Topics in Quantitative Macroeconomics, Fall 1998
D16
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Some (Corrected) Notes on Solving A Time to Build Model by Undetermined Coefficients (Elaboration on Example 4 in Christiano (1998).)

1 Model.

Consider a two period time-to-build version of the economy in homework #1 with a technology shock. In this economy, resources must be devoted for two periods, rather than just one, to augmenting the stock of capital. A planner maximizes $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$, subject to the resource constraint, $c_t + I_t \leq f(k_t, n_t, \theta_t)$, where θ_t has the following distribution:

$$\theta_t = \rho\theta_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim N(0, \sigma_\varepsilon),$$

and

$$f(k_t, n_t, \theta_t) = A_t k_t^\alpha \left[n_t \exp(\theta_t)\right]^{1-\alpha},\ \ \alpha = 0.36,\ \ A_t = Y_t^\gamma.$$

Also, I_t denotes investment, which is composed of the resources devoted to projects initiated in the current period and resources devoted to projects initiated in the previous period:

$$I_t = \phi x_t + (1-\phi)x_{t-1}. \tag{1}$$

The investment technology requires that if x_t units of gross investment is to occur during period $t+1$, i.e.,

$$k_{t+2} - (1-\delta)k_{t+1} = x_t,\ \ \delta = 0.028, \tag{2}$$

then resources in the amount ϕx_t must be applied in period t and $(1-\phi)x_t$ must be applied in period $t+1$, where $0 \leq \phi \leq 1$. In (2), δ is the rate of depreciation on a unit of capital. Once an investment project is initiated, its scale cannot be expanded or contracted. As a result, x_{t-1} (hence, k_{t+1}) is a state variable at time t, in addition to k_t and θ_t. Since the date t choice variable is k_{t+2}, the'minimal state variable' policy rule is $k_{t+2} = g(k_{t+1}, k_t, \theta_t)$.

2 Linearized Euler Equation

In solving the problem, it is convenient to substitute out x_t and I_t. Thus, substituting out I_t in the resource constraint using (1) and (2), we get:

$$c_t = f(k_t, n_t, \theta_t) - (\phi_1 k_{t+2} + \phi_2 k_{t+1} + \phi_3 k_t),$$

where

$$\phi_1 = \phi,\ \phi_2 = (1-\phi) - \phi(1-\delta),\ \phi_3 = -(1-\phi)(1-\delta).$$

The planner's problem is:

$$\max_{\{n_t, k_{t+2}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(f(k_t, n_t, \theta_t) - (\phi_1 k_{t+2} + \phi_2 k_{t+1} + \phi_3 k_t), n_t\right),$$

which leads to the following intertemporal Euler equation:

$$\phi_1 u_{c,t} + \phi_2 \beta u_{c,t+1} - \beta^2 u_{c,t+2}\left[f_{k,t+2} - \phi_3\right] = \zeta_{t+2},$$

or,

$$v^k(k_{t+4}, k_{t+3}, k_{t+2}, k_{t+1}, k_t, n_{t+2}, n_{t+1}, n_t, \theta_{t+2}, \theta_{t+1}, \theta_t) = \zeta_{t+2},$$

where

$$E\left[\zeta_{t+2} | -_t\right] = 0,\ -_t = \left\{k_{t+2-j}, c_{t-j}, n_{t-j}, \theta_{t-j}; j \geq 0\right\}.$$

Here, $u_{c,t}$ is the marginal utility of consumption. The dating, $t+2$, on ζ_{t+2} reflects the date when that variable is realized.[1] The intratemporal Euler equation can be solved in the same way as before:

$$v^n(k_{t+2}, k_{t+1}, k_t, n_t, \theta_t) = 0.$$

Define:

$$z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix},\ z_{1t} = \begin{bmatrix} k_{t+2} - k \\ n_t - n \end{bmatrix},\ z_{2t} = z_{1,t-1},$$

[1]Note that

$$\zeta_t \in -_t,\ \text{but}\ \zeta_{t+1} \notin -_t.$$

So, unlike in the one-period time to build case, the fact that ζ_{t+2} is orthogonal to the information set, $-_t$, does not imply that ζ_{t+2} is orthogonal to ζ_{t+1}. Thus,

$$cov(\zeta_{t+2}, \zeta_t) = 0,\ cov(\zeta_{t+2}, \zeta_{t+1}) \neq 0.$$

where k and n denote the steady state values of capital and labor, respectively. Here, z_{1t} are the date t endogenous variables that are to be determined. The linearized Euler equations can be written:

$$E_t\left[\alpha_0 z_{t+2} + \alpha_1 z_{t+1} + \alpha_2 z_t + \alpha_3 z_{t-1} + \beta_0 \theta_{t+2} + \beta_1 \theta_{t+1} + \beta_2 \theta_t\right] = 0, \tag{3}$$

where the α_i's are 2×4 matrices and the β_i's are 2×1 vectors. The right 2 columns of α_i, $i = 0, \ldots, 2$ are composed of zeros:

$$\alpha_i = [\underset{2\times 2}{\tilde{\alpha}_i} \vdots \underset{2\times 2}{0}],$$

for $i = 1, 2$. The first of the two equations in (3) corresponds to the intertemporal Euler equation, and the second, to the intratemporal one.

3 A Solution to the Linearized Euler Equation

A (minimal state) solution to the Euler equations is nested in:

$$z_t = A z_{t-1} + B \theta_t, \tag{4}$$

where A is a 4×4 matrix, and B is 4×1. There are restrictions on A, reflecting the identities linking z_t and z_{t-1}:

$$A = \left[\begin{array}{c} A_1 \\ \cdots \\ A_2 \end{array}\right],\ A_2 = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Here, A_1 is a 2×4 matrix of coefficients to be determined. Also,

$$B = \left[\begin{array}{c} B_1 \\ \cdots \\ B_2 \end{array}\right],$$

where B_1 is a 2×1 vector of coefficients to be determined and B_2 is a 2×1 vector of zeros.

4 Assigning Values to the Undetermined Coefficients

We can determine the unknown coefficients in A and B as follows. We know that A and B must have the property that they set (3) to zero. To see how this restricts A and B, simply substitute these into (3). For this, it is useful to first note that (4) implies:

$$z_{t+2} = A^3 z_{t-1} + A^2 B\theta_t + AB\theta_{t+1} + B\theta_{t+2},$$

or

$$z_{t+2} = A^3 z_{t-1} + A^2 B\theta_t + AB\rho\theta_t + B\rho^2\theta_t + \text{future } \varepsilon_t\text{'s}$$

Similarly,

$$\begin{aligned} z_{t+1} &= A^2 z_{t-1} + AB\theta_t + B\rho\theta_t + B\varepsilon_{t+1} \\ z_t &= Az_{t-1} + B\theta_t \end{aligned}$$

Substituting this into (3), we obtain:

$$E_t\left[\left(\alpha_0 A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3\right) z_{t-1} + F\theta_t + \text{future } \varepsilon_t\text{'s}\right] = 0, \tag{5}$$

where

$$F = \alpha_0(A^2 B + AB\rho + B\rho^2) + \alpha_1(AB + B\rho) + \alpha_2 B + \beta_0\rho^2 + \beta_1\rho + \beta_2. \tag{6}$$

Writing out (5), we get:

$$\alpha(A)z_{t-1} + F\theta_t = 0,$$

which, if this is to hold for all z_{t-1}, θ_t, requires:

$$\alpha(A) = 0_{2\times 4},\ \ F = 0_{2\times 1}.$$

We have 10 equations here. We have 8 unknowns in A (i.e., the elements of A_1) and 2 in B (i.e., the 2 elements in B_1). There is some hope that these 10 equations can be used to pin down the 10 unknowns. Finding A corresponds to finding the matrix zero of a matrix polynomial. We will use the state-space formulation to do this. Finding B conditional on A involves solving a linear system of equations.

4.1 Finding A

Consider A first. Since A and B in (4) are independent of σ_ε, we can find A by studying the deterministic version of the model. Accordingly, suppose $\sigma_\varepsilon = 0$, so that $\theta_t \equiv 0$. We can find A by setting up the deterministic version of the model in state space form:

$$aY_{t+1} + bY_t = 0, \;\; t = 0, 1, 2, ..., \tag{7}$$

where

$$Y_t = \begin{pmatrix} z_{1,t+1} \\ z_{1,t} \\ z_{t-1} \end{pmatrix}. \tag{8}$$

Also,

$$a = \begin{bmatrix} \tilde{\alpha}_0 & 0_{2\times 6} \\ 0_{6\times 2} & I_6 \end{bmatrix}, \; b = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 & \alpha_3 \\ -I_2 & 0_{2\times 2} & 0_{2\times 4} \\ 0_{4\times 2} & b_1 & b_2 \end{bmatrix}, \tag{9}$$

where

$$b_1 = \begin{bmatrix} -I_2 \\ 0_{2\times 2} \end{bmatrix}, \; b_2 = \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ -I_2 & 0_{2\times 2} \end{bmatrix}.$$

Finally,

$$z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}, \; z_{1t} = \begin{bmatrix} k_{t+2} - k \\ n_t - n \end{bmatrix}, \; z_{2t} = z_{1,t-1},$$

Here, the top four elements of Y_0 are free. Note that a is not invertible, so that the QZ decomposition must be used. One can find a minimal state solution as we discussed in class, by forming the 4×8 matrix D such that $DY_t = 0$ for $t = 0, 1, 2, ...$. Then, write

$$DY_t = D_1 \begin{pmatrix} z_{1,t+1} \\ z_{1,t} \end{pmatrix} + D_2 z_{t-1},$$

where D_1 is 4×4 and D_2 is 4×4. The matrix A_1 is formed from the third and fourth rows of $-D_1^{-1}D_2$. To verify that this is indeed the case, one needs only evaluate the matrix polynomial, α, at A and verify $\alpha(A) = 0$.

4.2 Finding B

Now consider the problem of computing B given A, from the equations, $F = 0$. Equation (6) can be written:

$$F = F_1 B + F_2 = 0,$$

where F_1 is 2×4 and F_2 is 2×1 :

$$\begin{aligned} F_1 &= \alpha_0 A^2 + \alpha_0 A \rho + \alpha_0 \rho^2 + \alpha_1 A + \alpha_1 \rho + \alpha_2, \\ F_2 &= \beta_0 \rho^2 + \beta_1 \rho + \beta_2. \end{aligned}$$

But, $B_2 = 0$, so this reduces to:

$$F_1^1 B_1 + F_2 = 0,$$

where F_1^1 denotes the first two columns of F_1. Then,

$$B_1 = -\left(F_1^1\right)^{-1} F_2.$$

As a check, it is always a good idea to directly verify that $F = 0$, where F is constructed as in (6).