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FINC 520, Spring 2010
Midterm Exam.

There are 100 points possible on this exam. The number of points allocated to each question are indicated, so you can allocate your time accordingly. The density function for a normal random variable, x, is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma_x^2}\right].$$

The mean of the random variable is μ and the variance, σ_x^2.

1. Consider the stationary $AR(1)$ process,

   $$y_t = \phi y_{t-1} + \varepsilon_t,\ \ \varepsilon_t\tilde{\ }N\left(0,\sigma^2\right),\ \ |\phi| < 1.$$

   Suppose you have T observations on this process, drawn every other period, *i.e.*, you have y_2, y_4, y_6, y_8, ..., y_{2T}. Obtain an expression for the joint density of the observations, conditional on the model parameters, ϕ and σ^2.

2. Consider a covariance stationary process with the following representation:

   $$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q},$$

   where ε_t is *iid* with mean zero and variance σ_ε^2. Define

   $$\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t.$$

   Derive a simple expression for V in terms of the ϕ_j's and θ_j's, where

   $$\lim_{T\to\infty} TE\left(\bar{y}_T - E\bar{y}_T\right)^2 = V.$$

3. Suppose two the bivariate stochastic process, $\begin{pmatrix} x_t & y_t \end{pmatrix}'$, has the following time series representation:

   $$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = G \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \varepsilon_t.$$

Here, G is a 2×2 matrix and ε_t is a 2×1 iid mean-zero process with variance, $E\varepsilon_t\varepsilon_t' = V$, which is orthogonal to $\begin{pmatrix} x_{t-s} & y_{t-s} \end{pmatrix}$, $s > 0$. Suppose you have $2T$ observations on y_t, $t = 1, 2, 3,$ ...,$2T$ while you only observe

$$\tilde{x}_t = x_t + x_{t-1}$$

every even period, $t = 2$, 4, 6, 8,...., $2(T-1)$, $2T$.

(a) Develop a state space/observer representation for this situation. (Hint: construct an observer vector, Y_t, a state, ξ_t, and matrices, F, Q and H such that

$$\begin{aligned} Y_t &= H'\xi_t \\ \xi_t &= G\xi_{t-2} + v_t, \end{aligned}$$

where v_t is orthogonal to past ξ_t's and Y_t, ξ_t and v_t are only defined for even values of t.)

b How can the Kalman smoother be used to make a guess about the 'missing' values of x_t defined over integer values of t?

4. Let Y and X denote 1×1 and $n \times 1$ mean-zero random variables, respectively, where $n > 1$. Define

$$P\left[Y|X\right] = \alpha' X,$$

where α is an $n \times 1$ vector of numbers that solves the following optimization problem:

$$\min_g E\left[Y - g'X\right]^2.$$

(a) Show that if γ is an $n \times 1$ vector that satisfies

$$E\left[Y - \gamma' X\right] X' = 0,$$

then $\gamma = \alpha$.

(b) Suppose y, x_1 and x_2 are scalar random variables. Show that

$$P\left[y|x_1, x_2\right] = P\left[y|x_1\right] + P\left(y - P\left[y|x_1\right]|x_2 - P\left[x_2|x_1\right]\right).$$

5. Let

$$\tilde{y}_t = F(L)\, y_t,$$

where

$$F(L) = \sum_{j=-\infty}^{\infty} F_j L^j,$$

and L denotes the lag operator and y_t is a zero mean, purely indeterministic, covariance-stationary stochastic process.

(a) Show that

$$S_{\tilde{y}}(z) = F(z)\, F\left(z^{-1}\right) S_y(z),$$

for $z = e^{-i\omega}$, where

$$S_y(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j,\ \ \gamma_k = E y_t y_{t-k},\ \ k = 0, \pm 1, \pm 2, \dots .$$

(b) The spectral representation theorem states that y_t has the following linear, orthogonal decomposition:

$$y_t = \int_0^{\pi} \left[a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\right] d\omega,$$

where $a(\omega)$, $a(\omega')$, $b(\omega)$, $b(\omega')$ are mean-zero, orthogonal random variables for all $\omega \neq \omega'$, $\omega, \omega' \in [0, \pi]$. Also, $Ea(\omega)^2 = Eb(\omega)^2 = \sigma^2$, all $\omega \in [0, \pi]$. We refer to

$$a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)$$

as the "ω-frequency component of y_t". Derive an expression that shows the impact of $F(L)$ on the phase and amplitude of the ω-frequency component of y_t.