Christiano
FINC 520 Midterm
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1. Suppose y_t is an ARMA(p,q) process, i.e.

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q},$$

where $E\left[\varepsilon_t^2\right] = \sigma_\varepsilon^2$. Write the ARMA(p,q) process in an VAR(1) process

$$\xi_t = F\xi_{t-1} + v_t.$$

Write down ξ_t, F, v_t, and $\Omega = E\left[v_t v_t'\right].$

2. Suppose that X_t follows an AR(p) process and v_t is a white noise process that is uncorrelated with X_{t-j} for all j.

   a. Show that
   $$Y_t = X_t + v_t$$
   follows an ARMA(p,p) process.

   b. Suppose that $p = 1$ and
   $$X_t = \rho X_{t-1} + \varepsilon_t,$$
   where $\rho = 0.99$ and ε_t is orthogonal to past X_t's and to all leads and lags of v_t. Provide a detailed description of how the Wold representation of Y_t is computed.

   c. Let the Wold innovation in Y_t be denoted w_t, *i.e*,
   $$w_t = Y_t - P\left[Y_t|Y_{t-1}, Y_{t-2}, ...\right].$$
   Prove that w_t is uncorrelated with itself at every lag.

   d. Derive a formula linking w_t to current and past v_t's and ε_t's.

   e. Suppose the variance of ε_t is very small relative to the variance of v_t. Show that a positive shock in ε_t triggers a long sequence of positive realizations of w_t. Carefully explain the intuition for this result. Why does this result not contradict the proposition proved in part (c) that w_t is iid?

3. Consider a process, y_t, such that

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t,$$

where

$$1 - \phi_1 z - \phi_2 z^2 = (1 - \lambda_1 z)(1 - \lambda_2 z),$$

where λ_1 and λ_2 are real and less than unity in absolute value.

a. Derive a and b in the partial fractions expansion:

$$\frac{1}{1 - \phi_1 z - \phi_2 z^2} = \frac{a}{1 - \lambda_1 z} + \frac{b}{1 - \lambda_2 z}.$$

b. Use the partial fractions expansion to derive expressions for the moving average coefficients, ψ_j, in the $MA(\infty)$ representation for y_t :

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}.$$

4. Suppose

$$y_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j},$$

where ε_t is serially uncorrelated with mean zero and variance σ_ε. Also, the b_j's are square summable. Let

$$\gamma_k = E y_t y_{t-k},\ S_y(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j,\ b(z) = \sum_{j=-\infty}^{\infty} b_j z^j.$$

Prove that

$$S_y(z) = b(z)\, b\left(z^{-1}\right) \sigma_\varepsilon^2.$$

(Hint: it is convenient to write

$$y_t = \sum_{j=-\infty}^{\infty} b_j \varepsilon_{t-j},$$

with the understanding, $b_j = 0$ for $j < 0$. The proof is tantamount to a demonstration that application of the z transform to a sequence converts convolution into multiplication.)

5. Consider the frequency ω component of a covariance stationary process

$$x_t = 2\cos(\omega t).$$

Suppose that

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j}.$$

a. Show that the ω frequency component of y_t is

$$y_t = 2s(\omega)\cos(\omega t + \theta(\omega)).$$

(Hint, use the facts $e^{i\omega} = \cos\omega + i\sin\omega$, $\cos(\omega) = \cos(-\omega)$, $\sin(\omega) = -\sin(-\omega)$.)

b. Display $\theta(\omega)$ in the case, $b_j = 0$ for all j except $j = -1$ and $b_{-1} = 1$. That is,

$$y_t = x_{t-1},$$

so that y_t is just x_t with a delay. Explain that the expression you get for $\theta(\omega)$ makes intuitive sense.