deterministic Case

Denote the dimension of Y_t by $m \equiv 2n$. Suppose that the rank of a is $m-l$, with $0 < l < m$. The QZ decomposition of matrices, a and b, is a set of orthonormal matrices Q and Z, and upper triangular matrices H_0 and H_1 with the properties:[8]

$$QaZ = H_0,\ QbZ = H_1. \tag{5.1}$$

It is possible to order the rows of H_0 so that the l zeros on its diagonal are located in the lower right part of H_0.[9] Denote the upper $(m-l)\times(m-l)$ block of H_0 by G_0. This matrix must be non-singular. Let the corresponding upper left $(m-l)\times(m-l)$ block in H_1 be denoted G_1. By construction, the l terms on the lower right part of the diagonal of H_0 are zero. I assume that the diagonal terms in the lower right $l\times l$ block of H_1 are non-zero. Also, it is useful to partition Z' as follows:

$$Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \tag{5.2}$$

where L_1 is $(m-l)\times m$ and L_2 is $l \times m$.

Inserting ZZ' $(= I)$ before Y_{t+1} and Y_t in (3.2), defining $\gamma_t \equiv Z'Y_t$, and pre-multiplying (3.2) by Q, (3.2) becomes:

$$H_0\gamma_{t+1} + H_1\gamma_t = 0,\ t = 0, 1, ... \ . \tag{5.3}$$

Partition γ_t as follows:

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}, \tag{5.4}$$

[8]This decomposition can be accomplished with MATLAB's qz command.

[9]This can be accomplished by processing the output of MATLAB's qz command by the code, qzdiv, originally written by Christopher Sims and maintained by Dynare. That code may be found at http://www.dynare.org/dynare-matlab-m2html/matlab/qzdiv.html

where γ_t^1 is $(m-l)\times 1$ and γ_t^2 is $l \times 1$. It is easy to verify that (5.3) implies $\gamma_t^2 = 0$, $t \geq 0$, i.e.,[10]

$$L_2 Y_t = 0, \; t = 0, 1, ... \; . \tag{5.5}$$

Thus, the 'full rank' part of the system appears in the upper left $(m-l)\times(m-l)$ block of (5.3) and the singular or 'static' part of the system appears in the lower right $l \times l$ block.

With (5.5) imposed, the last l equations in (5.3) are redundant, so (5.3) can be written

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = 0, \; t = 0, 1, ... \; . \tag{5.6}$$

In effect, we have reduced the size of the system in a way that puts us into the 'invertible a' case, by separating out the static part which is the source of the singularity. The set of solutions to the reduced sized system, (5.6), can be expressed as $\gamma_t^1 = (-G_0^{-1}G_1)^t \gamma_0^1$, $t \geq 0$, or,

$$P^{-1}\gamma_t^1 = \Lambda^t P^{-1} \gamma_0^1, \tag{5.7}$$

where $P\Lambda P^{-1} = -G_0^{-1}G_1$ is the eigenvector, eigenvalue decomposition of $-G_0^{-1}G_1$.

Here, we continue to define an MSV as in (3.8). We seek a candidate MSV by constructing a matrix, D, with the property that $DY_t = 0$ for all t. The matrix, D, corresponds to an actual MSV if D satisfies (3.10). In the present context, we already have l static restrictions, (5.5). So, to construct a candidate MSV, we require only $n-l$ additional restrictions. We find these by putting $n-l$ of the left eigenvectors of $-G_0^{-1}G_1$ into a $(n-l)\times(m-l)$ matrix $\tilde{p}$ and forming the $(n-l)\times m$ matrix, $\tilde{p}L_1$. Our candidate MSV is then given by:

$$D = \begin{bmatrix} \tilde{p}L_1 \\ L_2 \end{bmatrix}.$$

The number of candidate MSV's corresponds to the number of ways that D matrices like this can be constructed. That is determined by the number of ways that $n-l$ left eigenvectors can be selected from the set of $m-l$ eigenvectors of $-G_0^{-1}G_1$. Each of these D's is a candidate MSV solution. A candidate MSV solution is an actual MSV solution if D satisfies condition (3.10).

Now consider the set of non-explosive solutions, $Y_t \to 0$. The γ_t^1 that solve (5.7) converge to zero asymptotically if, and only if, $\tilde{p}\gamma_0^1 = 0$, where $\tilde{p}$ is composed of the rows of P^{-1} corresponding to diagonal terms in Λ that exceed 1 in absolute value. This condition is:

$$\tilde{p}L_1 Y_0 = 0. \tag{5.8}$$

[10] To see this, let us temporarily adopt a simpler notation. Let the lower right $l \times l$ block of H_0 be denoted Γ and let the corresponding block of H_1 be denoted W. Write $\Gamma = [\Gamma_{ij}]$ and $W = [W_{ij}]$. The matrices, Γ and W, are upper triangular, with the former having zeros along its diagonal and the latter having non-zero terms along its diagonal. Also, write $x_t = \gamma_t^2$, with $x_t = [x_{1t}, ..., x_{lt}]'$. Then we have $\Gamma x_{t+1} + W x_t = 0$ for $t = 0, 1, 2, ...$. Note that the last row of Γ is composed of zeros, so that the last row of this system of equations is $W_{l,l} x_{lt} = 0$ for all t. Since $W_{l,l} \neq 0$, this implies $x_{lt} = 0$ for all t. Now consider the $l-1^{th}$ equation:

$$\Gamma_{l-1,l} x_{l,t+1} + W_{l-1,l-1} x_{l-1,t} + W_{l-1,l} x_{l,t} = 0,$$

for $t = 0, 1, 2, ...$. But, since $x_{l,t} = 0$ for all t, this implies $W_{l-1,l-1} x_{l-1,t} = 0$ for all t. Since $W_{l-1,l-1} \neq 0$, this in turn implies $x_{l-1,t} = 0$ for $t = 0, 1, 2, ...$. Proceeding in this way, we establish recursively that $x_{j,t} = 0$ for all t, for $j = l, l-1, ..., 1$.

There are many candidate non-explosive solutions if q, the number of explosive eigenvalues of $-G_0^{-1}G_1$, is less than $n-l$. There is exactly one solution if $q=n-l$ and there are none if $q>n-l$.

Consider the case, $q=n-l$. Recall that the number of free elements in Y_0 is n. Equation (5.5) for $t=0$ represents l restrictions on Y_0, so that to pin Y_0 down uniquely, $n-l$ more restrictions are required. Construct the matrix $\tilde{p}$ using the $n-l$ left eigenvectors associated with the explosive eigenvalues in $-G_0^{-1}G_1$. Then, define

$$D=\begin{bmatrix} \tilde{p}L_1 \\ L_2 \end{bmatrix}. \tag{5.9}$$

Under the assumption that D satisfies (3.10), the unique non-explosive solution corresponds to the MSV associated with the Y_0 satisfying $DY_0=0$.

Now consider the case, $q<n-l$. In this case there are multiple, but finite, MSV solutions. But, there is actually a continuum of solutions, counting the non-MSV solutions. The latter are useful for thinking about the determinacy of the non-stochastic steady state equilibrium of the underlying nonlinear model. They can also be useful to gaining intuition about the economic source of the multiple solutions. So, we now discuss how to find a non-MSV solution that is close to the steady state solution.

Equation (5.5) provides l restrictions on Y_0. Suppressing q explosive eigenvalues from the system requires that γ_0^1 be orthogonal to the q left eigenvectors of Π associated with the explosive eigenvalues. Denote these by $\tilde{p}$ and define D as in (5.9), with the understanding that $\tilde{p}L_1$ is $q\times m$. Now, D is $(q+l)\times m<n\times m$. Let $\tilde{n}\equiv q+l$ and write

$$D=\begin{bmatrix} D_1 & \vdots & D_2 \end{bmatrix},$$

where D_1 is $\tilde{n}\times\tilde{n}$ and D_2 is $\tilde{n}\times(m-\tilde{n})$. Define

$$Y_0=\begin{pmatrix} z_0^1 \\ y \end{pmatrix},\ y=\begin{pmatrix} z_0^2 \\ z_{-1} \end{pmatrix},\ z_0=\begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix},$$

where Y_0 is $m\times 1$, z_0^1 is $\tilde{n}\times 1$, z_0^2 is $(n-\tilde{n})\times 1$. If we set z_{-1} to its steady state value of 0, then for z_0^2 close enough to zero, the computations reveal the local-to-steady-state properties of the underlying nonlinear equations that are approximated by (3.1).

We seek a sequence of Y_t's which respect the static restriction, (5.5), for all t and which suppress the explosive eigenvalues for all t, i.e., satisfy (5.8). That is, we seek Y_0, Y_1, with the property, $DY_t=0$ for $t=0,1,2,\ldots$. For $t=0$:

$$DY_0=D_1z_0^1+D_2y=0$$

implies (assuming D_1 is invertible and $D_1^{-1}D_2$ is real),

$$z_0^1=-D_1^{-1}D_2y.$$

Thus, we have a mapping from z_0^2 in y to z_0^1 and, hence, to Y_0. Given Y_0 we compute $\gamma_0^1=L_1Y_0$ and

$$\gamma_t^1=\Pi^t\gamma_0^1,\ t=1,2,\ldots\ . \tag{5.10}$$

Then,

$$Y_t = Z \left(\begin{array}{c} \underset{(m-l)\times 1}{\gamma_t^1} \\ \underset{l\times 1}{0} \end{array} \right), t = 1, 2, ...$$

Note that this sequence satisfies $L_2 Y_t = 0$ for $t = 0, 1, 2, ...$. In addition, all explosive eigenvalues have been suppressed, because $\tilde{p} L_1 Y_t = \tilde{p}\gamma_t^1 = 0$ for all t. In practice, $\tilde{p}\gamma_0^1$ is not exactly zero, and the effects of this can cumulate, especially if one or several of the explosive eigenvalues are particularly large. Thus, we expect that $\tilde{p}\gamma_t^1$ eventually begins to explode for large enough t, so that these higher values of t should be ignored. An alternative approach, which is mathematically the same as the one described above, is to replace Π^t by $P\tilde{\Lambda}^t P^{-1}$. Here, $\tilde{\Lambda}$ is the diagonal matrix of eigenvalues of Λ in which each explosive eigenvalue is replaced by 0. With this approach, the explosive eigenvalues truly are suppressed and Y_t must go to zero as $t \to \infty$.

The above algorithm, as long as $z_{-1} = 0$ and z_0^2 is close enough to zero, constructs an solution that is arbitrarily close to the steady state solution, $Y_t = 0$, $t \geq 0$. Because we can find solutions that are arbitrarily close to the steady state solution, we conclude that the steady state solution is indeterminate when $q < n - l$. The steady state is determinate when $q \geq n - l$. In that case, we can construct a region around the point in infinite-dimensional space corresponding to the steady state solution that is small enough that it does not contain another solution.

5.2. Stochastic Case

We now consider (4.5) with the structure on ω_{t+1} that is indicated in (4.4). We have $E_t\omega_{t+1} = 0$ which, among other things, requires that ω_{t+1} be orthogonal to all date t and earlier variables. Also, by comparing (4.1) and (4.2) we see that the elements of ξ_{t+1} are linear transformations on one-step-ahead forecast errors of endogenous variables in the system. The fact that we are in the non-invertible a case implies some restrictions on ξ_{t+1}. For example, a may be non-invertible because some equations are not forward looking, so that the corresponding row of α_0 is composed of zeros. This would be the case, for example, if one of the equations in the system included a resource constraint, or a static first order condition that did not involve expectations. When there are rows of α_0 that are composed of zeros, then the corresponding element of ξ_{t+1} is zero too. Another possibility (see section 7) is that all equations include forward-looking variables, but there are restrictions across the ξ_{t+1}'s. For example, it could be that the expectation of the future value of the same variable appears in two different equations, so that two of the ξ_{t+1}'s are proportional to each other. For an example, see section 7.2 below.

Premultiplying (4.5) by Q we obtain

$$H_0\gamma_{t+1} + H_1\gamma_t = Q\omega_{t+1}, \tag{5.11}$$

where $\gamma_t = Z'Y_t$. We adopt the partitioning used in the previous subsection and begin by establishing that $\gamma_t^2 = 0$ for all t and ω_{t+1} must satisfy a restriction that is specified below.

Express the bottom l equations in (5.11) as follows:

$$\begin{bmatrix} 0 & W_{1,2} & W_{1,3} \\ 0 & 0 & W_{2,3} \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \gamma^2_{1,t+1} \\ \gamma^2_{2,t+1} \\ \gamma^2_{3,t+1} \end{pmatrix} + \begin{bmatrix} H_{1,1} & H_{1,2} & H_{1,3} \\ 0 & H_{2,2} & H_{2,2} \\ 0 & 0 & H_{3,3} \end{bmatrix} \begin{pmatrix} \gamma^2_{1,t} \\ \gamma^2_{2,t} \\ \gamma^2_{3,t} \end{pmatrix} = \begin{pmatrix} Q^2_1\omega_{t+1} \\ Q^2_2\omega_{t+1} \\ Q^2_3\omega_{t+1} \end{pmatrix}, \quad (5.12)$$

where we set $l = 3$ to simplify the exposition. The first square matrix in (5.12) is the bottom right $l \times l$ block of H_0, the second square matrix is the bottom right block of H_1, and

$$Q^2 = \begin{pmatrix} Q^2_1 \\ Q^2_2 \\ Q^2_3 \end{pmatrix},$$

where Q^2 denotes the bottom l equations in Q. As before, we assume that the diagonal terms on the bottom right $l \times l$ block of H_1 are all non-zero. Finally, $\gamma^2_{i,t}$ denotes the i^{th} element, $i = 1, ..., l$, of γ^2_t. In addition to showing that $\gamma^2_t = 0$, we also establish that ω_{t+1} must satisfy $Q^2\omega_{t+1} = 0$ for all t.

Consider the last equation in (5.12):

$$\gamma^2_{3,t} = \frac{1}{H_{3,3}} Q^2_3\omega_{t+1}.$$

Because ω_{t+1} must be orthogonal to γ^2_t, we conclude that ω_{t+1} must have the property, $Q^2_3\omega_{t+1} = 0$. In the example in section 7 below, we found that given the restrictions on ξ_{t+1}, $Q^2_3\omega_{t+1} = 0$ is satisfied without further constraining ω_{t+1}. We conclude that $\gamma^2_{3,t} = 0$ for all t. Now, consider the second-to-last equation in (5.12):

$$W_{2,3}\gamma^2_{3,t+1} + H_{2,2}\gamma^2_{2,t} + H_{2,3}\gamma^2_{3,t} = Q^2_2\omega_{t+1},$$

after making use of $\gamma^2_{3,t} = 0$,

$$\gamma^2_{2,t} = \frac{Q^2_2}{H_{2,2}}\omega_{t+1}.$$

Orthogonality requires $Q^2_2\omega_{t+1} = 0$, a condition which may or may not require additional restrictions on ω_{t+1}. From this we conclude that $\gamma^2_{2,t} = 0$ for all t. A similar argument implies $Q^2_1\omega_{t+1} = 0$ and $\gamma^2_{1,t} = 0$ for all t. In this way, we can see that $\gamma^2_t = Q^2\omega_{t+1} = 0$ for all t, for any l. In practice, given the restrictions on ξ_{t+1} the condition, $Q^2\omega_{t+1}$ may be satisfied automatically. For an example, see section 7.2.

Given that $\gamma^2_t = 0$, we have that (5.11) can be expressed as

$$G_0\gamma^1_{t+1} + G_1\gamma^1_t = Q^1\omega_{t+1},$$

where G_0 and G_1 are the upper $(m - l) \times (m - l)$ blocks of H_0 and H_1, respectively. Also, Q^1 is the first $m - l$ rows of Q, where m is the length of the vector, Y_t.

6. 'Candidate' Versus 'Actual' MSV's

In the previous sections, we have frequently referred to candidate versus actual MSV solutions. Here, we develop an example to make concrete the difference between them. Consider the standard neo-Keynesian model[11]:

$$\begin{aligned} \delta \pi_{t+1} + \lambda y_t - \pi_t &= 0 \\ y_{t+1} - y_t - \frac{1}{\sigma}(r_t - \pi_{t+1}) &= 0 \\ \rho r_{t-1} + (1-\rho)\beta \pi_{t+1} + (1-\rho)\gamma y_t - r_t &= 0. \end{aligned} \tag{6.1}$$

The first equation is the neo-Keynesian Phillips curve, according to which current inflation, π_t, is a function of expected future inflation and current output, y_t. The second equation is a log-linear approximation of the household intertemporal Euler equation for holding bonds. It says that the expected growth rate in output, $y_{t+1} - y_t$, is proportional to the expected real rate of interest, equal to the nominal interest rate, r_t, minus expected inflation. The last equation is the monetary policy rule, which makes the nominal rate of interest a weighted average of a target rate of interest (this is a linear function of expected inflation and output) and the lagged nominal rate of interest. All the variables are expressed relative to their steady state values.

The system can be expressed in our canonical form as follows:

$$\begin{bmatrix} \delta & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 \\ (1-\rho)\beta & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \\ r_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & -\frac{1}{\sigma} \\ 0 & (1-\rho)\gamma & -1 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ r_t \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \\ r_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.2}$$

or,

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0,$$

in obvious notation. There is no uncertainty. A minimal state variable solution is a matrix A such that

$$z_t = A z_{t-1},$$

where A satisfies

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0. \tag{6.3}$$

We will now describe the set of candidate and actual MSV solutions. The parameter values that we adopt for the model are:

$$\delta = 0.99, \sigma = 1, \lambda = 0.3, \gamma = 0.15, \rho = 0.5, \beta = 1.5. \tag{6.4}$$

First, we set up the system in first-order form:

$$a Y_{t+1} + b Y_t = 0,$$

where

$$a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix},\ b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix},\ Y_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix}.$$

[11] See Clarida, Gali and Gertler, 2000, "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, February.

In this example, α_0 is 3×3 and has rank 2. As a result, the rank of a is 5. That is, $n = 3$, $m = 6$ and $l = 1$. To study the set of solutions, we must apply the QZ decomposition. Thus, we find Q and Z such that

$$QaZ = H_0,\ QbZ = H_1.$$

Here, $QQ' = I$, $ZZ' = I$, where here and throughout '$'$' denotes the Hermetian transpose (e.g., transposition and conjugation) and

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -0.22-0.0003i & -0.70-0.001i & -0.17-0.0002i & 0 & 0 & 0.65+0.0009i \\ 0.36+0.36i & 0.42-0.09i & 0.27+0.27i & 0 & 0 & 0.64+0.10i \\ 0.58-0.009i & -0.47+0.32i & 0.44-0.007i & 0 & 0 & -0.20+0.34i \\ -0.60 & 0. & 0.80 & 0 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & -0.25+0.0004i & 0.59-0.3i & 0.32-0.29i & -0.56 \\ 0 & 0 & -0.55+0.0008i & 0.26+0.34i & -0.65-0.28i & 0.11 \\ 0 & 0 & 0.27-0.0004i & 0.51-0.17i & 0.13-0.28i & 0.74 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.75-0.001i & 0.21+0.21i & -0.42-0.20i & -0.37 \end{bmatrix}.$$

Also, the upper 5×5 block of H_0, G_0, and the upper 5×5 block of H_1, G_1, are

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.14 & -0.66+0.21i & -0.15+0.37i \\ 0 & 0 & 0 & 0.96+0.26i & -0.09+0.37i \\ 0 & 0 & 0 & 0 & 0.77-0.21i \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0.25-0.0004i & -0.59+0.3i & -0.32+0.29i \\ 0 & 0 & 0.55-0.0008i & -0.26-0.34i & -.65+0.28i \\ 0 & 0 & -0.41 & 0.38+0.09i & -0.27-0.28i \\ 0 & 0 & 0 & -1.19 & -0.25+0.19i \\ 0 & 0 & 0 & 0 & -0.95 \end{bmatrix}.$$

Let

$$Z' = \begin{bmatrix} \underbrace{L_1}_{5\times 6} \\ \underbrace{L_2}_{1\times 6} \end{bmatrix},\ \gamma_t^1 = L_1 Y_t,\ \gamma_t^2 = L_2 Y_t.$$

As explained in the previous section, the system 'decouples' with $\gamma_t^2 = 0$ for $t \geq 0$:

$$L_2 Y_t = 0,\ \text{all } t, \tag{6.5}$$

and

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = 0, \tag{6.6}$$

or,

$$\gamma_{t+1}^1 = -G_0^{-1} G_1 \gamma_t^1 = \Pi \gamma_t^1.$$

Interestingly, the variables, γ_t^1, are complex and so is Π :

$$\Pi = \begin{bmatrix} 0 & 0 & -0.25+0.0004i & 0.59-0.30i & 0.32-0.29i \\ 0 & 0 & -0.55+0.0008i & 0.26+0.34i & -0.65-0.28i \\ 0 & 0 & 0.35 & 0.27-0.47i & 0.70-0.03i \\ 0 & 0 & 0 & 1.15-0.31i & 0.31+0.19i \\ 0 & 0 & 0 & 0 & 1.15+0.31i \end{bmatrix}$$

Still, we have no problem concluding that the class of solutions here is given by

$$\gamma_t^1 = \Pi^t \gamma_0^1, \tag{6.7}$$

where γ_0^1 has 2 'free parameters'. These are composed of the first three elements of Y_0, net of the one restriction on Y_0 implied by (6.5). Thus, (6.5) represents a 2 parameter space of solutions. The eigenvector-eigenvalue decomposition of Π,

$$\Pi = P\Lambda P^{-1}, \tag{6.8}$$

plays an important role in the dynamics of the solutions, (6.7). The eigenvalues of Π (i.e., the 5 terms on the diagonal of the diagonal matrix, Λ) are:

$$0, 0, 0.35, 1.15 \pm 0.31i.$$

Note that there are two explosive eigenvalues and three non-explosive. Also, although there are two repeating eigenvalues, it is nevertheless the case that the eigenvector-eigenvalue decomposition, (6.8) exists in this example.

We can compute candidate MSV solutions for the system as follows. Choose the 2×5 matrices, B, such that $B\gamma_0^1 = 0$ and B is composed of left eigenvectors of π. There are 10 ways to construct a B matrix in this way and each corresponds to a candidate MSV solution. This is because for any B constructed in this way we can construct a 3×6 matrix, D such that $DY_0 = 0$, from

$$D = \begin{bmatrix} BL_1 \\ L_2 \end{bmatrix}.$$

Whether a given candidate MSV solution is an actual solution requires that there exist a 3×3 real matrix A such that

$$A = -D_1^{-1} D_2,$$

where

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

That is, we require that D_1 be invertible and that $D_1^{-1} D_2$ be real. In this case,

$$z_t = A z_{t-1}$$

is an actual MSV solution.

It is easy to confirm that in each of the 10 candidate MSV solutions, D_1 is invertible. However, in only four cases $-D_1^{-1}D_2$ is real. Thus, among the 10 candidate MSV's there are just four actual MSV's. These are A_1, A_2, A_3, A_4 :

$$\underbrace{A_1}_{(0.35,1.15\pm0.31i)} = \begin{bmatrix} 1.01 & -0.3 & 0. \\ -1.01 & 1.30 & 1.0 \\ 0.92 & -0.43 & 0.35 \end{bmatrix}, \quad \underbrace{A_2}_{(0,1.15\pm0.31i)} = \begin{bmatrix} -61.85 & 0 & 45.18 \\ 269.30 & 0 & -193.28 \\ -87.87 & 0 & 64.16 \end{bmatrix},$$
$$\underbrace{A_3}_{(0,1.15\pm0.31i)} = \begin{bmatrix} 0 & -0.30 & 0.73 \\ 0 & 1.30 & 0.27 \\ 0 & -0.42 & 1.01 \end{bmatrix} \underbrace{A_4}_{(0,0,0.35)} = \begin{bmatrix} 0 & 0 & -0.34 \\ 0 & 0 & -0.74 \\ 0 & 0 & 0.31 \end{bmatrix},$$

where numbers in parentheses beneath A_j are the three eigenvalues of A_j, $j = 1, ..., 4$.

Note that the MSV solutions all look very different. The entries in A_2 are a couple of orders of magnitude different in size from the corresponding entries in the other A matrices. Note, too, that there is only one A matrix which has all its eigenvalues less than unity in absolute value. The latter is to be expected. There are two explosive eigenvalues in the decoupled system, (6.6). The fourth MSV extinguishes both eigenvalues by working with B constructed using the two associated left eigenvectors of Π.

An interesting special case of the model sets $\rho = 0$. In this case, (6.3) reduces to:

$$(\alpha_0 A + \alpha_1) A = 0.$$

We can see one MSV solution right away, without using the technology based on the QZ decomposition. In particular,

$$A = 0$$

represents a solution to the system. When we apply the QZ decomposition approach to this case, we find - like when $\rho \neq 0$ - that there are 10 candidate MSV's, but only 4 actual MSV's. Among these four MSV's, only the one in which $A = 0$ has all eigenvalues less than unity.

Another interesting special case occurs when $\beta = 0.8$ and ρ is held at its benchmark value of 0.5. In this case, Π has only one explosive eigenvalue. Thus, we can expect that there are many non-explosive solutions, possibly even many non-explosive MSV's. As before, there are 10 candidate MSV's. It turns out that each one is an actual MSV because each satisfies the invertibility condition and the requirement that A be real. A puzzling feature of the set of MSV's is that it does not include $A = 0$. Finally, four MSV's are non-explosive. It is of interest to display these:

$$A_1 = \begin{bmatrix} 0 & 0 & -0.40 \\ 0 & 0 & -0.85 \\ 0 & 0 & 0.38 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0.54 & 0.82 \\ 0 & 0.46 & 0.18 \\ 0 & 0.20 & 0.82 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 & 1.04 \\ 0 & 0 & 0.37 \\ 0 & 0 & 0.90 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0.65 & 0 & 0.29 \\ 0.54 & 0 & -0.26 \\ 0.24 & 0 & 0.63 \end{bmatrix}.$$

Since each matrix is different and each has only non-explosive eigenvalues, each represents a valid equilibrium. One may attempt to impose an equilibrium selection. For example, based on the economics of the model one might suppose it 'implausible' for the state to include lagged inflation or lagged output. But, two of the MSV solutions satisfy this plausibility criterion because the first two columns of A_1 and A_2 are zero. Another possible criterion is that A should 'resemble' the A computed when $\beta = 1.5$ and there is a unique, non-explosive solution. This criterion selects A_1 as the solution. However, it is not clear why this selection is appealing.

7. Sunspots in the New Keynesian Model that Does Not Satisfy the Taylor Principle

We now consider the version of (6.1) in which $0 < \beta < 1$, so that the monetary policy rule does not satisfy the Taylor principle. As a result, the steady state equilibrium is indeterminate and there exist sunspot equilibria. The intuition for this result is straightforward in the case $\rho = \gamma = 0$. Consider the following temporary deviation from the steady state equilibrium (i.e., the one in which all variables are constant). In the deviation, agents expect a higher inflation rate. Because $0 < \beta < 1$ the monetary authority raises r_t, but by less than the rise in expected inflation. Anticipating a decline in the real rate of interest, agents increase spending and this leads to an increase in output and marginal cost and, hence, actual inflation. In this way, the higher expected inflation is self-fulfilling. This logic suggests that there exist other equilibria in a neighborhood of the steady state equilibrium.

Because we wish to allow for the possibility of sunspot equilibria, we consider a version of (6.1) in which future variables are replaced by their expectation:

$$\begin{aligned} \delta E_t \pi_{t+1} + \lambda y_t - \pi_t &= 0 \\ E_t y_{t+1} - y_t - \frac{1}{\sigma}(r_t - E_t \pi_{t+1}) &= 0 \\ \rho r_{t-1} + (1-\rho)\beta E_t \pi_{t+1} + (1-\rho)\gamma y_t - r_t &= 0. \end{aligned} \tag{7.1}$$

We define

$$\begin{aligned} \pi_{t+1} &= E_t \pi_{t+1} + \eta_{t+1} \\ y_{t+1} &= E_t y_{t+1} + \psi_{t+1}, \end{aligned}$$

where η_{t+1} and ψ_{t+1} denote the unexpected components in π_{t+1} and y_{t+1}, respectively. The only restriction on η_{t+1} and ψ_{t+1} is that they be unpredictable as of time t.

If we mechanically follow the strategy of representing the equilibrium conditions in matrix form which was developed above (e.g., (3.1)), then the representation of (7.1) puts us in what we have called the 'non-invertible a case' (see, e.g., (6.2)). Given that we want to allow for the possibility of sunspots, this is somewhat more complicated to work with than if the a matrix in the VAR(1) representation of the equilibrium conditions were invertible. So, in the first section below, we adopt the 'trick' suggested in Clarida, Gali and Gertler (NBER working paper 6442), which puts the model in the invertible a form. Although this approach is conceptually more straightforward, it is somewhat idiosyncratic because it depends on a trick. So, in the second subsection below we apply our general strategy, which uses the QZ decomposition to put the system, (7.1), into the invertible a case.

7.1. Invertible a

So we instead represent the system in a slightly different format.[12] Define

$$Y_t = \begin{pmatrix} \pi_t \\ y_t \\ r_{t-1} \end{pmatrix}.$$

We assume that the system starts up in period 0, when the third element of Y_0 is given. The other two elements, π_0 and y_0, are to be determined. We write the system in the format of (4.5) as follows:

$$\begin{bmatrix} \delta & 0 & 0 \\ \frac{1}{\sigma} & 1 & -\frac{1}{\sigma} \\ (1-\rho)\beta & 0 & -1 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \\ r_t \end{pmatrix} + \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & 0 \\ 0 & (1-\rho)\gamma & \rho \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ r_{t-1} \end{pmatrix} = \begin{pmatrix} \delta\eta_{t+1} \\ \psi_{t+1} + \frac{1}{\sigma}\eta_{t+1} \\ (1-\rho)\beta\eta_{t+1} \end{pmatrix},$$

or,

$$aY_{t+1} + bY_t = \omega_{t+1},$$

in obvious notation. The system is now in the 'invertible a' form. With obvious modifications, the analysis after (4.5) can be applied. Thus, we can express a solution of the system as follows:

$$\tilde{Y}_{it} = \lambda_i^t \tilde{Y}_{i,0} + v_{i,t} + \lambda_i v_{i,t-1} + \cdots + \lambda_i^{t-1} v_{i,1}, \;\; v_{i,t} \equiv \tilde{P}_i a^{-1} \omega_t,$$

for $i = 1, 2, 3$. As before, $\Pi = P\Lambda P^{-1}$, where Λ is a diagonal matrix with the three eigenvalues of the system on the diagonal and the three columns of P are the right eigenvectors of Π. Also, $\tilde{P} \equiv P^{-1}$ and the three rows of $\tilde{P}$ are the left eigenvectors of Π. Finally, $\tilde{Y}_{i,t}$ denotes the i^{th} element of the column vector, $\tilde{P}Y_t$. Note that:

$$a^{-1}\omega_{t+1} = \begin{bmatrix} \frac{1}{\delta} & 0 & 0 \\ -\frac{1}{\sigma\delta}(\beta\rho - \beta + 1) & 1 & -\frac{1}{\sigma} \\ \frac{1}{\delta}(\beta - \beta\rho) & 0 & -1 \end{bmatrix} \begin{pmatrix} \delta\eta_{t+1} \\ \psi_{t+1} + \frac{1}{\sigma}\eta_{t+1} \\ (1-\rho)\beta\eta_{t+1} \end{pmatrix} = \begin{pmatrix} \eta_{t+1} \\ \psi_{t+1} \\ 0 \end{pmatrix}.$$

This expression is not surprising, because the object, $a^{-1}\omega_{t+1}$, is the one-step-ahead forecast error in Y_{t+1} given Y_t. We can see this by premultiplying the representation for Y_t by a^{-1} :

$$Y_{t+1} = \Pi Y_t + a^{-1}\omega_{t+1}.$$

The third element of $a^{-1}\omega_{t+1}$ is zero because the third element of Y_{t+1} is known at time t and so it has no forecast error.

We adopt the parameterization in (6.4), except that now $\beta = 0.5$. We have

$$\Pi = \begin{bmatrix} 1.0101 & -0.3030 & 0 \\ -0.7576 & 1.3023 & 0.5000 \\ 0.2525 & -0.0008 & 0.5000 \end{bmatrix} \tag{7.2}$$

$$\Lambda = \begin{bmatrix} 1.6216 & 0 & 0 \\ 0 & 0.8029 & 0 \\ 0 & 0 & 0.3879 \end{bmatrix}, \; \tilde{P} = \begin{bmatrix} -0.8034 & 0.7617 & 0.3395 \\ 0.6527 & 0.3971 & 0.6555 \\ -0.6427 & -0.2122 & 0.9468 \end{bmatrix}$$

[12] We follow the approach in Clarida, Gali and Gertler (NBER working paper 6442).

Note that we have one explosive eigenvalue. Non-explosiveness requires suppressing that eigenvalue, or,

$$\tilde{p}Y_0 = 0,\ v_{1,t} = 0 \text{ for } t = 1, 2, ...,$$

where $\tilde{p}$ is $\tilde{P}_1$, the first row of $\tilde{P}$. Also, $v_{1,t} \equiv \tilde{p}a^{-1}\omega_t$.[13] The latter requires

$$0.8034\eta_t = 0.7617\psi_t,\ t = 1, 2, ...,$$

where the elements of $\tilde{p}$ have been rounded. The requirement, $\tilde{p}Y_0 = 0$, can be accomplished in a variety of ways. Because the non-explosive eigenvalues die out it does not matter much how the two degrees of freedom in Y_0 are used to accomplish $\tilde{p}Y_0 = 0$. For convenience we do so by setting $Y_0 = 0$.

To do a sunspot simulation of length T periods, draw $\psi_1, \psi_2, ..., \psi_T$ from any distribution with the property, $E_t\psi_{t+1} = 0$. For our purposes the standard Normal distribution is good enough. Then, to impose $\tilde{p}a^{-1}\omega_t = 0$ set η_t as follows:

$$\eta_t = \frac{0.7617}{0.8034} \times \psi_t = 0.948 \times \psi_t,\ t = 1, ..., T. \tag{7.3}$$

By constructing the η_t's in this way, we ensure that $v_{1,t} = 0$, $t = 1, 2, ...\ T$. Then,

$$Y_t = \Pi Y_{t-1} + a^{-1}\omega_t = \Pi Y_{t-1} + \begin{pmatrix} \eta_t \\ \psi_t \\ 0 \end{pmatrix}.$$

for $t = 1, 2, ..., T$, where η_t satisfies (7.3).

Mathematically, the preceding simulation algorithm works. However, in practice the explosive eigenvalue will eventually make its appearance in a simulation that is long enough. An alternative strategy simulates the rotated system, $\tilde{Y}_t \equiv \tilde{P}Y_t$ and then 'unwinds' the $\tilde{Y}_t$'s at the end, that is, $Y_t = P\tilde{Y}_t$. The rotated system is given by (4.7), which is expressed in recursive form as follows:

$$\begin{aligned} \tilde{Y}_t &= \Lambda\tilde{Y}_{t-1} + P^{-1}a^{-1}\omega_t \\ &= \Lambda\tilde{Y}_{t-1} + \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{bmatrix} \begin{pmatrix} \eta_t \\ \psi_t \\ 0 \end{pmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{Y}_{t-1} + \begin{bmatrix} 0 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{bmatrix} \begin{pmatrix} \eta_t \\ \psi_t \\ 0 \end{pmatrix}, \end{aligned}$$

where the replacement of $\tilde{P}_1$ is designed to enforce $\tilde{P}_1\omega_t = 0$ and the replacement of λ_1 by 0 is designed to enforce the orthogonality, $\tilde{p}Y_t = 0$ for all t. In an example, we set $T = 20,000$, and found that the correlation between y_t and π_t is 0.98, 0.92, 0.81, 0.17 for $\beta = 0.5, 0.8,$ 0.9, 0.99, respectively.

[13] Here, we follow the previous convention in which $\tilde{p}$ is composed of the rows of $\tilde{P}$ which are associated with the explosive eigenvalues of Π. In the present example, $\tilde{p}$ simply a row vector.

7.2. Non invertible a

Now, the equilibrium conditions are represented in a stochastic version of (6.2):

$$\begin{bmatrix} \delta & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 \\ (1-\rho)\beta & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \\ r_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & -\frac{1}{\sigma} \\ 0 & (1-\rho)\gamma & -1 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ r_t \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \\ r_{t-1} \end{pmatrix} = \xi_{t+1}, \tag{7.4}$$

where

$$\xi_{t+1} \equiv \begin{pmatrix} \delta\eta_{t+1} \\ \frac{1}{\sigma}\eta_{t+1} + \psi_{t+1} \\ (1-\rho)\beta\eta_{t+1} \end{pmatrix}. \tag{7.5}$$

Expressing (7.4) in our canonical notation,

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = \xi_{t+1},$$

which is expressed in first order vector form as follows:

$$aY_{t+1} + bY_t = \omega_{t+1},$$

where

$$\omega_{t+1} = \begin{pmatrix} \xi_{t+1} \\ 0 \end{pmatrix}. \tag{7.6}$$

We transform the first order vector representation using the QZ decomposition as follows:

$$H_0\gamma_{t+1} + H_1\gamma_t = Q\omega_{t+1}, \tag{7.7}$$

where, as before,

$$H_0 = QaZ,\ H_1 = QbZ,\ \gamma_t = Z'Y_t.$$

Here, $m = 6$ and the rank of a is 5 so that $l = 1$. Also,

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -0.2524 & -0.7785 & -0.0637 & 0 & 0 & 0.5710 \\ 0.3886 & 0.4509 & 0.0981 & 0 & 0 & 0.7975 \\ 0.8517 & -0.4365 & 0.2151 & 0 & 0 & -0.1946 \\ -0.2448 & 0 & 0.9696 & 0 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & -0.3017 & 0.6217 & 0.6884 & -0.2203 \\ 0 & 0 & -0.6195 & 0.4282 & -0.6579 & 0.0007 \\ 0 & 0 & 0.2621 & 0.4119 & 0.0221 & 0.8724 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6757 & 0.5104 & -0.3044 & -0.4362 \end{bmatrix}.$$

Consider $Q^2\omega_{t+1}$ (this is discussed in section 5.2), where Q^2 corresponds to the last equation in Q. Note that $Q^2\omega_{t+1} = 0$ for all t, since

$$\begin{aligned}
& \left[\begin{array}{cccccc} -0.2448 & 0 & 0.9696 & 0 & 0 & 0 \end{array}\right] \xi_{t+1} \\
= & \left[\begin{array}{ccc} -0.2448 & 0 & 0.9696 \end{array}\right] \left(\begin{array}{c} 0.99 \times \eta_{t+1} \\ \eta_{t+1} + \psi_{t+1} \\ 0.25 \times \eta_{t+1} \end{array}\right) \\
= & \left[-0.2448 \times 0.99 + 0.9696 \times 0.25\right] \eta_{t+1} \\
= & \ 0.
\end{aligned}$$

Recall that

$$\gamma_t^1 = L_1 Y_t,\ \gamma_t^2 = L_2 Y_t,\ L = \left[\begin{array}{c} L_1 \\ L_2 \end{array}\right] = Z',$$

where L_1 is composed of the upper $m - l = 5$ rows of Z' and L_2 is composed of the bottom l rows (in this case, $l = 1$) of Z'. Also, γ_t^i, $i = 1, 2$ are $(m - l) \times 1$ and $l \times 1$ vectors, respectively.

Multiplying (7.7) by Q and taking into account $\gamma_t^2 = 0$ for all t,[14] we have that (7.7) reduces to

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = Q^1 \omega_{t+1}, \tag{7.8}$$

where Q^1 is the first $m - l$ rows of Q. With two exceptions, the system, (7.8), is a straight application of our discussion of the 'invertible a case' in 4. One exception is that there are only two degrees of freedom in how we select ξ_t, whereas in the invertible a discussion we assumed the number of degrees of freedom in ξ_t is equal to its dimension, which in this case is 3. The second exception has to do with the number of degrees of freedom in γ_0^1. Since $\gamma_0^1 = L_1 Y_0$, it might at first appear that the number of degrees of freedom in γ_0^1 is equal to the dimension of z_0, which is 3. But, we have an additional restriction on Y_0, which stems from $\gamma_t^2 = 0$ for all t, so that $L_2 Y_0 = 0$. The requirement that z_0 also satisfy this restriction reduces the number of degrees of freedom in γ_0^1 to two.

Thus, there are two degrees of freedom in computing a solution, $\{\gamma_t^1\}$, to (7.8): there are two choices available in γ_0^1 and two shocks in ξ_{t+1}, in (7.6). In the usual way, we try to absorb these degrees of freedom by limiting ourselves to non-explosive solutions. To this end, we examine the matrix, Π :

$$\Pi = -G_0^{-1} G_1 = P \Lambda P^{-1},$$

where

$$0, 0, 0.3879, 1.6216, 0.8029,$$

lie on the diagonal of the diagonal matrix, Λ. Non-explosiveness requires suppressing the explosive eigenvalue. This eliminates one of the degrees of freedom in γ_0^1, but not both. Since γ_0^1 does not have a substantial impact on the simulations when the explosive eigenvalue has been suppressed, we arbitrarily set $\gamma_0^1 = 0$.

Consistent with the discussion in 4, suppressing the explosive eigenvalue also pins down one of the free elements of ξ_{t+1}. To see how, rewrite (7.8):

$$\gamma_{t+1}^1 = \Pi \gamma_t^1 + G_0^{-1} Q^1 \omega_{t+1}.$$

[14] For the latter, recall the discussion in section 5.2.

Premultiplying by P^{-1} and defining $\tilde{\gamma}_t^1 \equiv P^{-1}\gamma_t^1$, we obtain

$$\tilde{\gamma}_{t+1}^1 = \Lambda \tilde{\gamma}_t^1 + P^{-1}\nu_{t+1}, \tag{7.9}$$

where

$$\nu_{t+1} \equiv G_0^{-1} Q^1 \omega_{t+1}.$$

Suppressing the explosive eigenvalue requires setting $\tilde{P}_4 \nu_t \equiv 0$, where $\tilde{P}_4$ is the fourth row of P^{-1}. We have

$$\begin{aligned}
\tilde{P}_4 \nu_t &= \tilde{P}_4 G_0^{-1} Q^1 \omega_t \\
&= \tilde{P}_4 G_0^{-1} Q^1 \begin{pmatrix} \delta \eta_{t+1} \\ \frac{1}{\sigma}\eta_{t+1} + \psi_{t+1} \\ (1-\rho)\beta \eta_{t+1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{bmatrix} 1.7602 & -0.9021 & 0.4445 \end{bmatrix} \begin{pmatrix} 0.99 \times \eta_{t+1} \\ \eta_{t+1} + \psi_{t+1} \\ 0.25 \times \eta_{t+1} \end{pmatrix} \\
&= \left[1.7602 \times 0.99 + 0.4445 \times 0.25 - 0.9021\right] \eta_{t+1} - 0.9021 \psi_{t+1},
\end{aligned}$$

so that

$$\eta_t = \frac{0.9021}{1.7602 \times 0.99 + 0.4445 \times 0.25 - 0.9021} \psi_t = 0.948 \times \psi_t.$$

Note that, as expected, the same relationship exists between η_t and ψ_t exists in this way of solving the system as we found in the previous subsection.

We can now simulate the system as follows. First, draw an iid sequence, $\psi_1, ..., \psi_T$, from a random number generator. Then, compute $\eta_1, ..., \eta_T$ using the previous expression. Given the ψ_t's and the η_t's, construct ω_t's using (7.5) and (7.6). Then, generate $\tilde{\gamma}_t^1$ using

$$\tilde{\gamma}_{t+1}^1 = \tilde{\Lambda} \tilde{\gamma}_t^1 + M G_0^{-1} Q^1 \omega_{t+1},$$

where $\tilde{\Lambda}$ is the matrix, Λ, with the fourth diagonal element replaced by 0 and M is P^{-1} with the fourth row replaced by a row of 0's. Zeroing out the explosive eigenvalue in Λ and the fourth row in P^{-1} is mathematically correct and ensures that the explosive eigenvalue cannot emerge as a result of rounding error. After generating $\tilde{\gamma}_1^1, ..., \tilde{\gamma}_T^1$, we obtain γ_t^1 from $\gamma_t^1 = P\tilde{\gamma}_t^1$, for $t = 1, 2, ..., T$. Finally, we obtain Y_t from

$$Y_t = Z \begin{pmatrix} \gamma_t^1 \\ 0 \end{pmatrix}.$$

If the calculations are done using the same sequence of realizations of $\psi_1, ..., \psi_T$ as in the previous subsection, then we also obtain the same sequence of realizations of π_t, y_t and r_t.