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FINC 520, Spring 2009
Homework 8, due Thursday, June 4.

1. Consider the iid Normal stochastic process, $\{x_t\}$, with $Ex_t = \mu$ and $E(x_t - \mu)^2 = \sigma^2$. Let $\mu_l = E(x_t - \mu)^l$ denote the l^{th} moment about the mean. A property of the Normal distribution is that odd-ordered moments are zero, and even-ordered moments satisfy:

$$\mu_{2k} = \frac{\sigma^{2k}}{2^k}\frac{(2k)!}{k!}.$$

Skewness of any distribution is defined as

$$s = \frac{E(x_t - \mu)^3}{\sigma^3}.$$

In the case of a Normal distribution, this is of course zero. Skewness is estimated as follows:

$$\hat{s}_T = \frac{\frac{1}{T}\sum_{t=1}^{T}(x_t - \hat{\mu})^3}{\hat{\sigma}^3},\ \hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}(x_t - \hat{\mu})^2,\ \hat{\mu} = \frac{1}{T}\sum_{t=1}^{T}x_t.$$

   (a) Set this estimator of skewness up as an exactly identified GMM estimator. Show that $\hat{s}_T$ is asymptotically normal with standard deviation,

$$\sqrt{\frac{6}{T}}.$$

   (b) Now suppose that the true value of the mean is known. Show that $\hat{s}_T$ is asymptotically normal with standard deviation,

$$\sqrt{\frac{15}{T}}.$$

   (Hint: to do each part of this question, you have to identify a different 'GMM stochastic process', $h_t(\theta, w_t)$, having the property, $Eh_t(\theta^0, w_t) = 0$, where θ^0 is the true value of the parameters, and $\theta = (\mu, \sigma, s)'$ in the case of part a while $\theta = (\sigma, s)'$ in the case of part b. When computing the matrices required by GMM, be sure to impose all the properties of the Normal distribution.)

(c) Does the sampling distribution of the estimator of σ depend on whether or not μ is known?

2. Consider the data from homework 7. Compute the skewness statistic for each of the four variables (inflation, the long rate, the short rate and GDP growth). To do this, you cannot rely on the assumptions of iidNormality used in the previous question. So, you must take a stand on how many lags to use in constructing the zero-frequency spectral density for the GMM error. For this, use the covariance, and the first, second and third lagged-covariances. Using the GMM sampling theory, compute the $p-$value of the empirical $\hat{s}$ statistic under the null hypothesis that the true value is zero and the sampling distribution of $\sqrt{T} \times \hat{s}$ is $N\left(0, \hat{V}\right)$, where $\hat{V}$ is the GMM estimator of the sampling variance of $\hat{s}$ (*i.e.*, the $p-$value of $\hat{s}$ is the probability that an $N\left(0, \hat{V}/T\right)$ random variable is larger than $\hat{s}$.

3. For question 2, you computed the $p-$value of $\hat{s}$ under the null hypothesis of no skewness, using asymptotic sampling theory. It is possible that T is not large enough for the asymptotic theory to be a good approximation. You can use a Monte Carlo sampling experiment to check this. The idea is to use a computer to directly compute the sampling distribution, in an empirically relevant sample size and under the null hypothesis of no skewness, of the skewness statistic. To do this, one needs to take a position on the mechanism that generated the data used to compute the skewness statistic. This step is important because, presumably, the finite sample distribution of the skewness statistic is sensitive to the details of the time series process generating the data. A natural choice for the data generating mechanism is the four variable VAR(4) you estimated for homework 3:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \phi_4 y_{t-4} + \varepsilon_t, \;\; E\varepsilon_t \varepsilon_t' = \Omega,$$

where Ω is estimated by

$$\begin{aligned} & \frac{1}{T} Y' M Y, \\ M & \equiv I - X\left(X'X\right)^{-1} X', \end{aligned}$$

and y_t, ε_t, X and Y are defined in homework 3 (and in class). Write

$$CC' = \Omega,$$

where C is lower triangular with positive elements on its diagonal.[1] Note that if $\tilde{\varepsilon}_t$ is $N(0, I)$, then $C\tilde{\varepsilon}_t$ has variance-covariance matrix, Ω. Generate N artificial observations, $y_1, y_2, ..., y_N$, using

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \phi_4 y_{t-4} + C\tilde{\varepsilon}_t,$$

where $\tilde{\varepsilon}_t$ is drawn independently from $N(0, I)$ for each t.[2] To start this simulation, you will need initial conditions, $y_0, ..., y_{-3}$. Set these to the value of Ey_t implied by the estimated VAR. Let $N = 200 + T$, where T is the number of rows in X, the matrix containing the actual data. Compute $\hat{s}$ for each of the four variables using $y_{201}, ..., y_{200+T}$. (Note that by beginning with y_{201}, the initial y_t's are in effect being drawn from the unconditional distribution of y_t.) Repeat this exercise 2,000 times. Obtain the $p-$value for each of your four skewness statistics, by computing the fraction of times that the simulated skewness statistics exceed their empirical counterparts. Note that the simulations impose the zero skewness statistic of the null hypothesis because of the way the VAR disturbances are drawn. Do you get a different result from what you found in question 2?

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[1] This matrix exists and is unique, given that Ω is positive definite. The routine, chol(B) in MATLAB computes the upper triangular Choleski decomposition. To obtain what we want, set C =chol(B')'.

[2] You can draw $\tilde{\varepsilon}_t$ by executing the MATLAB command, randn(4,1).