Christiano
Finance 520

Notes on Fuster, Hebert and Laibson, "Natural Expectations, Macroeconomic Dynamics, and Asset Pricing"

The paper by FHL argues that some economic variables over react to shock because people have 'natural' rather than rational expectation.[1] This note summarizes some of the econometric results that FHL point to in support for their position. The analysis is an application of the Sims approximation error formula.[2]

1. The Setup

Many theories in economics take the following form:

$$P_t = \sum_{j=0}^{\infty} \beta^j E_t d_{t+j},$$

where E_t is the linear projection operator on date t information and P_t is some endogenous variable. Also, β is the discount rate, a number that is close to unity. For example, imagine that P_t is a stock price and d_t is earnings. An object that is of interest is:

$$P_t - E_{t-1}P_t.$$

This is the move in P_t that is unexpected as of period $t-1$. There is a sense in economics that a lot of variables, for example, stock prices, over-react to new information. That is, $P_t - E_{t-1}P_t$ is somehow 'too volatile'.

Subtracting,

$$P_t - E_{t-1}P_t = \sum_{j=0}^{\infty} \beta^j \left[E_t d_{t+j} - E_{t-1}d_{t+j}\right]. \tag{1.1}$$

Thus, the unexpected move in the stock price is related to the revision in expectations about future earnings. Suppose that the time series representation for d_t has the following representation:

$$\begin{aligned} \Delta d_t &= \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + ... \\ &= \psi(L)\varepsilon_t, \end{aligned} \tag{1.2}$$

where the ψ_i's are additively summable and

$$\psi(L) = \frac{\theta(L)}{\phi(L)}, \tag{1.3}$$

where the roots of θ and of ϕ lie strictly outside the unity circle. As a result, Δd_t is covariance stationary and ε_t can be recovered from a linear combination of current and past Δd_t's. It follows that the ε_t's are serially uncorrelated and suppose its variance is not a function of t :

$$E\varepsilon_t^2 = \sigma^2.$$

[1] http://www.nber.org/confer/2011/Macro11/Fuster_Hebert_Laibson.pdf
[2] Sims, Christopher, 1972, 'The Role of Approximate Prior Restrictions in Distributed Lag Estimation,' Journal of the American Statistical Association, 67(337), pp. 169-175.

Note:

$$d_{t+j} = \Delta d_{t+j} + \Delta d_{t+j-1} + \Delta d_{t+j-2} + ... + \Delta d_t + d_{t-1},$$

for $j = 0, 1, 2, ...$ Then,

$$\begin{aligned} E_t d_t - E_{t-1} d_t &= E_t [\Delta d_t + d_{t-1}] - E_{t-1} [\Delta d_t + d_{t-1}] \\ &= E_t \Delta d_t - E_{t-1} \Delta d_t = \varepsilon_t. \end{aligned}$$

Also,

$$\begin{aligned} E_t d_{t+1} - E_{t-1} d_{t+1} &= E_t [\Delta d_{t+1} + \Delta d_t] - E_{t-1} [\Delta d_{t+1} + \Delta d_t] \\ &= [E_t \Delta d_{t+1} - E_{t-1} \Delta d_{t+1}] + [E_t \Delta d_t - E_{t-1} \Delta d_t] \\ &= [(\psi_1 \varepsilon_t + \psi_2 \varepsilon_{t-1} + ...) - (\psi_2 \varepsilon_{t-1} + ...)] + \varepsilon_t \\ &= [1 + \psi_1] \varepsilon_t. \end{aligned}$$

Similarly,

$$E_t d_{t+j} - E_{t-1} d_{t+j} = \left[1 + \psi_1 + \psi_2 + ...\psi_j\right] \varepsilon_t. \tag{1.4}$$

It follows from (1.4) that

$$\lim_{j \to \infty} E_t d_{t+j} - E_{t-1} d_{t+j} = \psi(1) \varepsilon_t.$$

That is, the sum of the moving average coefficients of Δd_t determines the impact of ε_t on the long-run forecast of d_t. It's not completely surprising that this forecast should involve a sum, since the level in the distant future is a result of the sum of the increments between now and then.

Substituting (1.4) into (1.1):

$$\begin{aligned} P_t - E_{t-1} P_t &= E_t d_t - E_{t-1} d_t + \beta [E_t d_{t+1} - E_{t-1} d_{t+1}] \\ &\quad + \beta^2 [E_t d_{t+2} - E_{t-1} d_{t+2}] + ... \\ &= \varepsilon_t + \beta [1 + \psi_1] \varepsilon_t + \beta^2 [1 + \psi_1 + \psi_2] \varepsilon_t + ... \ . \end{aligned}$$

Collecting terms,

$$\begin{aligned} P_t - E_{t-1} P_t &= \left[\frac{1}{1-\beta} + \beta \frac{1}{1-\beta} \psi_1 + \beta^2 \frac{1}{1-\beta} \psi_2 + ...\right] \varepsilon_t \\ &= \frac{1}{1-\beta} \left[1 + \beta \psi_1 + \beta^2 \psi_2 + ...\right] \varepsilon_t \\ &= \frac{\psi(\beta)}{1-\beta} \varepsilon_t. \end{aligned} \tag{1.5}$$

Note that since β is close to unity, the response of the stock price to a shock, ε_t, depends (roughly, because β is only 'close' to unity, not exactly unity) on the sum of the moving average coefficients in Δd_t. Note too, that this in turn corresponds (roughly) to the spectral density of Δd_t at $\omega = 0$, for, recall

$$S_{\Delta d}\left(e^{-i\omega}\right) = \psi\left(e^{-i\omega}\right) \psi\left(e^{i\omega}\right) \sigma^2, \tag{1.6}$$

and note that $e^{-i \times 0} = 1$.

FHL posit that people simplify a complex reality, and this results in an overstatement of the long run impact of shocks, i.e., objects like $\psi(1)$ or $\psi(\beta)$. They find support for this idea in their analysis of data on corporate earnings. They fit a 40 lag univariate autoregressive representation on the first difference of the log of corporate earnings, Δd_t :

$$\Delta d_t = B(L)\,\Delta d_{t-1} + \varepsilon_t, \tag{1.7}$$

where ε_t is orthogonal to all past Δd_t's. Evidently,

$$\theta(L) = 1,\ \phi(L) = 1 - B(L)\,L,\ \psi(L) = \frac{1}{1 - B(L)\,L}.$$

FHL then treat this representation as the 'true', complex, reality. The stock market participant, putting on his/her econometrician hat on, fits a representation,

$$\Delta d_t = \hat{B}(L;\gamma)\,\Delta d_{t-1} + \hat{\varepsilon}_t,$$

where γ is a set of parameters. The kind of specification FHL have in mind is a low order AR representation, such as:

$$\hat{B}(L;\gamma) = \phi_1 + \phi_2 L + ... + \phi_4 L^3,$$

so that

$$\gamma = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{pmatrix}'. \tag{1.8}$$

The agent selects the parameters of $\hat{B}(L;\gamma)$ to minimize the variance of $\hat{\varepsilon}_t$:

$$\begin{aligned} \hat{\varepsilon}_t &= \Delta d_t - \hat{B}(L;\gamma)\,\Delta d_{t-1} \\ &= \left[B(L) - \hat{B}(L;\gamma)\right]\Delta d_{t-1} + \varepsilon_t, \end{aligned}$$

using (1.7).[3] Using the results for a variance developed in class, we obtain Sims' approximation formula:

$$Var(\hat{\varepsilon}_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[B\left(e^{-i\omega}\right) - \hat{B}\left(e^{-i\omega};\gamma\right)\right] S_{\Delta d}\left(e^{-i\omega}\right)\left[B\left(e^{i\omega}\right) - \hat{B}\left(e^{i\omega};\gamma\right)\right] d\omega + \sigma^2, \tag{1.9}$$

where $S_{\Delta d}\left(e^{-i\omega}\right)$ is the spectral density of Δd_t (see (1.6)). The econometrician's problem is to choose γ to minimize $Var(\hat{\varepsilon}_t)$ in (1.9). If there is no specification error, then the problem is solved by choosing γ so that $B\left(e^{-i\omega}\right) = \hat{B}\left(e^{-i\omega};\gamma\right)$ for all ω. But, if there is specification error then $B\left(e^{-i\omega}\right) = \hat{B}\left(e^{-i\omega};\gamma\right)$ is not possible for all ω. What least squares does, according to (1.9), is to make $B\left(e^{-i\omega}\right)$ close to $\hat{B}\left(e^{-i\omega};\gamma\right)$ in frequency bands where $S_{\Delta d}\left(e^{-i\omega}\right)$ is large and let the two be different otherwise. To get the impact of ε_t on P_t right it is crucial that $B\left(e^{-i\omega}\right)$ be close to $\hat{B}\left(e^{-i\omega};\gamma\right)$ for ω in a neighborhood of $\omega = 0$. But, if much of the power of Δd lies in other frequency bands, then the econometrician is likely to obtain a poor estimate of $\psi\left(e^{-i\omega}\right)$ in a neighborhood of $\omega = 0$.

[3] Imagine that the broader economy is an endowment economy where P_t is the price of a Lucas tree and d_t is the fruit falling off the tree. In this way, the actual law of motion of Δd_t is exogenous and not influenced by the fact that agents have a mistaken belief about it.

2. The Computations

Although (1.9) provides insight about the consequences of misspecification in regression, actual implementation of regression is better done by the Yule Walker equations (with some work, you can show that these are the first order conditions associated with (1.9)).

It is interesting to have a look at the 'true' value of $\psi(L)$. For this, fit a 40th order AR fit to 254 first differences of the log of real corporate earnings (including a constant term in the regression, but then forgetting about that term later). Then,

$$\psi(L) = \frac{1}{1 - \phi_1 L - \phi_2 L^2 - ... - \phi_{40} L^{40}},$$

where the ϕ_j's are the OLS estimates. Also

$$\sigma^2 = \frac{1}{254} \sum_{t=1}^{254} \hat{\varepsilon}_t^2.$$

It is not necessary to compute the ψ_j's to evaluate (1.6). However, it is interesting to look at and graph the ψ_j's. An easy way to obtain the ψ_j's by applying the inverse Fourier transform:

$$\begin{aligned} \psi_j &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi\left(e^{-i\omega}\right) e^{i\omega j} d\omega \\ &= \frac{1}{2\pi} \int_{0}^{\pi} \left[\psi\left(e^{-i\omega}\right) e^{i\omega j} + \psi\left(e^{i\omega}\right) e^{-i\omega j}\right] d\omega. \end{aligned} \tag{2.1}$$

Note that in the second equality we have exploited the symmetry in the integrand to shorten the range of the integral by 1/2. This is useful for computational purposes because $\psi\left(e^{-i\omega}\right)$ is fairly non-smooth (it has 40 parameters!) and so many points are required in the Riemann approximation for the approximation to be accurate. Let

$$\omega_j = \frac{2\pi}{N} j, \; j = 0, ..., \frac{N}{2},$$

where N is large and even. Then,

$$\begin{aligned} \psi_\tau &\simeq \frac{1}{2\pi} \sum_{j=1}^{N/2} \left[\psi\left(e^{-i\omega_j}\right) e^{i\omega_j \tau} + \psi\left(e^{i\omega_j}\right) e^{-i\omega_j \tau}\right] (\omega_j - \omega_{j-1}) \\ &= \frac{1}{N} \sum_{j=1}^{N/2} \left[\psi\left(e^{-i\omega_j}\right) e^{i\omega_j \tau} + \psi\left(e^{i\omega_j}\right) e^{-i\omega_j \tau}\right]. \end{aligned}$$

Note that the $j = 0$ term appears to be left off. In fact we're measuring the height of each rectangle by the height of the integrand on the right side of the rectangle. The actual ψ_τ's implied by this formula are discussed below.

To begin the calculations implemented by the econometrician, we require the covariances of Δd_t. Assuming $\hat{B}$ corresponds to a fourth order AR representation, we require

$$\gamma_\tau = E \Delta d_t \Delta d_{t-\tau} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi\left(e^{-i\omega}\right) \psi\left(e^{i\omega}\right) \sigma^2 e^{i\omega\tau} d\omega,$$

for $\tau = 0, 1, ..., 4$. Again, exploit the symmetry in this integral:

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)\sigma^2 e^{i\omega\tau}d\omega \\
= & \frac{1}{2\pi}\left[\int_{0}^{\pi}\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)\sigma^2 e^{i\omega\tau}d\omega + \int_{-\pi}^{0}\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)\sigma^2 e^{i\omega\tau}d\omega\right] \\
= & \frac{\sigma^2}{2\pi}\left[\int_{0}^{\pi}\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)\left(e^{i\omega\tau}+e^{-i\omega\tau}\right)\right]d\omega \\
= & \frac{\sigma^2}{\pi}\int_{0}^{\pi}\cos\left(\omega\tau\right)\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)d\omega.
\end{aligned}$$

Here, we have made use of the fact that $\psi\left(e^{-i\omega}\right)\psi\left(e^{i\omega}\right)$ is real. Then,

$$\begin{aligned}
\gamma_\tau & \simeq \frac{\sigma^2}{\pi}\sum_{j=1}^{N/2}\cos\left(\omega_j\tau\right)\psi\left(e^{-i\omega_j}\right)\psi\left(e^{i\omega_j}\right)\left(\omega_j-\omega_{j-1}\right) \\
& = 2\frac{\sigma^2}{N}\sum_{j=1}^{N/2}\cos\left(\omega_j\tau\right)\psi\left(e^{-i\omega_j}\right)\psi\left(e^{i\omega_j}\right),
\end{aligned}$$

for $\tau = 0, 1, 2, 3, 4$. The following orthogonality conditions are necessary and sufficient to solve the econometrician's problem:

$$E\left[\Delta d_t - \phi_1\Delta d_{t-1} - \phi_2\Delta d_{t-2} - \phi_3\Delta d_{t-3} - \phi_4\Delta d_{t-4}\right]\Delta d_{t-j} = 0,$$

for $j = 1, 2, 3, 4$, or,

$$\begin{aligned}
\gamma_1 & = \phi_1\gamma_0 + \phi_2\gamma_1 + \phi_3\gamma_2 + \phi_4\gamma_3 \\
\gamma_2 & = \phi_1\gamma_1 + \phi_2\gamma_0 + \phi_3\gamma_1 + \phi_4\gamma_2 \\
\gamma_3 & = \phi_1\gamma_2 + \phi_2\gamma_1 + \phi_3\gamma_0 + \phi_4\gamma_1 \\
\gamma_4 & = \phi_1\gamma_3 + \phi_2\gamma_2 + \phi_3\gamma_1 + \phi_4\gamma_0.
\end{aligned}$$

This is expressed in matrix form as follows:

$$d = X\gamma,$$

where γ is given in (1.8) and[4]

$$d = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix},\ X = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix}.$$

Solving for γ:

$$\gamma = X^{-1}d.$$

[4] The matrix X is a Toeplitz matrix obtained in MATLAB with the command, toeplitx(d), where $d = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}'$.

We also require the econometrician's estimated innovation variance. We obtain this as follows. The econometrician's believed representation is:

$$\Delta d_t = \phi_1 \Delta d_{t-1} + \phi_2 \Delta d_{t-1} + \phi_3 \Delta d_{t-1} + \phi_4 \Delta d_{t-1} + u_t,$$

where $Eu_t^2 = \sigma_u^2$. Note

$$\begin{aligned} \gamma_0 &= E(\Delta d_t)^2 = E[\phi_1 \Delta d_{t-1} + \phi_2 \Delta d_{t-2} + \phi_3 \Delta d_{t-3} + \phi_4 \Delta d_{t-4} + u_t] \Delta d_t \\ &= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \phi_3 \gamma_3 + \phi_4 \gamma_4 + \sigma_u^2, \end{aligned}$$

because $Eu_t \Delta d_t = \sigma_u^2$. The econometrician then infers the long run effect of a shock using

$$\psi^{spec.\ error}(\beta) = \frac{1}{1 - \hat{B}(\beta, \gamma)\beta},$$

where the superscript, *spec. error*, indicates that the estimate is based on a specification error (the econometrician has falsely simplified a complex reality by misspecifying what is in fact an order 40 ar with an order 4 ar).

3. The Results

The key results are presented in the following graph. The top panel displays ψ_j and $\psi_j^{spec.\ error}$, for $j = 0, ..., 30$ as 'actual' and 'estimated', respectively.[5] The bottom panel displays the corresponding spectral densities over the frequencies, $\omega \in [0, \pi]$. Note how un-smooth the curves associated with the 40^{th} order autoregressive representation are. This is to be expected, given the large number of parameters.

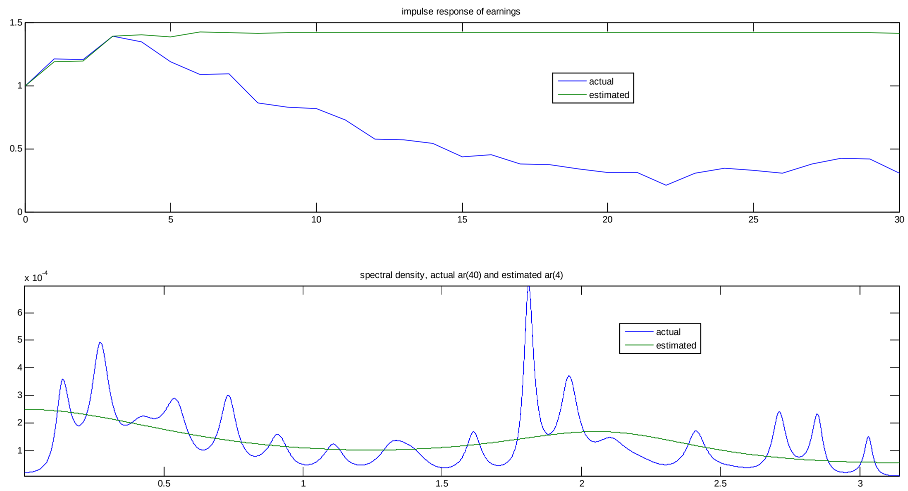
Consider first the bottom panel. The true spectral density is quite wobbly, and it suddenly takes a plunge towards zero near $\omega = 0$.[6] The spectral density implied by the econometrician's model is smooth because it has few parameters and it does its best to cut a smooth line through the actual spectral density. That spectral density rises a little in a neighborhood below $\omega = 0.5$ and the spectral density of the estimated model faithfully reproduces that. However, the estimated spectral density does not have the flexibility to then capture the plunge towards zero near $\omega = 0$. Moreover, as we see in (1.9), there is not much penalty for missing the plunge to zero because the actual spectral density is so small there. This is too bad for our poor econometrician, because the only thing he really cares about is getting things right in a neighborhood of $\omega = 0$.

The error made by the econometrician is to overstate the spectrum at and near $\omega = 0$. Equivalently, this means he overstates the long run impact on earnings of a shock to earnings (see the

[5] In the calculations, I arbitrarily set $\sigma^2 = 0.01^2$, rather than using the estimated one. This has no effect on the reported results, as long as σ_u^2 is computed as indicated in the text.

[6] Incidentally, note the spike in the estimated spectral density in a neighborhood of the frequency corresponding to period 4. There is an important annual component in these quarterly data.

discussion surrounding (1.6) above). We can see this in the top panel of the figure.



The true and estimated values of $\psi(\beta)/(1-\beta)$ are 52 and 192, respectively. There are two notable things about these results. First, the econometrician over estimates the impact of an earnings shock by a factor of 4. Because the econometrician as trader is the one whose actions actually move the stock price, that price will move too much. Also, it is interesting how just how much stock prices move in this model. The variables are to be interpreted as the log of the stock price and the log of earnings. So a one percent shock to earnings ought to move the stock price by 52 percent. In reality it will move the stock price by 192 percent, if all traders make the same specification error when they do their data analysis.