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Zero Nominal Interest Rates: Why They're Good and How to Get Them

Harold L. Cole
Senior Economist
Research Department
Federal Reserve Bank of Minneapolis

Narayana Kocherlakota
Senior Economist
Research Department
Federal Reserve Bank of Minneapolis

Abstract

This study shows that in a standard one-sector neoclassical growth model, in which money is introduced with a cash-in-advance constraint, zero nominal interest rates are optimal. Milton Friedman argued in 1969 that zero nominal rates are necessary for efficient resource allocation. This study shows that they are not only necessary but sufficient. The study also characterizes the monetary policies that will implement zero rates. The set of such policies is quite large. The only restriction these policies must satisfy is that asymptotically money shrinks at a rate no greater than the rate of discount.

In a classic essay, Milton Friedman (1969, p. 34) states that only monetary policies that generate a zero nominal interest rate will lead to optimal resource allocations. He argues that "it costs . . . no physical resources to add to real cash balances," and hence it follows that "*the optimum quantity of money . . . will be attained by a rate of price deflation that makes the nominal rate of interest equal to zero*" (italics in original). This prescription of zero nominal interest rates has come to be known as the *Friedman rule.*

Friedman's argument convincingly shows that zero nominal interest rates are necessary for efficient resource allocation. However, Friedman leaves three key questions unanswered. First, are zero nominal interest rates not only necessary, but *sufficient* to ensure an optimal allocation of resources? For example, suppose there is a severe price deflation at the same time that nominal interest rates are zero. Individuals might (inefficiently) lower their capital holdings to take advantage of the high real rate of return offered by money.

Second, what kinds of monetary policies *implement* zero nominal interest rates, in the sense that the policies are consistent with the existence of an equilibrium with zero nominal interest rates? If money growth and inflation rates are equal in equilibrium, then one way to implement zero nominal interest rates would seem to be to shrink the money supply at the efficient rate of return on capital (net of depreciation). Is this true? And, if so, is it the only possible monetary policy that produces zero nominal interest rates?

Finally, we must confront the question of *unique* implementation. For a particular specification of monetary policy, while there may be one equilibrium in which nominal rates are always zero, there may also be one or more equilibria in which they are not. A central bank cannot force individuals to coordinate on its desired equilibrium if other, less desirable equilibria are possible. Hence, we would like to know, What are the characteristics of monetary policies which only implement zero nominal interest rates?

In this article, we use a simple economic model to address these questions of optimality, implementation, and unique implementation of monetary policy. The model is a standard one-sector neoclassical growth model that has one main friction: a *cash-in-advance constraint* that requires households to use cash balances accumulated before each period to buy consumption goods in that period.[1] The cash-in-advance constraint is a simple way to motivate a transactions demand for money: when interest rates are positive, households do not hold money as a store of value, but rather only because they need money to purchase consumption goods. Similarly, the cash-in-advance constraint is generally viewed as a clean way to incorporate the quantity theory of money into a decision-theoretic framework. In particular, if nominal rates are positive, then in each period, households hold only enough money to fund their purchases of consumption goods in the next period. This implies that (consumption) velocity is constant at one, so the inflation rate in any period is equal to the difference between the rates of money growth and consumption growth (which is the essence of the quantity theory of money).

We first use the model to assess the characteristics of interest rates when monetary policy is optimal. The cash-in-advance constraint implies that households have to wait until next period to use their current wage earnings to buy goods. Consequently, households equate their marginal rate of substitution between consumption and leisure not to their marginal product of labor, but rather to their marginal product of labor discounted by the time value of money. We show that this wedge can be eliminated if and only if the time value of money—that is, the nominal interest rate—is zero in every period.

Next, we completely characterize the set of monetary policy rules that implement zero nominal interest rates. Interestingly, the set is defined only by the long-run behavior of monetary policy; even extreme contractions and expansions of the money supply are consistent with zero nominal interest rates as long as such movements do not last for an infinite amount of time. Correspondingly, in these equilibria, real balances may vary considerably and, in fact, can grow exponentially.

Finally, we show that, at least when households have utility functions that are logarithmic in consumption and additively separable in consumption and leisure, there is a large set of policies that uniquely implement zero nominal interest rates. An example of such a policy is one that leads money to shrink for a finite number of periods at a rate no slower than households' psychic discount rate and to shrink thereafter exactly at the psychic discount rate. The intuitive explanation for this example is simple: if the nominal interest rate is positive in any period in this kind of economy, households hold only enough money to buy their desired level of consumption goods. Hence, if the nominal interest rate is to be positive, then the rate of price deflation has to equal the rate of money shrinkage; but this in turn implies a nonpositive nominal interest rate.

Our results have a key theoretical implication. Most economists' intuition about the (long-run) effects of changes in the supply of money is shaped by Friedman's (1963 [1968, p. 39]) famous dictum that "inflation is always and everywhere a monetary phenomenon." Our main message is that while inflation is a monetary phenomenon for any suboptimal monetary policy, inflation is entirely a real phenomenon for any optimal monetary policy (because the rate of deflation equals the real rate of interest).

Our results also have a striking policy implication. Zero nominal interest rates are consistent with a large set of monetary policies. This means that the optimality of monetary policy can be verified only by looking at interest rates, not by looking at the growth rates of the money supply.

The Environment Without Money . . .

In this section, we set out the physical environment in which agents interact, and we characterize efficient allocations in that environment.

We consider an infinite-horizon environment with a continuum of identical households. Each household has a unit of time in every period; this time can be split between leisure l_t and work n_t. There is a single consumption good. In period t, the typical household ranks streams of consumption and leisure $(c_{t+s},l_{t+s})_{s=0}^{\infty}$ according to the utility function

$$(1) \qquad \sum_{s=0}^{\infty}\beta^s u(c_{t+s},l_{t+s}).$$

The utility function u is strictly concave and continuously differentiable and satisfies the conditions that $u_c(0,l) = \infty$ for all l and $u_l(c,0) = \infty$ for all c.

At the beginning of period 1, there are $k_0 > 0$ units of capital. (All quantities are written in per capita terms.) In period t, capital and labor can be used to produce output according to the production function

(2) $$y_t = f(k_{t-1}, n_t).$$

The production function f is continuously differentiable, homogeneous of degree one, and concave.

Output y_t can be split between consumption c_t and investment x_t:

(3) $$y_t = c_t + x_t.$$

Capital accumulates according to this law of motion:

(4) $$k_t = (1-\delta)k_{t-1} + x_t.$$

Capital must satisfy the nonnegativity restriction that

(5) $$k_t \geq 0.$$

Given this description of the environment, what is the symmetric Pareto optimal allocation of resources (in which households all have the same consumption and leisure sequences)? This can be calculated by solving the social planner's problem:

(6) $$\max_{\{c_t,k_t,n_t\}} \sum_{t=1}^{\infty} \beta^t u(c_t, 1-n_t)$$

subject to the physical resource constraints:

(7) $$c_t + k_t \leq f(k_{t-1}, n_t) + (1-\delta)k_{t-1}$$

(8) $$k_t \geq 0$$

(9) $$k_0 \text{ given.}$$

(Note that we have substituted out investment and output in this representation of the planner's problem.)

The *unique optimum* $(c_t, k_t, n_t)_{t=1}^{\infty}$ of the social planner's problem is the unique solution to the following set of equations:

(10) $$c_t + k_t = f(k_{t-1}, n_t) + (1-\delta)k_{t-1}$$

(11) $$u_{l,t} = u_{c,t} f_{n,t}$$

(12) $$-u_{c,t} + \beta(f_{k,t+1} + 1 - \delta)u_{c,t+1} = 0$$

(13) $$\liminf_{t\to\infty} \beta^t u_{c,t} k_t = 0$$

(where *lim inf* represents the *limit infima,* or the greatest lower bounds). Here, and throughout the article, $u_{c,t} = u_c(c_t, l_t)$, $u_{l,t} = u_l(c_t, l_t)$, $f_{n,t} = f_n(k_{t-1}, n_t)$ and $f_{k,t+1} = f_k(k_t, n_{t+1})$.

Henceforth, we use the term *optimal allocation* to refer to the above unique solution to the social planner's problem. We assume that the utility function u and the production function f are such that the optimal allocation is globally stable: for any k_0, the solution to the social planner's problem has the property that (c_t, k_t, n_t) converges to a strictly positive steady state (c_{ss}, k_{ss}, n_{ss}) as t goes to infinity.

. . . And With Money

Here we add to the physical environment just described a particular monetary trading arrangement that households use to allocate resources among themselves, and we characterize the equilibria that arise under this arrangement for different monetary policies. The key feature of the trading arrangement is that households are required to use previously accumulated money balances to buy consumption goods. This cash-in-advance feature generates a transactions demand for money.

Money itself adds no new possibilities for resource reallocations to the environment, so no equilibrium with money can make all households better off relative to the optimal allocation characterized above. In fact, because households must use a low-yield asset (money) for their purchases of consumption goods, equilibrium allocations are typically Pareto inferior to the above optimal allocations.

To describe the monetary trading arrangement, we first specify the ownership of the various goods. There is a continuum of *firms,* each of which is endowed with a constant returns-to-scale technology that allows the firm to produce output according to the above production function (2). *Households* begin life with equal claims to the profits of these firms. (In equilibrium, the profits are zero, so we will ignore them.) Households also own their time endowment, k_0 units of capital, and M_0 units of money. Finally, there is an entity called the *government* which can give money to or take it from households. Before trade begins, the government specifies a sequence of monetary taxes and transfers $\{\tau_t\}_{t=1}^{\infty}$; this transfer sequence implies a sequence of money supply levels by the accumulation equation

(14) $$M_t = \tau_t + M_{t-1}.$$

Trading works as follows. Each household starts period t with m_{t-1} units of money, b_{t-1} units of bonds, and k_{t-1} units of capital. At the beginning of the period, a competitive goods market opens. Let money be the numeraire good in this market. Firms buy labor at wage rate w_t and rent capital at rental rate r_t from households and use these inputs to produce consumption and investment goods. Households buy consumption and investment goods from firms at price p_t.

In the goods market, households face two restrictions on their ability to purchase goods. One is that households do not receive their wage and rental payments until after the goods market has closed. (This can be understood intuitively: a firm cannot pay its workers until the firm has sold its goods.) The other restriction is that households cannot use credit or bonds to purchase consumption goods (although households can use credit to buy investment goods). These two restrictions together imply that all consumption purchases have to be made using the original money holdings m_{t-1}. This restriction is termed a *cash-in-advance constraint;* it is meant to capture the idea that money can be used to buy more goods than can be bought with credit.[2]

After the goods market closes, the asset market opens. In the asset market, households receive $1 + i_{t-1}$ units of money for every unit of bonds with which they started the period. A household also receives its nominal labor income $w_t n_t$ and capital income $r_t k_{t-1}$ less its expenditures on new capital $p_t x_t$ and receives a net transfer of money from the government τ_t. The household divides its nominal wealth in the asset market among money holdings and one-period bonds. Then the asset market closes, and the period ends.

Given the trading arrangement, the problem of a representative household is to

$$(15)\quad \max_{\{c_t,k_t,n_t,m_t,b_t\}} \sum_{t=1}^{\infty} \beta^t u(c_t, 1-n_t)$$

subject to

$$(16)\quad m_{t-1} \geq p_t c_t$$

$$(17)\quad m_t + b_t \leq r_t k_{t-1} + w_t n_t + b_{t-1}(1+i_{t-1}) + m_{t-1} + \tau_t - p_t(c_t+x_t)$$

$$(18)\quad k_t = (1-\delta)k_{t-1} + x_t$$

$$(19)\quad k_t \geq 0,\ m_t \geq 0, \text{ and } b_t \geq -B.$$

The household's first constraint (16) says that all consumption purchases must be financed with cash brought into the goods market. The second constraint (17) says that available wealth can be split between money and bonds in the asset market. The third constraint (18) is the transition equation for the capital stock (4). The last constraint (19) guarantees that capital and money holdings are both nonnegative and imposes a lower bound on debt which rules out Ponzi schemes in which the household borrows an ever-increasing amount over time. We assume that B is sufficiently large so that this constraint never binds in equilibrium.

We use the capital transition equation (18) to substitute out for x_t in the household's budget constraint in the asset market (17). We use μ_t and λ_t to denote the Lagrangian multipliers on constraints (16) and (17), respectively. Since the household's objective function is concave and its constraint set is convex, the household's problem has a unique solution.

This optimum is in turn the unique solution to the first-order conditions and the transversality conditions on the stock variables. The *first-order conditions* consist of (16), (17), and

$$(20)\quad \beta^t u_{c,t} - p_t(\mu_t+\lambda_t) = 0$$

$$(21)\quad \beta^t u_{l,t} - w_t\lambda_t = 0$$

$$(22)\quad -\lambda_t + (1+i_t)\lambda_{t+1} = 0$$

$$(23)\quad \lambda_{t+1}[r_{t+1} + (1-\delta)p_{t+1}] - \lambda_t p_t = 0$$

$$(24)\quad -\lambda_t + \mu_{t+1} + \lambda_{t+1} = 0.$$

These conditions ensure that in any solution to the household's problem, there are no finitely lived deviations which are welfare-improving for the household. The *transversality conditions* consist of

$$(25)\quad \liminf_{t\to\infty} \lambda_t p_t k_t = 0$$

$$(26)\quad \liminf_{t\to\infty} \lambda_t (b_t+B) = 0$$

$$(27)\quad \liminf_{t\to\infty} \lambda_t m_t = 0.$$

Note that the transversality conditions are restrictions only on the limit infima of the relevant sequences, not on the limits. [See the Appendix for a proof of the sufficiency of these five first-order conditions and the three (apparently weak) transversality conditions.]

The problem of the representative firm is a sequence of static maximization problems, since the firm simply seeks to maximize profits in each period by renting labor and capital to produce output which it sells to households. The static problem of the firm, then, is to

$$(28)\quad \max_{N_t,K_t} p_t f(K_t,N_t) - w_t N_t - r_t K_t.$$

The firm's first-order conditions are

$$(29)\quad p_t f_{K,t} = r_t$$

$$(30)\quad p_t f_{N,t} = w_t.$$

Under this trading arrangement, there are five commodities traded in each period: consumption, capital, labor, money, and bonds. The *market-clearing conditions* for the first four of these commodities are

$$(31)\quad f(K_t,N_t) = c_t + x_t$$

$$(32)\quad k_{t-1} = K_t$$

$$(33)\quad n_t = N_t$$

$$(34)\quad m_t = M_t.$$

Since bonds are private assets traded between households, bonds are in zero net supply. Hence, the bond market-clearing condition is

$$(35)\quad b_t = 0.$$

We define an *equilibrium* for the monetary trading arrangement as a sequence of prices and quantities

$$(36)\quad \{p_t,r_t,w_t,i_t,c_t,k_t,n_t,m_t,b_t,K_t,N_t\}_{t=1}^{\infty}$$

such that (i) given these prices, the choice variables of the household and the firm solve their respective problems and (ii) the market-clearing conditions are satisfied.

Consider a sequence $\{p_t,i_t,c_t,k_t,n_t\}$ that satisfies the following set of equations:

$$(37)\quad \beta^{t+1}u_{c,t+1}f_{n,t}(p_t/p_{t+1}) - \beta^t u_{l,t} = 0$$

$$(38)\quad 1 + i_t - (\beta^{t+1}u_{c,t+1}p_{t+2})/(\beta^{t+2}u_{c,t+2}p_{t+1}) = 0$$

$$(39)\quad \beta^{t+2}u_{c,t+2}(p_{t+1}/p_{t+2})(f_{k,t+1}+1-\delta) - \beta^{t+1}u_{c,t+1}(p_t/p_{t+1}) = 0$$

$$(40)\quad f(k_{t-1},n_t) + (1-\delta)k_{t-1} - k_t - c_t = 0$$

$$(41)\quad p_t c_t \leq M_{t-1}$$

(with equality if $i_{t-1} > 0$);

$$(42)\quad \liminf_{t\to\infty} \beta^t u_{c,t} M_{t-1}/p_t = 0$$

$$(43)\quad \liminf_{t\to\infty} \beta^t u_{c,t} k_{t-1} = 0.$$

With such a sequence, we can use the firm's first-order conditions and the market-clearing conditions to figure out values for $\{r_t,w_t,K_t,N_t\}$ such that $\{p_t,i_t,c_t,k_t,n_t,r_t,w_t,K_t,N_t\}$ is an equilibrium. Consequently, hereafter, when we refer to an *equilibrium*, we will be referring to a sequence $\{p_t, i_t,c_t,k_t,n_t\}$ that satisfies equations (37)–(43).

Implementing Optimal Policy

As stated in the introduction, the article is about three questions: Are zero nominal interest rates both necessary and sufficient conditions for optimality of monetary policy? What kinds of monetary policies implement zero nominal interest rates? And what kinds of monetary policies uniquely implement zero nominal interest rates? In this section, we answer these three questions in the following three propositions.

Optimality

The first proposition demonstrates that zero nominal interest rates are both necessary and sufficient conditions for optimality of monetary policy.

PROPOSITION 1. *Equilibrium quantities are Pareto optimal if and only if $i_t = 0$ for all t.*

Proof. Suppose that $i_t = 0$ for all t. This fact implies, from condition (22) of the household's problem, that $\lambda_t = \lambda_{t+1}$. This in turn, along with condition (24) of the household's problem, implies that $\mu_{t+1} = 0$ for all t. This result, along with condition (20), implies that in the solution to the household's problem,

$$u_{c,t+1}/\beta u_{c,t+2} = p_{t+1}/p_{t+2}. \tag{44}$$

Hence, we have

$$\beta u_{c,t+1}(f_{k,t+1}+1-\delta) - u_{c,t} = 0 \tag{45}$$

$$u_{c,t}/u_{l,t} = f_{n,t}. \tag{46}$$

Our equilibrium thus satisfies the optimality conditions (10)–(13) and so is optimal.

Now suppose that in an equilibrium $i_t \neq 0$ for some t. Then our optimality condition (11) is not satisfied since

$$u_{l,t} = \beta u_{c,t+1} f_{n,t} p_t/p_{t+1} \tag{47}$$

$$= u_{c,t} f_{n,t}/(1+i_t). \tag{48}$$

In words, the marginal rate of substitution between consumption and leisure is not equal to the marginal product of labor. If $i_t \neq 0$, then, quantities are not Pareto optimal.

Q.E.D.

No matter what the tax and transfer scheme is, as long as interest rates are equal to zero, the equilibrium outcome satisfies (10)–(13) and so is Pareto optimal. What creates a distortion here is the lag between households' working and their being able to use their wage income to buy consumption goods. If nominal interest rates are zero, then households are indifferent between being paid today or being paid in the future, and the distortion associated with the trading arrangement is eradicated.

Implementation

Proposition 1 shows that the Friedman rule is optimal. The next proposition answers our second question by characterizing the set of monetary policy choices that implement this rule.[3]

PROPOSITION 2. *An equilibrium such that $i_t = 0$ forever exists if and only if both*

(i) $\liminf_{t\to\infty} M_t = 0$

(ii) $\inf_t M_t\beta^{-t} = \kappa > 0$

are true.

Proof. First, we show that these conditions are sufficient to guarantee the existence of such an equilibrium. We start by assuming that the money supply satisfies the two conditions. Set $i_t = 0$, and suppose that $p_t = \beta^{t-1}u_{c,t}\,p_1/u_{c,1}$ for all $t > 1$, where p_1 is a constant to be specified later. Suppose that the equilibrium quantities are equal to the sequence $\{c_t,k_t,n_t\}_{t=1}^{\infty}$ which satisfies (10)–(13), where $K_t = k_{t-1}$ and $N_t = n_t$. We can set $b_t = 0$ and the input prices to satisfy (29) and (30). To see that the transversality condition with regard to money is satisfied, note that

$$\liminf_{t\to\infty} \beta^t u_{c,t} M_{t-1}/p_t = 0 \tag{49}$$

since $\beta^t u_{c,t}/p_t$ is constant (because $i_t = 0$) and since $\liminf_{t\to\infty} M_{t-1} = 0$.

To complete the proof of the sufficiency of conditions (i) and (ii) of Proposition 2, we need to pick p_1 so that the cash-in-advance constraint is always satisfied. We know that the Pareto optimal sequence $\{c_t\}$ converges to a positive value c_{ss}. Hence, there is a bound c^* such that $c_t \leq c^*$ for all t. To ensure that the cash-in-advance constraint is satisfied, pick $p_0 \leq \kappa/c^*$. Then

$$M_t/(p_t c_t) \geq M_t/(p_t c^*) = M_t\beta^{-t}/(p_0 c^*) \geq \kappa/(p_0 c^*). \tag{50}$$

Next we show that Proposition 2's two conditions are necessary. First note that if $i_t = 0$, then $\beta^t u_{c,t}/p_t$ is constant; thus, the transversality condition on money can only be satisfied if condition (i) is satisfied. Next, to prove the necessity of (ii), recall from Proposition 1 that if $i_t = 0$, then the equilibrium quantities are determined by (10)–(13); hence, $c_t \to c_{ss} > 0$. Therefore, as t goes to infinity,

$$\beta^{t-1}/p_t \to u_c(c_1,n_1)/[p_1 u_c(c_{ss},n_{ss})]. \tag{51}$$

Because $u_c(0,n) = \infty$, Pareto optimal quantities are always positive. By combining that result with the fact that c_t goes to a positive limit, we can conclude that c_t is bounded away from zero. Thus, $\beta^{-t}c_t p_t$ is bounded from below by some positive number κ. The cash-in-advance constraint tells us that

$$1 \leq M_t/(p_t c_t) = M_t\beta^{-t}/(\beta^{-t}p_t c_t) \leq M_t\beta^{-t}/\kappa \tag{52}$$

which in turn implies condition (ii). Q.E.D.

Proposition 2 completely characterizes the wide class of monetary policies for which some equilibrium exhibits zero nominal interest rates. The key restrictions are on the long-run behavior of money. Condition (i) says that for some subsequence of periods $\{t_1,t_2,t_3,...\}$, M_{t_n} converges to 0 as t_n goes to infinity. Intuition tells us that as long as condition (i) is satisfied, households cannot increase current consumption by permanently lowering their money holdings by a discrete amount. Note that for some monetary policies that satisfy condition (i), real balances may be growing exponentially (although not faster than interest rates), and nonetheless, households are at an optimum. Condition (ii) says that if money falls faster than β asymptotically and nominal interest rates are zero, then prices eventually fall at rate β, so the cash-in-advance constraint will eventually be violated.

The asymptotic restrictions in Proposition 2 have surprisingly little bite for short- or intermediate-run behavior. Even though the money supply is growing or shrinking at any rate over any finite period of time, nominal interest rates may still always be zero. Moreover, the money supply can be oscillating aperiodically between an exponential growth path and an exponential decline path forever, and nominal interest rates may still always be zero. In any of these equilibria, the quantity theory is no longer valid because the behavior of prices over these arbitrarily long periods of time is dictated solely by the behavior of real quantities, not by the behavior of money supplies.

In our model, the initial price level is endogenous, but that assumption is not driving Proposition 2. Suppose the initial price level were exogenously specified to be p_1. Then, if nominal rates are to be zero, the entire sequence of prices is pinned down by the resultant equilibrium condition that $u_{c,t}\beta^t/p_t$ is constant over time. Despite this determinacy of the price level, there is still a large set of money supplies consistent with zero nominal interest rates. As long as the money supply is such that the cash-in-advance constraint is satisfied in every period, and the money supply converges to zero along some subsequence of periods (requirements which are not mutually exclusive because prices are converging to zero over time), the money supply is consistent with zero nominal interest rates in every period. Thus, even with an exogenous initial price level, there is a large (infinite-dimensional) set of monetary policies consistent with zero nominal interest rates.

Unique Implementation

Proposition 2 guarantees only that if monetary policy satisfies the two conditions, some equilibrium will deliver zero nominal interest rates. We can easily show that if $M_{t+1}/M_t = \delta$, where $1 > \delta > \beta$, another (suboptimal) equilibrium exists in which the cash-in-advance constraint binds and nominal rates are positive. As we stressed in the introduction, we want to be able to uniquely implement zero nominal interest rates in order to rule out the kinds of monetary policies which could lead to either optimal or suboptimal equilibrium quantities. The following proposition provides a set of monetary policies that uniquely implement zero nominal interest rates (at least when preferences are logarithmic in consumption).

PROPOSITION 3. *Let $u(c,l) = \ln(c) + v(l)$. Suppose that for some $T \geq 1$, $M_{t+1}/M_t \leq \beta$ for all $t \leq T$ and $M_{t+1}/M_t = \beta$ for all $t > T$. Then, in all equilibria, $i_t = 0$ for all t.*

Proof. Assume otherwise—that $i_t \neq 0$. If $i_t > 0$, then $\mu_{t+1} > 0$ and $p_{t+1}c_{t+1} = M_t$. Since $p_{t+2} \leq M_{t+1}/c_{t+2}$, condition (38) implies that

$$(53) \qquad 1 + i_t = p_{t+2}c_{t+2}/\beta p_{t+1}c_{t+1} \leq M_{t+1}/\beta M_t \leq 1$$

which is a contradiction. Since $i_t \geq 0$ (or households would strictly prefer to borrow in order to hold money), it follows that $i_t = 0$. Q.E.D.

Standard quantity theory logic, along with the Fisher equation, implies that the way to generate zero nominal interest rates is for the money supply to shrink at the real rate of interest. However, Proposition 2 makes clear that contrary to the simple logic of the quantity theory and the Fisher equation, this is not the only way to achieve zero nominal interest rates. Proposition 3 does partially support the common intuition by showing that shrinking the money supply at the rate of discount will uniquely implement zero interest rates. But Proposition 3 also shows that many other money supply paths (those that feature temporarily faster rates of shrinkage) will uniquely implement zero rates.

These results are surprising because the cash-in-advance model is widely viewed as providing an intellectual underpinning for the quantity theory: in any period in which the interest rate is positive, the inflation rate equals the difference between the growth rates of money and consumption. We have seen here, though, that this feature fails to hold exactly when monetary policy is optimal. Along equilibrium paths in which nominal interest rates are always zero, the inflation rate is independent of the growth rate of the money supply—which is hardly consistent with typical presentations of the quantity theory.

A Puzzle

Now we consider what happens if the government chooses taxes and transfers so that money shrinks faster than the rate of discount in every period. We find that this situation presents something of a puzzle since, at least for log utility, it has no equilibrium.

PROPOSITION 4. *If $u(c,n) = \ln(c) + v(n)$, then if $M_{t+1}/M_t \leq \delta < \beta$ for all t, there is no equilibrium.*

Proof. If $M_{t+1}/M_t \leq \delta < \beta$ for all t, then $\inf_t M_t\beta^{-t} = 0$. Then, from Proposition 2 we know that there is no equilibrium in which $i_t = 0$ for all t. Assume that $i_t > 0$ in some period. Note that this implies that $\mu_t > 0$ and, hence, that the cash-in-advance constraint holds with equality in period t (that is, $p_{t+1}c_{t+1} = M_t$):

$$(54) \qquad 1 + i_t = \lambda_t/\lambda_{t+1} = (\beta^{t+1}u_{c,t+1}/p_{t+1})/(\beta^{t+2}u_{c,t+2}/p_{t+2})$$

$$(55) \qquad = \beta^{-1}p_{t+2}c_{t+2}/p_{t+1}c_{t+1} = \beta^{-1}p_{t+2}c_{t+2}/M_t$$

$$(56) \qquad \leq \beta^{-1}M_{t+1}/M_t \leq 1.$$

This contradicts our assumption that $i_t > 0$. Q.E.D.

The intuition behind this proposition is simple. We know that the cash-in-advance constraint does not bind in any period. If it did, then the nominal interest rate in that period would be bounded above by the sum of the rate of discount and the rate of money shrinkage; this sum is negative because money is shrinking so fast. However, if the cash-in-advance constraint does not bind in any period, then equilibrium quantities are Pareto optimal and nominal interest rates are zero. Asymptotically, prices must grow at rate β, and this implies that the cash-in-advance constraint will eventually be violated.

We call this a *puzzle* because our (standard) notion of equilibrium in our (standard) trading arrangement in our (standard) environment does not tell us what happens for a wide class of monetary policies that governments might contemplate using. The question facing researchers is, What notions of equilibrium, trading arrangements, or environments should we be examining instead to understand the effects of these policies?

Concluding Comments

We have shown that in a standard one-sector neoclassical growth model, in which money is introduced with a cash-in-advance constraint, zero nominal interest rates are optimal; and we have characterized the monetary policies that will implement zero rates. Surprisingly, we have found that the set of such policies is quite large. The only restriction that these policies must satisfy is that asymptotically money shrinks at a rate no greater than the rate of discount.

The intuition behind this result is simple. When the nominal interest rate is zero, the rate of growth of prices is pinned down to equal the rate of deflation, but individuals do not care how much real balances they hold, as long as the amount is at least as large as their consumption needs. Because the demand for real balances is indeterminate when interest rates are zero, the set of nominal money supply paths that intersect with the money demand function at zero nominal interest rates is large.

Our results can be extended to generalizations of our physical environment. For example, trivially, they can be extended to a multisector neoclassical growth model since none of our results hinge on the existence of a single consumption or capital good. Also, versions of our results can be obtained for environments in which total factor productivity is stochastic, though for those environments, the results do have to be amended to respect the stochastic version of the transversality condition.

We have proven our results for a particular monetary trading arrangement. However, our results apply to any monetary trading arrangement that satisfies the following *satiation property:* For any given level of consumption, there exists a finite level of real money balances such that households with real balances above that level are indifferent between using money and bonds as a way of accumulating additional wealth if the two assets earn the same rate of return. This property holds for the *cash-credit goods* arrangement considered by Robert Lucas and Nancy Stokey (1987); they allow for a type of consumption good which, like capital in our model, can be purchased on credit. The satiation property also holds for versions of *shopping time* models, in which money allows agents to conserve on transaction costs, and *money-in-the-utility-function* models, in which households derive a direct benefit from holding money.

Our results should also carry over directly if we extend the environment to include government debt. Then the Friedman rule can be interpreted as pegging the interest rate on government debt to zero. Our characterizations (in Propositions 2 and 3) of the money supply sequences that implement and uniquely implement the Friedman rule apply immediately.

Much of the recent literature concerning the Friedman rule focuses on environments in which governments must raise taxes through distortionary means. The arguments we have made about implementing zero nominal interest rates can be extended to environments with distortionary taxes if the monetary trading arrangements satisfy the above satiation property. Of course, zero interest rates will not be necessary and sufficient conditions for optimal monetary policy in all such environments. However, V. V. Chari, Lawrence Christiano, and Patrick Kehoe (1996) consider the monetary arrangements we have discussed and show that zero nominal interest rates are necessary and sufficient for optimal monetary policy if preferences satisfy certain homotheticity and separability conditions which are generally considered natural.

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[1]The *cash-in-advance constraint* is a commonly used device to motivate a demand for money in otherwise frictionless economic models. It is a feature of models used, for example, by Robert Clower (1967), Jean Michel Grandmont and Yves Younes (1972), Charles Wilson (1979), and Robert Lucas (1984).

[2]We do not explain why households use a trading arrangement with these two restrictions. Some informational imperfections can be embedded into the physical environment described above to produce the two restrictions; for examples, see the work of Robert Townsend (1987) and Harold Cole and Alan Stockman (1992). We suspect (but have not proven) that our results are robust to making these imperfections explicit.

[3]Wilson (1979) proves a result similar to Proposition 2.

Appendix

Sufficiency of the First-Order and Transversality Conditions for Household Optimality

In this appendix, we demonstrate that the first-order conditions and the transversality conditions described in the preceding paper are sufficient for household optimality.

Let $\{c_t,l_t\}$ be part of a sequence of vectors that satisfy the first-order conditions and transversality conditions, and suppose that $\{c'_t,l'_t\}$ gives more utility to the household. Then

$$\text{(A1)}\quad 0 < \lim_{T\to\infty}\sum_{t=1}^{T}\beta^t\{u(c'_t,l'_t) - u(c_t,l_t)\}$$

$$\text{(A2)}\quad \le \liminf_{T\to\infty}\sum_{t=1}^{T}\beta^t\{u_c(c_t,l_t)(c'_t-c_t) + u_l(c_t,l_t)(l'_t-l_t)\}$$

by concavity;

$$\text{(A3)}\quad = \liminf_{T\to\infty}\sum_{t=1}^{T}\{\lambda_t p_t(c'_t-c_t) + \lambda_t w_t(l'_t-l_t) + \mu_t p_t(c'_t-c_t)\}$$

by the first-order conditions;

$$\text{(A4)}\quad \le \liminf_{T\to\infty}\sum_{t=1}^{T}\big[\lambda_t\{(m'_{t-1}-m_{t-1}) + (b'_{t-1}-b_{t-1})(1+i_{t-1}) + [r_t + p_t(1-\delta)](k'_{t-1}-k_{t-1})\} + \lambda_t\{(m_t-m'_t) + (b_t-b'_t) + (k_t-k'_t)p_t\} + \mu_t p_t(m'_{t-1}/p_t)-(m_{t-1}/p_t)\big]$$

by the wealth and cash-in-advance constraints;

$$\text{(A5)}\quad = \liminf_{T\to\infty}\lambda_T[(m_T-m'_T) + (b_T-b'_T) + p_T(k_T-k'_T)]$$

by the first-order conditions;

$$\text{(A6)}\quad \le \liminf_{T\to\infty}\lambda_T[m_T + (b_T+B) + k_T p_T] - \liminf_{T\to\infty}\lambda_T[m'_T + (b'_T+B) + k'_T p_T]$$

$$\text{(A7)}\quad \le \liminf_{T\to\infty}\lambda_T[m_T + (b_T+B) + k_T p_T]$$

by the nonnegativity constraints; and

$$\text{(A8)}\quad = 0$$

by the transversality condition. This assumption generates a contradiction. We can therefore conclude that if a sequence of quantities satisfies the first-order conditions and the transversality conditions, then it must be optimal for the household.

References

Chari, V. V.; Christiano, Lawrence J.; and Kehoe, Patrick J. 1996. Optimality of the Friedman rule in economies with distorting taxes. *Journal of Monetary Economics* 37 (April): 203–23.

Clower, Robert W. 1967. A reconsideration of the microfoundations of monetary theory. *Western Economic Journal* 6 (December): 1–8.

Cole, Harold L., and Stockman, Alan C. 1992. Specialization, transactions technologies, and money growth. *International Economic Review* 33 (May): 283–98.

Friedman, Milton. 1963. *Inflation: Causes and consequences.* Bombay: Asia Publishing House (for Council for Economic Education). Reprinted 1968. In *Dollars and deficits,* pp. 21–71. Englewood Cliffs, N.J.: Prentice-Hall.

___________. 1969. The optimum quantity of money. In *The optimum quantity of money and other essays,* pp. 1–50. Chicago: Aldine.

Grandmont, Jean Michel, and Younes, Yves. 1972. On the role of money and the existence of a monetary equilibrium. *Review of Economic Studies* 39 (July): 355–72.

Lucas, Robert E., Jr. 1984. Money in a theory of finance. *Carnegie-Rochester Conference Series on Public Policy* 21 (Autumn): 9–45.

Lucas, Robert E., Jr., and Stokey, Nancy L. 1987. Money and interest in a cash-in-advance economy. *Econometrica* 55 (May): 491–513.

Townsend, Robert M. 1987. Asset return anomalies in a monetary economy. *Journal of Economic Theory* 41 (April): 219–47.

Wilson, Charles. 1979. An infinite horizon model with money. In *General equilibrium, growth, and trade,* ed. Jerry R. Green and José Alexandre Scheinkman, pp. 81–104. New York: Academic Press.