Christiano
FINC 520, Spring 2007
Homework 3, due Wednesday, April 25.

1. Consider the four variable VAR you estimated for the previous homework. Define the 'business cycle frequencies' as the component of data corresponding to frequencies of fluctuation between 1 and 8 years (i.e., 4 and 32 quarters). Let 'high frequencies' denote the component of data corresponding to fluctuations between 2 quarters and 1 year. Let 'low frequencies' denote the component corresponding to fluctuations longer than 8 years.

   (a) For the short rate, the predicted (i.e., using the expectations hypothesis) long rate and the actual short rate, compute and display the share of variance in the low frequencies, the business cycle frequencies and the high frequencies. How well does the expectations hypothesis do in predicting the variance decomposition, in frequency domain, of the long rate? What is the variance of the long rate predicted by the expectations hypothesis, divided by the variance of the short rate?

   (b) Instead of using the ϕ_1, ϕ_2, ϕ_3, ϕ_4 computed from the data, use $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. Thus, all the data are now a white noise (about a constant). What is the effect of this change on the distribution of variance of the short and predicted long rates across different frequency bands? What is the effect of this change on the relative variance of the predicted long rate to the variance of the short rate? Explain, using economic intuition.

   (c) Now set $\phi_2 = \phi_3 = \phi_4 = 0$, $\phi_1 = 0.999 \times I$. Repeat the calculations in (b) above. Explain your results, using economic intuition.

2. Suppose that two time series processes are related as follows:

$$y_t = \sum_{k=-\infty}^{\infty} h_j x_{t-j} + \varepsilon_t,$$

   where all variables have mean zero and ε_t is uncorrelated with x_{t-j} for all j. Evidently, the above relationship is the projection of y_t onto

$\{..., x_{-1}, x_0, x_1, ...\}$. Let

$$\Gamma_{yx,k} \equiv E y_t x_{t-k},\ \Gamma_{x,k} \equiv E x_t x_{t-k},$$

and show that

$$g_{yx}\left(e^{-i\omega}\right) = h\left(e^{-i\omega}\right) g_x\left(e^{-i\omega}\right),$$

where

$$g_{yx}\left(e^{-i\omega}\right) = \sum_{k=-\infty}^{\infty} \Gamma_{yx,k} e^{-i\omega k},\ g_x\left(e^{-i\omega}\right) = \sum_{k=-\infty}^{\infty} \Gamma_{x,k} e^{-i\omega k},\ h\left(e^{-i\omega}\right) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k}.$$

Conclude that the Fourier transform of the projection coefficients is given by:

$$h\left(e^{-i\omega}\right) = \frac{g_{yx}\left(e^{-i\omega}\right)}{g_x\left(e^{-i\omega}\right)},$$

so that the individual coefficients may be recovered from the relation:

$$h_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g_{yx}\left(e^{-i\omega}\right)}{g_x\left(e^{-i\omega}\right)} e^{i\omega k} d\omega,$$

which may be approximated by a Riemann sum.

3. (Optimal seasonal adjustment). Suppose that a time series, $\{X_t\}$, has the following representation:

$$X_t = x_t + u_t,$$

where x_t and u_t are purely indeterministic and ergodic, covariance stationary processes. Suppose that u_t is the source of seasonality in X_t. That is, the spectrum of u_t, $S_u\left(e^{-i\omega}\right)$, has much power concentrated in the seasonal frequencies (i.e., those near $\omega = 2\pi/4$ in quarterly data). The spectrum of x_t, $S_x\left(e^{-i\omega}\right)$, is smooth and does not display a peak in the seasonal frequencies. The econometrician seeks to estimate the 'seasonally adjusted data', x_t, by projecting x_t onto a complete realization of X_t (i.e., $\{..., X_{-1}, X_0, X_1, ...\}$):

$$x_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k} + v_t,$$

where v_t is uncorrelated with X_{t-k} for all k. Let $\hat{x}_t$ denote the 'seasonally adjusted' data:

$$\hat{x}_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k}.$$

(a) Derive the formula for $h\left(e^{-i\omega}\right)$ in terms of the known objects, $S_u\left(e^{-i\omega}\right)$ and $S_x\left(e^{-i\omega}\right)$.

(b) Show that $S_{\hat{x}}\left(e^{-i\omega}\right) < S_x\left(e^{-i\omega}\right)$ for all ω.

(c) Show that if $S_x\left(e^{-i\omega}\right)$ is smooth across all frequencies, while $S_u\left(e^{-i\omega}\right)$ has sharp peaks at the seasonal frequencies, then $S_{\hat{x}}\left(e^{-i\omega}\right)$ will have substantial *dips* at the seasonal frequencies. (It is ironic that optimal seasonal adjustment produces a series, $\hat{x}_t$, that itself displays seasonality.)

4. (Hodrick-Prescott filter). Suppose we a partial realization, $y_1, ..., y_T$, from a covariance stationary, indeterministic and ergodic time series, $\{y_t\}$. The HP filter solves the problem:

$$\min_{\left\{y_t^T\right\}} \sum_{t=1}^{T-1} \left\{ \left(y_t - y_t^T\right)^2 + \lambda \left[\left(y_{t+1}^T - y_t^T\right) - \left(y_t^T - y_{t-1}^T\right) \right]^2 \right\},$$

and the 'HP-filtered' data are

$$y_t^c \equiv y_t - y_t^T.$$

Suppose T is large and the start of the data set is arbitrarily far in the past.

(a) Construct the filter, $g(L)$, which expresses

$$y_t^c = g(L)\, y_t,$$

where $g(L)$ is called 'the HP filter'. (Hint: first compute the first order necessary condition for optimality satisfied by y_t^T. In lag operator form, this has the representation,

$$y_t = B(L)\, y_t^T,$$

where $B(L)$ is symmetric in positive and negative powers of L, i.e., $B(L) = B(L^{-1})$. Note that since $y_t^T = y_t - y_t^c$, this implies:

$$y_t = B(L) y_t - B(L) y_t^c,$$

or,

$$y_t^c = g(L) y_t,$$

where the HP filter, $g(L)$ is

$$g(L) = \frac{B(L) - 1}{B(L)}.$$

End of hint!)

(b) To see what the HP filter does to a time series, graph $g(e^{-i\omega})$ for $\omega \in (0, \pi)$. Note that it looks like a 'high pass filter'. That is, it looks like a band pass filter which lets higher frequencies of oscillation through and zeros out the lower frequencies. What is the (approximate) cutoff between frequencies allowed through and frequencies set to zero? Here, imagine you are working with quarterly data and $\lambda = 1600$.

5. Suppose that the data are generated by a true (scalar) autoregressive representation of the following form:

$$y_t = \phi(L) y_{t-1} + \varepsilon_t,$$

where $\phi(L)$ is a polynomial in non-negative powers of L and the polynomial coefficients are square-summable. Also, ε_t is a white noise, uncorrelated with y_{t-s}, $s > 0$. Suppose the econometrician estimates $\phi(L)$ by running a regression of y_t on p lags of itself. The econometrician is assumed to have an entire (i.e., doubly infinite) realization of data. The econometrician may commit some form of specification error, for example by choosing a value of p smaller than the true value (the true lag length may actually be infinite). By 'running a regression', the econometrician is assumed to choose coefficients, $\hat{\phi}_1, ..., \hat{\phi}_p$, for the AR polynomial, $\hat{\phi}(L)$, so that

$$y_t - \hat{\phi}(L) y_{t-1}$$

has the smallest possible variance in the (infinite!) sample.

(a) Argue carefully (be clear when you use ergodicity and covariance stationarity) that the econometrician's choice of $\hat{\phi}(L)$ solves

$$\min_{\hat{\phi}_1,\ldots,\hat{\phi}_p} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\phi\left(e^{-i\omega}\right)-\hat{\phi}\left(e^{-i\omega}\right)\right]g_y\left(e^{-i\omega}\right)\left[\phi\left(e^{i\omega}\right)-\hat{\phi}\left(e^{i\omega}\right)\right]' d\omega,$$

where $g_y\left(e^{-i\omega}\right)$ is the covariance generating function of $\{y_t\}$. Note that if the econometrician commits no specification error, then

$$\phi\left(e^{-i\omega}\right)=\hat{\phi}\left(e^{-i\omega}\right), \text{ for all } \omega\in(-\pi,\pi).$$

(b) Suppose the econometrician does commit specification error, so that the previous equality is not possible over all frequencies, $\omega\in(-\pi,\pi)$. Suppose the econometrician is particularly interested in the sum of the AR coefficients, $\phi(1)$. Explain why the econometrician's estimator of this object, $\hat{\phi}(1)$, is likely to be a good one if there is an important low-frequency component in the data, $\{y_t\}$. Alternatively, if the data are primarily driven by high frequency components, then $\phi(1)$ is likely to be badly estimated.