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FINC 520, Spring 2009
Homework 7, due Thursday, May 28.

1. We describe a model economy in terms of its equilibrium conditions, expressed in linearized form:

$$\beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \quad (1)$$

$$-\left[r_t - E_t \pi_{t+1} - rr_t^* \right] + E_t x_{t+1} - x_t = 0 \quad (2)$$

$$u_t + \phi_\pi \pi_t + \phi_x x_t - r_t = 0 \quad (3)$$

$$rr_t^* - \rho \Delta a_t - \frac{1}{1+\varphi} (1 - \lambda) \tau_t = 0 \quad (4)$$

Here, π_t denotes the net rate of change in prices from $t-1$ to t, x_t denotes equilibrium output relative to its first-best level, and r_t denotes the nominal rate of interest. Equation (1) summarizes the implications of price setting in the model. It represents the recursive representation of

$$\pi_t = \kappa x_t + \beta E_t \overbrace{[\kappa x_{t+1} + \beta \kappa x_{t+2} + ...]}^{\pi_{t+1}},$$

which says that current inflation is a function of current and expected future output. The idea is that the firms setting prices in the current period do so as a function of marginal costs in the present and the future periods. In the model, marginal cost of production are related to x_t, the deviation of output from its first best level ('the output gap'). Equation (2) represents the intertemporal Euler equation in the actual economy, minus the intertemporal Euler equation in the first best equilibrium. This explains why it is the output gap, x_t, which appears in this expression rather than output. (Note: normally, it would be consumption that appears in this equation, rather than output. However, the two are the same in this model economy, because there is no investment.) Also, the expression in square brackets is the deviation between the real interest rate in the actual equilibrium, $r_t - E_t \pi_{t+1}$, and the interest rate that would prevail in the first best equilibrium,

rr_t^*. Equation (3) represents the monetary policy rule, which expresses the nominal interest rate, r_t, as a function of inflation, the output gap and a random monetary policy shock, u_t. Finally, equation (4) shows how rr_t^* is determined as a function of a technology shock in the model, a_t, and a preference shock. Here, Δ means the first difference operator, so that $\Delta a_t \equiv a_t - a_{t-1}$. Note that if monetary policy set $r_t = rr_t^*$, then the first best equilibrium would result, since $x_t = \pi_t = 0$ solves the equations in this case.[1] However, monetary policy does not set the interest rate in this way. Instead, it sets the interest rate according to (3).[2]

To solve the model, it is useful to express the equilibrium conditions, (1)-(4), and the laws of motion of the shocks in matrix form. The shocks can be written as follows:

$$
\begin{aligned}
s_t &= \begin{pmatrix} \Delta a_t \\ u_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ u_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \varepsilon_t^{\tau} \end{pmatrix} \\
s_t &= P s_{t-1} + \epsilon_t,
\end{aligned}
$$

say. Equations (1)-(4) can be expressed as follows:

$$
E_t \begin{bmatrix} \beta & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ rr_{t+1}^* \end{pmatrix} + \begin{bmatrix} -1 & \kappa & 0 & 0 \\ 0 & -1 & -1 & 1 \\ \phi_\pi & \phi_x & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr_t^* \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\rho & 0 & -\frac{1}{1+\varphi}(1-\lambda) \end{pmatrix} s_t,
$$

or,

$$
\alpha_0 E_t z_{t+1} + \alpha_1 z_t + \beta_1 s_t = 0,
$$

where z_t summarizes the period t endogenous variables:

$$
z_t = \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr_t^* \end{pmatrix}.
$$

[1] Zero inflation is part of the first-best equilibrium in this economy, because there are Calvo-style pricing frictions and the effects of these are eliminated in a zero-inflation environment.

[2] One possibility is that monetary policy cannot set the interest rate to rr_t^* because the monetary authority does not see the shocks that determine rr_t^*.

We search for a solution to the model of the following form:

$$z_t = Bs_t.$$

We pin down the value of B by the requirement that the equilibrium conditions must be satisfied:

$$\alpha_0 E_t B s_{t+1} + \alpha_1 B s_t + \beta_1 s_t = 0,$$

or, using the law of motion for s_t :

$$\left[\alpha_0 BP + \alpha_1 B + \beta_1\right] s_t = 0,$$

so that for B to be a solution we require

$$\alpha_0 BP + \alpha_1 B + \beta_1 = 0.$$

We can solve this system of equations for B using the linearity of the *vec* operator and the fact,

$$vec\,(AQC) = (C' \otimes A)\, vec\,(Q)\,.$$

Parameterize the model as follows:

$$\delta = 0.2,\ \ \lambda = 0.4,\ \ \rho = 0.8,\ \ \kappa = 0.2,\ \ \phi_\pi = 1.5,\ \ \phi_x = 0.5,\ \ \varphi = 1,\ \ \beta = 0.99.$$

Let all the innovation standard deviations be 0.01.

The econometrician is provided with data on inflation and the output gap. In addition, the econometrician is provided with survey data on expectations of inflation over the next two periods:

$$\hat{\pi}_{t,2}.$$

The econometrician interprets this survey data as corresponding to

$$\hat{\pi}_{t,2} = E_t\left[\pi_{t+1} + \pi_{t+2}\right],$$

where the expectation is evaluated relative to the information set of the agents in the model:

$$\hat{\pi}_{t,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \left[Bs_{t+1} + Bs_{t+2}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} B\left[P + P^2\right] s_t.$$

Let y_t denote the date t values of the variables available to the econometrician:

$$y_t = \begin{pmatrix} \pi_t \\ x_t \\ \hat{\pi}_{t,2} \end{pmatrix}.$$

all past data inflation expectations, $\hat{\pi}_{t,2}$. nominal rate of interest.

(a) Express the system in state-space, observer form. You need not include any measurement error in the observer equation. Generate $T = 200$ artificial observations on y_t by drawing innovations from a normal random number generator. Use $y_1, ..., y_T$ to estimate the parameters, ϕ_π, λ and κ, using unconditional Gaussian maximum likelihood (let the econometrician know the true values of all the other parameters). You could use the MATLAB optimization routine, fminsearch.m, to do the optimization. To build confidence that you have found the global optimum of the likelihood, graph that likelihood with respect to each estimated parameter, holding the other estimated parameters at their estimated values. Do the graphs suggest that you have found the global optimum? The degree of curvature of the likelihood in the neighborhood of the estimated value of a particular parameter is an indicator of the precision with which that parameter is pinned down by the data. Do the parameters all look equally well pinned down?

(b) Generate $T = 4,000$ observations on y_t and redo (a). Consistency of the ML estimator suggests that the estimated value of the parameters should be virtually equal to the corresponding true value. Redo the calculations with $T = 40$ observations. Does the estimation performance deteriorate much with such a reduction in the number of observations? Does precision - measured by the degree of curvature in the likelihood - drop substantially?

(c) The previous part of this question shows how estimation procedures can be adapted to include expectation data. Another feature of the data that is useful to take into account is the fact that different variables are available at different frequencies of observation. For example, inflation data are typically available monthly, while aggregate output data are available quarterly. Consider the case in which gap data are only available for even values of t, and for each even value of t, what is provided is $(x_t + x_{t+1})/2$. That is, every other period the econometrician receives the two-period average of output. Inflation expectations and inflation continue to be available for each integer, t. As before, the model is defined for integer values of t. Adapt the strategy outlined on pages 14-19 of http://faculty.wcas.northwestern.edu/~lchrist/course/estimationhandout.pdf to cover our mixed data frequency situation. Redo (a) and (b). Compare the graph of the likelihood in the cases covered in (a)-(b) versus (c). Does the missing data considered in this question lead to a significant loss of estimation precision, as measured by the curvature in the likelihood function?

2. Suppose the data are generated by the following autoregressive representation:

$$y_t = \rho y_{t-1} + \varepsilon_t,$$

where $\rho = 0.9$ and ε_t is iid over time and distributed according to a uniform distribution over the interal, $[-0.01, 0.01]$. Also consider the case $\rho = 0$ and adjust the interval of the uniform distribution so that Ey_t^2 is the same between the two cases.

   (a) For each of the two cases, draw 1,000 artificial samples of 100 observations each. Compute the sample mean in each data set and draw its histogram. Superimpose on this diagram a graph of the asymptotic normal distribution predicted by the Central Limit Theorem (CLT) discussed in class (the variance in the CLT requires the spectral density of the process, and for this you can use the true spectral density). How well does the asymptotic theory perform?

   (b) Do the same as in (a) twice. Once for the case in which each artificial data set has 1,000 observations and once for the case in which each artificial data set has 10 observations. What can you infer about the role of serial correlation and quantity of observations in determining the adequacy of the central limit theorem in small samples?