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FINC 520, Spring 2009
Homework 4, due Thursday, April 30.

1. Consider the four variable VAR you estimated for homework #3. Let

$$\xi_t = \begin{pmatrix} Y_t - Y \\ Y_{t-1} - Y \\ Y_{t-2} - Y \\ Y_{t-3} - Y \end{pmatrix},$$

where Y denotes the unconditional mean for the VAR variables, implied by your estimated VAR(4). The estimated VAR(4) can also be used to construct F and Q in:

$$\xi_t = F\xi_{t-1} + v_t, \; Ev_t v_t' = Q.$$

The actual long term rate corresponds to $R_t^l = \tau' \xi_t$ for a particular vector, τ. The predicted long rate, based on the term structure hypothesis, corresponds to $\hat{R}_t = p'\xi_t$ for a different vector p.

   (a) Use the Riemann approximation to the inverse Fourier transform to compute

   $$\gamma_j = E R_t^l \hat{R}_{t-j}^l,$$

   for $j = -5, -4, ..., 4, 5$. Graph these. Does the term structure hypothesis look good based on these calculations?

   (b) Redo the previous graph, based on three different frequency components of R_t^l and $\hat{R}_t^l$, isolated using appropriately constructed band pass filters. Let the three frequency components be the 'business cycle frequencies' (the component of data corresponding to frequencies of fluctuation between 1 and 8 years (i.e., 4 and 32 quarters)), 'high frequencies' (the component of the data corresponding to fluctuations between 2 quarters and 1 year) and 'low frequencies' (the component corresponding to fluctuations longer than 8 years). Does the expectations hypothesis do better on some frequencies than on others?

(c) Compute the share of variance in $\hat{R}_t^l$ and R_t^l in low, business cycle and high frequencies. How does the expectations hypothesis do on this dimension? (Hint: recall from the spectral representation theorem that the variance of a process is the sum of the variances of that process in different frequency components.)

2. This question explores whether there is a 'phase shift' between the actual and predicted long rates. Obviously, for the expectations hypothesis to be a good one requires that the phase shift between the two series be small. As we will see, the phase shift between two series is a function of the complex part of their cross-spectrum (i.e., the Fourier transform of γ_j in the previous question.) That is, consider two mean zero, covariance stationary, purely indeterministic variables, x_t and y_t. Denote their covariance by $\Gamma_{yx}(j) = Ey_t x_{t-j}$. The cross-spectrum between the two series is defined by:

$$S_{yx}\left(e^{-i\omega}\right) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma_{yx}(j)\, e^{-i\omega j}.$$

In general, the cross spectrum is complex, and we express it in polar form as follows:

$$S_{yx}\left(e^{-i\omega}\right) = r(\omega)\, e^{i\theta(\omega)},$$

where S_{xy} is complex if and only if $\theta \neq 0$. Also, $r(\omega) \geq 0$. We will argue that the phase relationship between the frequency ω components of x_t and y_t is controlled by $\theta(\omega)$, with the frequency ω components of these two series reaching a maximum at the same date if, and only if, $\theta(\omega) = 0$. In the latter case, we say there is no phase shift between the frequency ω components of the two series.

To develop the above results, it is useful to follow Sargent (1979) in considering the projection of y_t onto x_t infinitely far in the future and the past:

$$y_t = \sum_{k=-\infty}^{\infty} h_j x_{t-j} + \varepsilon_t,$$

where, by the orthogonality property of projections, ε_t is uncorrelated with x_{t-j} for all j. Let

$$\Gamma_x(k) \equiv E x_t x_{t-k}.$$

The projection represents y_t in terms of x_t at all leads and lags and a component, ε_t, that is orthogonal to x_t at all leads and lags.

(a) Is ε_t autocorrelated? Explain.

(b) Show that the Fourier transforms of these objects and the projection coefficients are related as follows:

$$S_{yx}\left(e^{-i\omega}\right) = h\left(e^{-i\omega}\right) S_x\left(e^{-i\omega}\right),$$

where

$$S_x\left(e^{-i\omega}\right) = \sum_{k=-\infty}^{\infty} \Gamma_x(k)\, e^{-i\omega k},\ h\left(e^{-i\omega}\right) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k}.$$

(Hint: use a version of the argument used in class about how Fourier transforms convert convolution operators into multiplication.) An implication of this result, not of specific interest here, is that the Fourier transform of the projection coefficients, $\{h_j\}$, is given by:

$$h\left(e^{-i\omega}\right) = \frac{S_{yx}\left(e^{-i\omega}\right)}{g_x\left(e^{-i\omega}\right)},$$

so that the individual projection coefficients may be recovered using the inverse Fourier transform:

$$h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g_{yx}\left(e^{-i\omega}\right)}{g_x\left(e^{-i\omega}\right)} e^{i\omega k} d\omega.$$

This may be approximated by a Riemann sum.

(c) By the spectral representation theorem, x_t is the integral of sinusoidal functions across different frequencies. Suppose the ω component of x_t is as follows:

$$x_t = 2\cos(\omega t),\ t = 0, \pm1, \pm2,\ \ldots .$$

Show that the frequency ω component of y_t 'due to' the frequency ω component of x_t (i.e., ignoring ε_t) is as follows:

$$y_t = 2s(\omega)\cos(\omega t + \theta(\omega)),$$

and display an expression for $s(\omega)$. Explain why $s(\omega)$ must be non-negative. This establishes the sense in which $\theta(\omega)$, the complex part of the cross spectrum, controls the phase relationship between y_t and x_t.

(d) Compute $\theta(\omega)$ for $\omega \in (0, \pi)$ when $y_t = \hat{R}_t^l$ and $x_t = R_t^l$ in question (1), and the data are generated by your estimated VAR(4).

3. (Optimal seasonal adjustment). Suppose that a time series, $\{X_t\}$, has the following representation:

$$X_t = x_t + u_t,$$

where x_t and u_t are purely indeterministic and ergodic, covariance stationary processes and $x_t \perp u_s$ for all s, t. Suppose that u_t is the source of seasonality in X_t. That is, the spectrum of u_t, $S_u(e^{-i\omega})$, has much power concentrated in the seasonal frequencies (i.e., those near $\omega = 2\pi/4$ in quarterly data). The spectrum of x_t, $S_x(e^{-i\omega})$, is smooth and does not display a peak in the seasonal frequencies. The econometrician seeks to estimate the 'seasonally adjusted data', x_t, by projecting x_t onto a complete realization of X_t (i.e., $\{..., X_{-1}, X_0, X_1, ...\}$):

$$x_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k} + v_t,$$

where v_t is uncorrelated with X_{t-k} for all k. Let $\hat{x}_t$ denote the 'seasonally adjusted' data:

$$\hat{x}_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k}.$$

(a) Derive the formula for $h(e^{-i\omega})$ in terms of the known objects, $S_u(e^{-i\omega})$ and $S_x(e^{-i\omega})$.

(b) Show that $S_{\hat{x}}(e^{-i\omega}) < S_x(e^{-i\omega})$ for all ω.

(c) Show that if $S_x(e^{-i\omega})$ is smooth across all frequencies, while $S_u(e^{-i\omega})$ has sharp peaks at the seasonal frequencies, then $S_{\hat{x}}(e^{-i\omega})$ will have substantial *dips* at the seasonal frequencies. (It may seem ironic that optimal seasonal adjustment produces a series, $\hat{x}_t$, that itself displays seasonality.)

4. Suppose that the data are generated by a true (scalar) autoregressive representation of the following form:

$$y_t = \phi(L)\, y_{t-1} + \varepsilon_t,$$

where $\phi(L)$ is a polynomial in non-negative powers of L and the polynomial coefficients are square-summable. Also, ε_t is a white noise, uncorrelated with y_{t-s}, $s > 0$. Suppose the econometrician estimates $\phi(L)$ by running a regression of y_t on p lags of itself. The econometrician is assumed to have an entire (i.e., doubly infinite) realization of data. The econometrician may commit some form of specification error, for example by choosing a value of p smaller than the true value (the true lag length may actually be infinite). By 'running a regression', the econometrician is assumed to choose coefficients, $\hat{\phi}_1, ..., \hat{\phi}_p$, for the AR polynomial, $\hat{\phi}(L)$, so that

$$y_t - \hat{\phi}(L)\, y_{t-1}$$

has the smallest possible variance in the (infinite!) sample.

   (a) Argue carefully (be clear when you use ergodicity and covariance stationarity) that the econometrician's choice of $\hat{\phi}(L)$ solves

$$\min_{\hat{\phi}_1,\ldots,\hat{\phi}_p} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\phi\left(e^{-i\omega}\right) - \hat{\phi}\left(e^{-i\omega}\right)\right] g_y\left(e^{-i\omega}\right)\left[\phi\left(e^{i\omega}\right) - \hat{\phi}\left(e^{i\omega}\right)\right]' d\omega,$$

   where $g_y\left(e^{-i\omega}\right)$ is the covariance generating function of $\{y_t\}$. Note that if the econometrician commits no specification error, then

$$\phi\left(e^{-i\omega}\right) = \hat{\phi}\left(e^{-i\omega}\right), \text{ for all } \omega \in (-\pi, \pi).$$

   (b) Suppose the econometrician does commit specification error, so that the previous equality is not possible over all frequencies, $\omega \in (-\pi, \pi)$. Suppose the econometrician is particularly interested in the sum of the AR coefficients, $\phi(1)$. Explain why the econometrician's estimator of this object, $\hat{\phi}(1)$, is likely to be a good one if there is an important low-frequency component in the data, $\{y_t\}$. Alternatively, if the data are primarily driven by high frequency components, then $\phi(1)$ is likely to be badly estimated. Explain.