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FINC 352, Spring 2008
Homework 3, due Wednesday, April 30.

1. Consider the following time series representation for x_t :

$$x_t = \rho_x x_{t-1} + \varepsilon_t,\ \rho_x = 0.9,\ \sigma_\varepsilon = 0.01.$$

In class we have worked out that the variance of x_t is:

$$E\left(x_t\right)^2 = \frac{\sigma_x^2}{1-\rho_x^2},$$

and the τ^{th} lag covariance is

$$C\left(\tau\right) = Ex_t x_{t-\tau} = \rho_x^{|\tau|} E x_t^2,\ \tau = 0, \pm 1, \pm 2, \ldots\ .$$

According to the discussion in class, the spectral density of x_t is:

$$S_x\left(e^{-i\omega}\right) = \frac{\sigma_x^2}{\left(1-\rho_x e^{-i\omega}\right)\left(1-\rho_x e^{i\omega}\right)}.$$

Applying the inverse Fourier transform to this yields the covariances:

$$C\left(\tau\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x\left(e^{-i\omega}\right) e^{i\omega\tau} d\omega.$$

(a) Consider two parameterizations of the x_t process. One has the parameter values given above, and the other has $\rho_x = 0.0$ and σ_ε set so that the variance of the two x_t representations is the same. On the same figure, graph the spectral densities associated with the two representations of x_t for $\omega \in (0, \pi)$. Also, generate 100 artificial time series observations from each representation. Note how the one with the higher value of ρ_x generates a pattern of x_t's that wanders around more lazily than does the representation with the smaller value of ρ_x. That 'lazy' pattern in the high-ρ_x representation shows up as relatively more power in the lower frequencies of its spectrum.

(b) Approximate the integral for $C(\tau)$ by the sum of the area of N rectangles, based on the Riemann sum interpretation of the integral. That is, if $\omega_j = 2\pi j/N$, with $j = -(N/2)+1, ..., N/2$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_x\left(e^{-i\omega}\right)e^{i\omega\tau}d\omega \approx \frac{1}{2\pi}\sum_{j=-(N/2)+1}^{N/2} S_x\left(e^{-i\omega_j}\right)e^{i\omega_j\tau}\left(\omega_j - \omega_{j-1}\right).$$

Is correct to work with the lower value of $-(N/2)+1$ for j, or with the lower value of $-(N/2)$?

Evaluate $C(0)$, $C(2)$, $C(4)$ using the exact formulas above and the Riemann approximation for various values of N. What is the smallest values of N for which the Riemann approximation is good?

2. Generate 10,000 observations from the first order autoregressive time series representation above. Then, HP filter the data. Consider the so-called HP filter:

$$\min_{\left\{y_t^T\right\}}\sum_{t=1}^{T-1}\left\{\left(y_t - y_t^T\right)^2 + \lambda\left[\left(y_{t+1}^T - y_t^T\right) - \left(y_t^T - y_{t-1}^T\right)\right]^2\right\},$$

and the 'HP-filtered' data are

$$y_t^c \equiv y_t - y_t^T.$$

Note that as $\lambda \to \infty$, y_t^T converges to a linear time trend.

(a) Calculate $C(0)$, $C(2)$, $C(4)$ using the artificial data, where $C(\tau)$ is the lag τ covariance of y_t^c. To compute y_t^c you will have to use the software provided with this homework.

(b) We now repeat (a) using the frequency domain. For this, you will first need the Fourier transform of the HP-filter. To find this, compute the first order necessary condition for optimality satisfied by y_t^T. In lag operator form, this has the representation,

$$y_t = B(L)\,y_t^T,$$

where $B(L)$ is symmetric in positive and negative powers of L, i.e., $B(L) = B(L^{-1})$. Note that since $y_t^T = y_t - y_t^c$, this implies:

$$y_t = B(L)\, y_t - B(L)\, y_t^c,$$

or,

$$y_t^c = g(L)\, y_t,$$

where the HP filter, $g(L)$ is

$$g(L) = \frac{B(L) - 1}{B(L)}.$$

i. To see what the HP filter does to a time series, graph $g(e^{-i\omega})$ for $\omega \in (0, \pi)$. Note that it looks like a high pass filter: it lets higher frequencies of oscillation in y_t through and zeros out the lower frequencies. What is the cutoff between frequencies allowed through and frequencies set to zero? Here, imagine you are working with quarterly data and $\lambda = 1600$.

ii. Use the spectral formulas together with the Riemann approximation to calculate $C(0)$, $C(2)$, $C(4)$. For N sufficiently larger, your answer should be the same as in (a).

3. Measurement error. Suppose a statistical agency receives data in this form:

$$y_t = x_t + u_t,$$

where u_t has variance σ_u^2, x_t has variance σ_y^2. Here, x_t denotes the true value of a variable and u_t denotes an uncorrelated measurement error, where both are Normally distributed.

(a) Suppose the statistical agency reports $\hat{y}_t$,

$$\hat{y}_t = E[x_t | y_t].$$

Write

$$\hat{y}_t = x_t + \hat{u}_t.$$

Compute the expression for $\hat{y}_t = E[x_t | y_t]$ and the correlation between $\hat{u}_t$ and x_t.

(b) In practice, researchers adopt the following conventional model of measurement error: the data are the sum of the truth and a measurement error that is uncorrelated with truth. Is this consistent with our model of the statistical agency?