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FINC 520, Spring 2008
Final Exam.

This is a closed book exam. Points associated with each question are provided in parentheses. Good luck!

1. Prove the law of iterated mathematical expectations:

$$Ex = E\left[Ex|y\right],$$

where x and y are two random variables with joint density, $f\left(x,y\right)$.

2. Define a martingale difference sequence and prove that it is serially uncorrelated.

3. Consider the process:

$$y_t = \rho y_{t-1} + u_t,$$

where

$$\begin{aligned} u_t &= \varepsilon_t \sigma_t, & (1) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2. & (2) \end{aligned}$$

Here, ε_t is iid over time with $\varepsilon_t \tilde{} N\left(0,1\right)$ and ε_t is independent of u_{t-s}, $s>0$. Also, $0<\alpha_1+\beta<1$, $0<\beta<1$.

(a) Derive the non-linear function that allows one to recover σ_t^2 from $u_2, u_3, ..., u_{t-1}$ and σ_2^2, for $t=3,...,T$. Show that, for t sufficiently large,

$$\sigma_t^2 = f\left(u_{t-1}, u_{t-2}, ...\right),$$

and derive the expression for f.

ans:

$$\begin{aligned} \sigma_3^2 &= \alpha_0 + \alpha_1 u_2^2 + \beta\sigma_2^2 \\ \sigma_4^2 &= \alpha_0 + \alpha_1 u_3^2 + \beta\left[\alpha_0 + \alpha_1 u_2^2 + \beta\sigma_2^2\right] \\ &= \alpha_0 + \alpha_1 u_3^2 + \beta\alpha_1 u_2^2 + \beta\left[\alpha_0 + \beta\sigma_2^2\right] \\ \sigma_5^2 &= \alpha_0 + \alpha_1 u_4^2 + \beta\alpha_1 u_3^2 + \beta^2\alpha_1 u_2^2 + \beta\left[\alpha_0 + \beta\alpha_0 + \beta^2\sigma_2^2\right] \end{aligned}$$

$$
\begin{aligned}
&= \alpha_0\left[1+\beta+\beta^2\right]+\alpha_1\left[u_4^2+\beta u_3^2+\beta^2 u_2^2\right]+\beta^3\sigma_2^2 \\
&\;\;.... \\
\sigma_t^2 &= \alpha_0\left[1+...+\beta^{t-3}\right]+\alpha_1\left[u_{t-1}^2+\beta u_{t-2}^2+...+\beta^{t-3}u_2^2\right]+\beta^{t-2}\sigma_2^2 \\
&= \alpha_0\frac{1-\beta^{t-2}}{1-\beta}+\alpha_1\left[u_{t-1}^2+\beta u_{t-2}^2+...+\beta^{t-3}u_2^2\right]+\beta^{t-2}\sigma_2^2
\end{aligned}
$$

etc. Using the restriction on the magnitude of β, we conclude

$$
\sigma_t^2=\frac{\alpha_0}{1-\beta}+\alpha_1\left[u_{t-1}^2+\beta u_{t-2}^2+...\right]
$$

(b) Show that u_t is a martingale difference sequence.
answer: To verify that the process is an m.d.s., note

$$
\begin{aligned}
& E\left[u_t|u_{t-1},u_{t-2},...\right] \\
&= E\left[\varepsilon_t\sigma_t|\varepsilon_{t-1}\sigma_{t-1},\varepsilon_{t-2}\sigma_{t-2},...\right] \\
&= E\left[\varepsilon_t|\varepsilon_{t-1}\sigma_{t-1},\varepsilon_{t-2}\sigma_{t-2},...\right]E\left[\sigma_t|\varepsilon_{t-1}\sigma_{t-1},\varepsilon_{t-2}\sigma_{t-2},...\right] \\
&= 0.
\end{aligned}
$$

(c) Obtain an expression for the variance of u_t, conditional on $u_{t-1},u_{t-2},...$. Show that the unconditional variance of u_t satisfies:

$$
Eu_t^2=E\sigma_t^2.
$$

answer:The conditional variance of u_t is:

$$
\begin{aligned}
& E\left[u_t^2|u_{t-1},u_{t-2},...\right] \\
&= E\left[\varepsilon_t^2\sigma_t^2|u_{t-1},u_{t-2},...\right] \\
&= E\left[\varepsilon_t^2|u_{t-1},u_{t-2},...\right]E\left[\sigma_t^2|u_{t-1},u_{t-2},...\right] \\
&= E\left[\sigma_t^2|u_{t-1},u_{t-2},...\right] \\
&= \alpha_0+\alpha_1 u_{t-1}^2+\beta E\left[\sigma_{t-1}^2|u_{t-1},u_{t-2},...\right] \\
&= \alpha_0+\alpha_1 u_{t-1}^2+\beta\sigma_{t-1}^2,
\end{aligned}
$$

where the last equality reflects the derivations in (a) above. One implication of the third equality is, using LIME:

$$
Eu_t^2=E\sigma_t^2.
$$

(d) Prove that the (unconditional) variance of u_t has the following first order autoregressive representation:

$$Eu_t^2 = \alpha_0 + (\alpha_1 + \beta) Eu_{t-1}^2.$$

answer: Take the unconditional expectation of the expression for the conditional variance, impose LIME and:

$$E\sigma_t^2 = \alpha_0 + \alpha_1 Eu_{t-1}^2 + \beta E\sigma_{t-1}^2.$$

Then, imposing,

$$Eu_t^2 = E\sigma_t^2,$$

we obtain the result sought.

(e) Suppose we define the stochastic process, u_t, $t = 1, 2, \ldots$, as follows. Let $u_0 \tilde{} N(A, B)$. Do there exist values for A, B such that if $\sigma_0^2 = u_0^2$, the stochastic process is constructed as in (1)-(2), $\{u_t, t = 1, 2, \ldots\}$ is covariance stationary? Explain carefully.

ans: covariance stationarity requires that the mean and variance be independent of t, and that the covariance between u_t and u_{t+j} only be a function of j. The indicated covariance is zero because u_t is a m.d.s. This is true even for small t. For example,

$$E[u_1|u_0] = E[\varepsilon_1\sigma_1|\varepsilon_0\sigma_0] = E[\varepsilon_1|\varepsilon_0\sigma_0] E[\sigma_1|\varepsilon_0\sigma_0] = 0.$$

From the latter result, and LIME, we have that $Eu_1 = 0$. In terms of variance, we know that the variance evolves as follows:

$$Eu_t^2 = \alpha_0 + (\alpha_1 + \beta) Eu_{t-1}^2.$$

Thus, if

$$B = Eu_0^2 = \frac{\alpha_0}{1 - (\alpha_1 + \beta)},$$

then the variance of u_t is the same for each of $t = 1, 2, \ldots$.

$$\begin{aligned} u_t &= \varepsilon_t \sigma_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

(f) Suppose we have a (very short!) set of data, y_1, y_2, y_3, y_4. Write out the likelihood function of y_4 conditional on y_1, y_2, y_3 and σ_2^2. (Recall, the normal density for a variable, z_t, with mean μ and variance σ, is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\frac{(z-\mu)^2}{\sigma^2}\right].)$$

answer:

$$f\left(y_4 | \left[y_1, y_2, y_3\right]\right) = \frac{1}{\sqrt{2\pi\left[\alpha_0 + \alpha_1\left(y_3 - \rho y_2\right)^2 + \beta\sigma_3^2\right]}} \exp\left(-\frac{1}{2}\frac{\left(y_4 - \rho y_3\right)^2}{\left[\alpha_0 + \alpha_1\left(y_3 - \rho y_2\right)^2 + \beta\sigma_3^2\right.}\right.$$

where (from the first part of this question),

$$\sigma_3^2 = \alpha_0 + \alpha_1\left(y_2 - \rho y_1\right)^2 + \beta\sigma_2^2.$$

4. Suppose we have the following two time series representations:

$$\begin{aligned} x_t &= \rho x_{t-1} + \varepsilon_t,\ E\varepsilon_t^2 = \sigma_\varepsilon^2 \\ z_t &= \gamma z_{t-1} + u_t,\ Eu_t^2 = \sigma_u^2, \end{aligned}$$

where $|\rho|, |\gamma| < 1$ and ε_t, u_t are each mean zero and serially uncorrelated. Also, ε_t is uncorrelated with y_{t-s}, $s > 0$, and u_t is uncorrelated with z_{t-s}, $s > 0$. Finally, ε_t and u_t are uncorrelated with each other at all leads and lags. Consider

$$y_t = x_t + z_t.$$

Display a formula for computing the Wold representation of y_t for the given representations of x_t and z_t. Display an expression relating Wold errors in y_t to the ε_t's and u_t's. You may present that expression in lag-operator form.

answer: premultiply y_t by $(1-\rho L)(1-\gamma L)$:

$$(1-\rho L)(1-\gamma L)\, y_t = (1-\gamma L)\,\varepsilon_t + (1-\rho L)\, u_t = (1+\theta L)\, w_t,$$

where w_t is the Wold error in y_t and $|\theta| < 1$ is a parameter to be determined. The object to the right of the first equality has the covariance function of an MA(1). It's variance is:

$$\begin{aligned}
V_0 &= E\left[\varepsilon_t - \gamma\varepsilon_{t-1} + u_t - \rho u_{t-1}\right]\left[\varepsilon_t - \gamma\varepsilon_{t-1} + u_t - \rho u_{t-1}\right] = \left(1+\gamma^2\right)\sigma_\varepsilon^2 + \left(1+\rho^2\right)\sigma_u^2 \\
V_1 &= E\left[\varepsilon_t - \gamma\varepsilon_{t-1} + u_t - \rho u_{t-1}\right]\left[\varepsilon_{t-1} - \gamma\varepsilon_{t-2} + u_{t-1} - \rho u_{t-2}\right] = -\gamma\sigma_\varepsilon^2 - \rho\sigma_u^2 \\
V_\tau &= E\left[\varepsilon_t - \gamma\varepsilon_{t-1} + u_t - \rho u_{t-1}\right]\left[\varepsilon_{t-\tau} - \gamma\varepsilon_{t-\tau-1} + u_{t-\tau} - \rho u_{t-\tau-1}\right] = 0,\ \tau > 1.
\end{aligned}$$

We determine θ by solving

$$\begin{aligned}
V_0 &= \left(1+\theta^2\right)\sigma_w^2 \\
V_1 &= \theta\sigma_w^2
\end{aligned}$$

for θ and σ_w^2, subject to the constraint, $|\theta| < 1$. The mapping from the Wold innovations in x_t and z_t to the Wold innovation in y_t is given by

$$w_t = \frac{(1-\gamma L)\,\varepsilon_t + (1-\rho L)\,u_t}{1+\theta L}.$$

5. Consider the time series representations in the previous question. Suppose the econometrician observes $y_t = x_t + z_t$, as well as a survey of time t forecasts of y_{t+1} by people who are known to be aware of the history of past observations on x_t and z_t. That is, the econometrician has data on $\hat{y}_t^1 = E\left[x_{t+1}|x_t\right] + E\left[z_{t+1}|z_t\right] + \eta_t$. Here, η_t is a disturbance that captures the possibility that people don't accurately report their forecast. The disturbance is orthogonal to x_t and y_t at all leads and lags. The data observed by the econometrician are

$$Y_t = \begin{pmatrix} y_t \\ \hat{y}_t^1 \end{pmatrix}.$$

Express this system in the form of a state-space/observer system.

answer: Define a state as follows:

$$\begin{aligned}
\xi_t &= \begin{pmatrix} x_t \\ z_t \end{pmatrix} \\
\begin{pmatrix} x_t \\ z_t \end{pmatrix} &= \begin{bmatrix} \rho & 0 \\ 0 & \gamma \end{bmatrix}\begin{pmatrix} x_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ u_t \end{pmatrix} \\
\xi_t &= F\xi_{t-1} + v_t.
\end{aligned}$$

Then,

$$
\begin{aligned}
Y_t &= \begin{bmatrix} 1 & 1 \\ \rho & \gamma \end{bmatrix} \begin{pmatrix} x_t \\ z_t \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_t \end{pmatrix} \\
Y_t &= H'\xi_t + w_t \\
w_t &= \begin{pmatrix} 0 \\ \eta_t \end{pmatrix}, \; H' = \begin{bmatrix} 1 & 1 \\ \rho & \gamma \end{bmatrix}.
\end{aligned}
$$

6. Consider two zero-mean time series, x_t and y_t, which may be correlated. Under the null hypothesis, the standard deviation of each time series is the same. Use GMM to estimate the standard deviation of each time series and to test the null hypothesis that the two standard deviations are the same. (Hint: one of the parameters that you estimate must be the difference between the two standard deviations, and you must test the null hypothesis that this parameter is zero, and you may find the following identity convenient: $\gamma(\gamma + 2\sigma_y) = \sigma_x - \sigma_y$.)

   answer: The variance of x_t and y_t be denoted σ_x^2 and σ_y^2, respectively. Let γ denote the difference between the two standard deviations. Note

$$\gamma(\gamma + 2\sigma_y) = \sigma_x - \sigma_y.$$

   Let

$$h_t(\gamma, \sigma_y) = \begin{pmatrix} \gamma(\gamma + 2\sigma_y) - \left[(x_t)^2 - (y_t)^2\right] \\ (\sigma_y)^2 - (y_t)^2 \end{pmatrix}.$$

   Note that when evaluated at the true values of γ, σ_y,

$$Eh_t(\gamma, \sigma_y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Define

$$g_T(\gamma, \sigma_y) = \frac{1}{T}\sum_{t=1}^{T} h_t(\gamma, \sigma_y),$$

   Let $\hat{\gamma}$, $\hat{\sigma}_y$ denote the values of γ, σ_y which set $g_T(\gamma, \sigma_y) = \begin{pmatrix} 0 & 0 \end{pmatrix}'$. Then, we use the asymptotic sampling theory to determine the standard deviation of $\hat{\gamma}$. Define the 2 by 2 matrix, D :

$$D = \frac{\partial g_T(\gamma, \sigma_y)}{\partial(\gamma, \sigma_y)} = 2\begin{bmatrix} \gamma + \sigma_y & \gamma \\ 0 & \sigma_y \end{bmatrix},$$

evaluated at the true values of γ, σ_y. Denote the zero frequency spectral density of h_t, when evaluated at the true values of γ and σ_y, by

$$S = \sum_{k=-\infty}^{\infty} E h_t h'_{t-k}.$$

Then, standard GMM theory (see, e.g., Hamilton) implies

$$\sqrt{T} \begin{pmatrix} \hat{\gamma} - \gamma \\ \hat{\sigma}_y - \sigma_y \end{pmatrix} \tilde{} N(0, V),$$

where

$$V = D^{-1} S \left(D^{-1} \right)'.$$

Use the 2,2 element of $\hat{V}$, the estimated V, to for a $t-$statistic, $\hat{\gamma} / \sqrt{\hat{V}_{2,2}}$. If the absolute value of this statistic is less than 1.96, then you fail to reject the null hypothesis at the 95 percent significance level.