Six Results from the Frequency Domain

- Suppose $\{Y_t\}$ is a covariance stationary process with no deterministic component. By Wold's Decomposition Theorem (see, e.g., Sargent, *Macroeconomic Theory*, chapter XI, section 11) this stochastic process has the following representation:

$$\begin{aligned} Y_t &= D(L)e_t,\ Ee_te_t' = V \\ &= \left[D_0 + D_1L + D_2L^2 + ...\right] e_t, \end{aligned}$$

  where $\{e_t\}$ is serially uncorrelated and

$$\sum_{i=0}^{\infty} D_iD_i' < \infty.$$

- Define:

$$S_Y(z) = D(z)VD(z^{-1}),\ \text{where } z \text{ is a variable.}$$

- **Result #1**:

$$S_Y(z) = C(0) + C(1)z + C(2)z^2 + C(3)z^3 + ...$$

$$+C(-1)z^{-1} + C(-2)z^{-2} + C(-3)z^{-3} + ...$$

where

$$C(\tau) \equiv EY_tY'_{t-\tau}$$

- Proof: do the multiplication and collect terms in powers of z.
- Example:

$$Y_t = D_0e_t + D_1e_{t-1},$$

$$C(0) \equiv EY_tY'_t = D_0VD'_0 + D_1VD'_0$$

$$C(\tau) \equiv EY_tY'_{t-\tau} = \begin{cases} D_1VD'_0 & \tau = 1 \\ 0 & \tau > 1 \\ D_1VD'_0 & \tau = -1 \\ 0 & \tau < -1 \end{cases}.$$

- Note,

$$\begin{aligned}S_Y(z) &= [D_0 + D_1 z]\, V \left[D_0' + D_1' z^{-1}\right] \\ &= D_0 V D_0' + D_1 V D_1' \\ &\quad + D_1 V D_0' z + D_0 V D_1 z^{-1} \\ &= C(0) + C(1) z + C(-1) z^{-1},\end{aligned}$$

  consistent with Result #1.

- **Result #2**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega h} d\omega = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0. \end{cases}$$

- Proof:
  - Result Obvious for $h = 0$. Consider $h \neq 0$
  - Note:

$$\begin{aligned} e^{-i\omega h} &= \cos(-\omega h) + i\sin(-\omega h) \\ &= \cos(\omega h) - i\sin(\omega h) \end{aligned}$$

  - Note:

$$\sin(\pi k) = 0, \text{ for all integer } k.$$

  - Then,

$$\begin{aligned} \int_{-\pi}^{\pi} e^{-i\omega h} d\omega &= \frac{-1}{ih}\left[e^{-i\pi h} - e^{i\pi h}\right] \\ &= \frac{-1}{ih}\left[\cos(\pi h) - i\sin(\pi h) - (\cos(\pi h) + i\sin(\pi h))\right] \\ &= \frac{2}{h}\sin(\pi h) = 0 \end{aligned}$$

- Result #1 and #2 Imply **Result #3**:

$$C(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_Y(e^{-i\omega})e^{i\omega\tau}d\omega.$$

- Proof:

$$\int_{-\pi}^{\pi} S_Y(e^{-i\omega})e^{i\omega\tau}d\omega$$

$$\overbrace{=}^{\text{by Result \#1}} \int_{-\pi}^{\pi}\left[C(0) + C(1)e^{-i\omega} + C(2)e^{-2i\omega} + ... + C(1)' e^{i\omega} + C(2)' e^{2i\omega}\right] e^{i\omega\tau}d\omega$$

$$\overbrace{=}^{\text{by Result \#2}} \int_{-\pi}^{\pi} C(\tau) e^{-\tau i\omega}e^{\tau i\omega}d\omega$$

$$= 2\pi C(\tau).$$

- The Effects of Filtering

  – Consider the Filtered Data:

$$\tilde{Y}_t = F(L)Y_t, \; F(L) = \sum_{j=-\infty}^{\infty} F_j L^j$$

  – The logic that establishes Result #1 implies

$$\begin{aligned} & F(z)D(z)VD\left(z^{-1}\right)' F\left(z^{-1}\right)' \\ = \; & \tilde{C}(0) + \tilde{C}(1)z + \tilde{C}(2)z^2 + ... + \tilde{C}(1)' z^{-1} + \tilde{C}(2)' z^{-2} + ... \end{aligned}$$

  so that, by Result #3

$$E\tilde{Y}_t\tilde{Y}'_{t-\tau} = C_{\tilde{Y}}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{-i\omega})S_Y(e^{-i\omega})F(e^{i\omega})'e^{i\omega\tau}d\omega$$

Six Results from the Frequency Domain ...

– A Filter of Particular Interest (the Band Pass Filter):

$$F_D(L), \text{ such that } F_D\left(e^{-i\omega}\right) = \begin{cases} 1 & \omega \in D \equiv \{\omega : \omega \in [a,b] \cup [-b,-a]\} \\ 0 & \text{otherwise} \end{cases}$$

– Then,

$$C_{\tilde{Y}}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{-i\omega})S_Y(e^{-i\omega})F(e^{i\omega})'e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-b}^{-a} S_Y(e^{-i\omega})e^{i\omega\tau}d\omega + \int_a^b S_Y(e^{-i\omega})e^{i\omega\tau}d\omega\right]$$

$\Rightarrow$Band Pass Filter ‘Shuts Off Power’ for Frequencies Outside D.

– Suppose $D \cap \tilde{D} = \varnothing$, Then **Result #4:**

$$F_D(L)Y_t \perp F_{\tilde{D}}(L)Y_t.$$

∗ Proof: Consider:

$$Z_t = \begin{pmatrix} F_D(L)Y_t \\ F_{\tilde{D}}(L)Y_t \end{pmatrix} = \begin{pmatrix} F_D(L) \\ F_{\tilde{D}}(L) \end{pmatrix} D(L)e_t,$$

∗ By Result #3

$$C_Z(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{bmatrix} F_D(e^{-i\omega})S_Y(e^{-i\omega})F_D(e^{i\omega})' & F_D(e^{-i\omega})S_Y(e^{-i\omega})F_{\tilde{D}}(e^{i\omega})' \\ F_{\tilde{D}}(e^{-i\omega})S_Y(e^{-i\omega})F_D(e^{i\omega})' & F_{\tilde{D}}(e^{-i\omega})S_Y(e^{-i\omega})F_{\tilde{D}}(e^{i\omega})' \end{bmatrix} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{bmatrix} F_D(e^{-i\omega})S_Y(e^{-i\omega})F_D(e^{i\omega})' & 0 \\ 0 & F_{\tilde{D}}(e^{-i\omega})S_Y(e^{-i\omega})F_{\tilde{D}}(e^{i\omega})' \end{bmatrix} d\omega$$

∗ Note that the Upper Right and Lower Left Blocks Are Zero.

– Now Supppose

$$(**)\ D \cap \tilde{D} \ = \ \varnothing \text{ and } D \cup \tilde{D} = [-\pi, \pi].$$

∗ Then, **result #5**

$$F_D(L)Y_t + F_{\tilde{D}}(L)Y_t \ = \ Y_t.$$

The equality means that the stochastic process on the left of the equality has the same covariance function as the stochastic process on the right.

∗ Proof of result #5 - let

$$\eta_t = \tau Z_t,\ \tau = [I \vdots I],$$

where I is the identity matrix with the same dimension as Y_t.

To establish the result, we must establish that the covariance function of $\{\eta_t\}$ and $\{Y_t\}$ coincide.

* Proof of Result #5. Let

$$S_\eta(z) = \tau \begin{pmatrix} F_D(z) \\ F_{\tilde{D}}(z) \end{pmatrix} D(z) V D(z^{-1})' \begin{pmatrix} F_D(z^{-1})' & F_{\tilde{D}}(z^{-1})' \end{pmatrix} \tau'$$

$$= [F_D(z) + F_{\tilde{D}}(z)] \, D(z) V D(z^{-1})' \left[F_D(z^{-1})' + F_{\tilde{D}}(z^{-1})' \right]$$

* By Result #3

$$C_\eta(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[F_D(e^{-i\omega}) + F_{\tilde{D}}(e^{-i\omega}) \right] D(e^{-i\omega}) V D(e^{i\omega})' \left[F_D(e^{i\omega})' + F_{\tilde{D}}(e^{i\omega})' \right] e^{i\omega\tau} d\omega$$

* By $(**)$:

$$F_D(e^{-i\omega}) + F_{\tilde{D}}(e^{-i\omega}) = 1, \text{ for all } \omega \in (-\pi, \pi)$$

so

$$C_\eta(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(e^{-i\omega}) V D(e^{i\omega})' e^{i\omega\tau} d\omega = C_Y(\tau),$$

which establishes the result sought.

- Orthogonal decomposition result.

  Let D_1, D_2, ...,D_N denote a partition of the interal, $(-\pi, \pi)$, into N pieces, with (a_i, b_i) corresponding to each D_i, ω_i an arbitrary element in $[a_i, b_i]$ and

$$D_i \cap D_j = \varnothing \text{ for all } i \neq j,\ D_1 \cup D_2 \cup \cdots \cup D_N = [-\pi, \pi] \,.$$

- Write, $Y_{it} = F_{D_i}(L)\, Y_t$. Then an obvious extension of Result #5 yields

$$\sum_{i=1}^{N} Y_{it} = Y_t,\ Y_{it} \perp Y_{jt} \text{ for all } i \neq j.$$

  Thus, Y_{it}, $i = 1, ..., N$ represents an orthogonal decomposition of Y_t. Note that the variance of Y_{it} is

$$C_{Y_i}(0) = \frac{1}{2\pi}\left[\int_{-b_i}^{-a_i} S_Y(e^{-i\omega})d\omega + \int_{a_i}^{b_i} S_Y(e^{-i\omega})d\omega\right]$$

- By a simple change-of-variable argument,

$$\int_{-b_i}^{-a_i} S_Y(e^{-i\omega})d\omega = \int_{a_i}^{b_i} S_Y(e^{i\omega})d\omega,$$

so that

$$C_{Y_i}(0) = \frac{1}{\pi}\int_{a_i}^{b_i} S_Y(e^{-i\omega})d\omega$$

- Suppose we have a very fine partition, $D_1, ..., D_N$, with N large. By continuity of $S_Y(e^{-i\omega})$ w.r.t. ω

$$\begin{aligned} C_{Y_i}(0) &= \frac{1}{\pi}\int_{a_i}^{b_i} S_Y(e^{-i\omega})d\omega \\ &\simeq \frac{1}{\pi}S_Y(e^{-i\omega_i})\,(b_i - a_i) \end{aligned}$$

- We have established **Result #6**:

  A stationary stochastic proces, $\{Y_t\}$, can be decomposed into orthogonal frequency components each with variance proportional to $S_Y(e^{-i\omega})$, for $\omega \in (0, \pi)$.

- Alternative route to Result #6.

  – Spectral Decomposition Theorem: Any Covariance Stationary Process Can Be Written:

$$Y_t = \int_0^{\pi} [a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)]\, d\omega.$$

  where the four random variables, $a(\omega)$, $a(\omega')$, $b(\omega)$, $b(\omega')$, are independent of each other for all $\omega, \omega' \in (0, \pi)$.

  – Note: Spectral Decomposition Theorem also provides an additive decomposition of Y_t across frequencies.

  – cos and sin are periodic with period 2π

$$\begin{aligned} \cos(\omega t) &= \cos(\omega t'), \\ \omega t' &= \omega t + 2\pi, \\ t' - t &= \frac{2\pi}{\omega}. \end{aligned}$$

  – So, Frequency ω Corresponds to Period $2\pi/\omega$.

- This Completes 'Tour' of Frequency Domain!