Technical Appendix to "Risk Shocks",

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A Appendix A: Scaling and Miscellaneous Variables

To solve our model, we require that the variables be stationary. To this end, we adopt a particular scaling of the variables. Because our model satisfies sufficient conditions for balanced growth, when the equilibrium conditions of the model are written in terms of the scaled variables, only the growth rates and not the levels of the stationary shocks appear. In this appendix we describe the scaling of the model that is adopted. In addition, we describe the mapping from the variables in the scaled model to the variables measured in the data.

Let

$$\begin{aligned} q_t &= \Upsilon^t \frac{Q_{\bar{K}',t}}{P_t}, y_{z,t} = \frac{Y_t}{z_t^+},\ i_t = \frac{I_t}{z_t^+ \Upsilon^t},\ \tilde{w}_t \equiv \frac{W_t}{z_t^+ P_t}, \\ \bar{k}_t &= \frac{\bar{K}_t}{z_{t-1}^+ \Upsilon^{t-1}},\ r_t^k = \Upsilon^t \tilde{r}_t^k,\ \mu_{z,t}^* = \frac{z_t^+}{z_{t-1}^+},\ c_t = \frac{C_t}{z_t^+}, \end{aligned}$$

where $\tilde{r}_t^k P_t$ denotes the nominal rental rate on capital. The rate of inflation in the nominal wage rate is:

$$\pi_{w,t} \equiv \frac{W_t}{W_{t-1}} = \frac{\tilde{w}_t \mu_{z,t}^* \pi_t}{\tilde{w}_{t-1}}.$$

Consider gdp growth, according to the model.

$$\frac{Y_t^{gdp}}{z_t^+} \equiv y_t = c_t + \frac{i_t}{\mu_{\Upsilon,t}} + g_t,$$

or,

$$Y_t^{gdp} = y_t z_t^+,$$

so that

$$\begin{aligned} \Delta \log Y_t^{gdp} &= \log Y_t^{gdp} - \log Y_{t-1}^{gdp} = \log(y_t) - \log(y_{t-1}) + \log\left(z_t^+\right) - \log\left(z_{t-1}^+\right) \\ &= \log(y_t) - \log(y_{t-1}) + \log \frac{\mu_{z,t}^*}{\mu_z^*}. \end{aligned}$$

Note that we have subtracted the steady state value of $\log \mu_{z,t}^*$ from this expression. This is because $\Delta \log Y_t^{gdp}$ is the growth rate of GDP, after subtracting its steady state value.

Let N_{t+1} denote period t nominal net worth, so that

$$n_{t+1} = \frac{N_{t+1}}{P_t z_t^+}.$$

Then,

$$\Delta \log \frac{N_{t+1}}{P_t} = \log n_{t+1} - \log n_t + \log \frac{\mu_{z,t}^*}{\mu_z^*}.$$

Again, this variable is expressed in deviation from its steady state.

Another variable is investment. There is an issue about what units to measure investment in. Investment times its relative price is given by:

$$inv_t \equiv \frac{I_t}{\Upsilon^t \mu_{\Upsilon,t}} = \frac{i_t z_t^+ \Upsilon^t}{\Upsilon^t \mu_{\Upsilon,t}} = \frac{i_t z_t^+}{\mu_{\Upsilon,t}},$$

so that, in deviation from steady state:

$$\Delta \log inv_t \equiv \log inv_t - \log inv_{t-1} = \log i_t - \log i_{t-1} + \log \frac{\mu_{z,t}^*}{\mu_z^*} - \left(\log \mu_{\Upsilon,t} - \log \mu_{\Upsilon,t-1}\right).$$

The relative price of investment goods is given by

$$P_{I,t} \equiv \frac{1}{\Upsilon^t \mu_{\Upsilon,t}},$$

so that

$$\begin{aligned}
\Delta \log P_{I,t} &= -t \log \Upsilon + (t-1) \log \Upsilon - \log \mu_{\Upsilon,t} + \log \mu_{\Upsilon,t-1} + \log \Upsilon \\
&= -\log \mu_{\Upsilon,t} + \log \mu_{\Upsilon,t-1},
\end{aligned}$$

in deviation from steady state.

$$\Delta \log C_t = \log c_t - \log c_{t-1} + \log \frac{\mu_{z,t}^*}{\mu_z^*}$$

Real credit growth (in deviation from steady state) for entrepreneurs is computed as follows:

$$\begin{aligned}
Credit_t^e &= \left[q_t \bar{k}_{t+1} - n_{t+1}\right] z_t^+ \\
\Delta Credit_t^e &= \log\left[q_t \bar{k}_{t+1} - n_{t+1}\right] - \log\left[q_{t-1} \bar{k}_t - n_t\right] + \log \frac{\mu_{z,t}^*}{\mu_z^*}
\end{aligned}$$

To obtain total credit growth, we need to add the credit by intermediate good firms for working capital. From (2.40), this credit, scaled by $P_t z_t^*$ is:

$$\psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\Upsilon \mu_{z^*,t}} + \psi_{l,t}\tilde{w}_t.$$

The real amount of this credit is:

$$Credit_t^f = \left[\psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\Upsilon \mu_{z^*,t}} + \psi_{l,t}\tilde{w}_t\right] z_t^*.$$

So total credit, $Credit_t$, is:

$$Credit_t = \left[q_t \bar{k}_{t+1} - n_{t+1} + \psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\Upsilon \mu_{z^*,t}} + \psi_{l,t}\tilde{w}_t\right] z_t^*,$$

and its growth rate (in deviation from steady state) is:

$$\begin{aligned}\Delta Credit_t \quad = \quad & \log\left[q_t \bar{k}_{t+1} - n_{t+1} + \psi_{k,t}\frac{r_t^k u_t \bar{k}_t}{\Upsilon \mu_{z^*,t}} + \psi_{l,t}\tilde{w}_t\right] \\ & - \log\left[q_{t-1}\bar{k}_t - n_t + \psi_{k,t-1}\frac{r_{t-1}^k u_{t-1}\bar{k}_{t-1}}{\Upsilon \mu_{z^*,t-1}} + \psi_{l,t-1}\tilde{w}_{t-1}\right] + \log\frac{\mu_{z,t}^*}{\mu_z^*}.\end{aligned}$$

The growth rate of the real wage is:

$$\Delta \log\frac{W_t}{P_t} = \log\tilde{w}_t - \log\tilde{w}_{t-1} + \log\frac{\mu_{z,t}^*}{\mu_z^*}$$

B Appendix B: Dynamic Equations

B.1 Equilibrium Conditions

B.1.1 Prices

The equations pertaining to prices are:

$$(1) p_t^* - \left[(1-\xi_p)\left(\frac{K_{p,t}}{F_{p,t}}\right)^{\frac{\lambda_f}{1-\lambda_f}} + \xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]^{\frac{1-\lambda_f}{\lambda_f}} = 0 \qquad (2.21)$$

and

$$(2) E_t\left\{\zeta_{c,t}\lambda_{z,t}y_{z,t} + \left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_f}}\beta\xi_p F_{p,t+1} - F_{p,t}\right\} = 0, \qquad (2.22)$$

where $\lambda_{z,t}$ denotes $\lambda_t z_t^* P_t$. Also,

$$(3)\zeta_{c,t}\lambda_{z,t}\lambda_f y_{z,t} s_t + \beta\xi_p \left(\frac{\tilde{\pi}_{t+1}}{\pi_{t+1}}\right)^{\frac{\lambda_f}{1-\lambda_f}} K_{p,t+1} - K_{p,t} = 0. \tag{2.23}$$

Note that both these equations involve $F_{p,t}$. This reflects that a lot of equations have been substituted out. In particular, we have

$$(4)F_{p,t}\left[\frac{1-\xi_p\left(\frac{\tilde{\pi}_t}{\pi_t}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f} = K_{p,t},\ \tilde{p}_t = \frac{K_{p,t}}{F_{p,t}},$$

where $\tilde{p}_t$ is the price set by price-optimizing firms in period t. In addition, $\tilde{p}_t$ is substituted out using the equilibrium condition relating the aggregate price level to the prices of intermediate goods.

B.1.2 Wages

The demand for labor is the solution to the following problem:

$$\max W_t \overbrace{\left[\int_0^1 (h_{t,i})^{\frac{1}{\lambda_w}} di\right]^{\lambda_w}}^{=l_t} - \int_0^1 W_{t,i} h_{t,i} di,$$

where $W_{t,i}$ is the wage rate of $i-$type workers and W_t is the wage rate for homogeneous labor, l_t. The first order condition is:

$$h_{t,i} = l_t \left(\frac{W_t}{W_{t,i}}\right)^{\frac{\lambda_w}{\lambda_w - 1}}.$$

The wages of non-optimizing households evolve as follows:

$$W_{j,t} = \tilde{\pi}_{w,t}\left(\mu_{z^*,t}\right)^{\iota_\mu}\left(\mu_{z^*}\right)^{1-\iota_\mu} W_{j,t-1},\ \tilde{\pi}_{w,t} \equiv \left(\pi_t^*\right)^{\iota_{w1}}\left(\pi_{t-1}\right)^{\iota_{w2}}\bar{\pi}^{1-\iota_{w1}-\iota_{w2}}. \tag{2.24}$$

Nominal wage growth, $\pi_{w,t}$, is:

$$\pi_{w,t} = \frac{\tilde{w}_t \mu_{z,t}^* \pi_t}{\tilde{w}_{t-1}},$$

where $\tilde{w}_t$ denotes the scaled wage rate:

$$\tilde{w}_t \equiv \frac{W_t}{z_t^* P_t}.$$

The labor input variable that we treat as observed is the sum over the various different types of labor:

$$
\begin{aligned}
h_t &= \int_0^1 h_{it} di \\
&= l_t W_t^{\frac{\lambda_w}{\lambda_w - 1}} \int_0^1 \left(W_{t,i}\right)^{\frac{\lambda_w}{1-\lambda_w}} di \\
&= l_t W_t^{\frac{\lambda_w}{\lambda_w - 1}} \left(W_t^*\right)^{\frac{\lambda_w}{1-\lambda_w}},
\end{aligned}
$$

where

$$
\begin{aligned}
W_t^* &\equiv \left[\int_0^1 \left(W_{t,i}\right)^{\frac{\lambda_w}{1-\lambda_w}} di\right]^{\frac{1-\lambda_w}{\lambda_w}} \\
&= \left[\left(1-\xi_w\right)\tilde{W}_t + \int_{\xi_w \text{ monopolists that do not reoptimize}} \left(\tilde{\pi}_{w,t}\left(\mu_{z^*,t}\right)^{\iota_\mu}\left(\mu_{z^*}\right)^{1-\iota_\mu} W_{i,t-1}\right)^{\frac{\lambda_w}{1-\lambda_w}} di\right]^{\frac{1-\lambda_w}{\lambda_w}} \\
&= \left[\left(1-\xi_w\right)\tilde{W}_t + \xi_w\left(\tilde{\pi}_{w,t}\left(\mu_{z^*,t}\right)^{\iota_\mu}\left(\mu_{z^*}\right)^{1-\iota_\mu} W_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}.
\end{aligned}
$$

Let $w_t^* \equiv W_t^*/W_t$, and use linear homogeneity:

$$
w_t^* = \left[\left(1-\xi_w\right)\frac{\tilde{W}_t}{W_t} + \xi_w\left(\frac{\tilde{\pi}_{w,t}\left(\mu_{z^*,t}\right)^{\iota_\mu}\left(\mu_{z^*}\right)^{1-\iota_\mu}}{\pi_{w,t}} w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}},
$$

$\tilde{W}_t$ is the nominal wage set by the $1-\xi_w$ wage optimizers in the current period. Rewriting,

$$
w_t^* = [(1-\xi_w) w_t^{\frac{\lambda_w}{1-\lambda_w}} + \xi_w\left(\frac{\tilde{\pi}_{w,t}\left(\mu_{z,t}^*\right)^{\iota_\mu}\left(\mu_z^*\right)^{1-\iota_\mu}}{\pi_{wt}} w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}]^{\frac{1-\lambda_w}{\lambda_w}}, \tag{2.25}
$$

where

$$
w_t \equiv \frac{\tilde{W}_t}{W_t}. \tag{2.26}
$$

We conclude:

$$
h_t = l_t \left(w_t^*\right)^{\frac{\lambda_w}{1-\lambda_w}}. \tag{2.27}
$$

For purposes of evaluating aggregate utility, it is also convenient to have an expression for the following:

$$
\begin{aligned}
&\int_0^1 h_{it}^{1+\sigma_L} di \\
&= l_t^{1+\sigma_L} W_t^{-\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \int_0^1 \left(W_{t,i}\right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} di \\
&= l_t^{1+\sigma_L} W_t^{-\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \ddot{W}_t^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}},
\end{aligned}
$$

where

$$\ddot{W}_t \equiv \left[\int_0^1 (W_{t,i})^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} di \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}}.$$

Then,

$$\begin{aligned}
\ddot{W}_t &= \left[\int_0^1 (W_{t,i})^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} di \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}} \\
&= \left[(1-\xi_w) \left(\tilde{W}_t\right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} + \int_{\xi_w \text{ that change}} (W_{t,i})^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} di \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}} \\
&= \left[(1-\xi_w) \left(\tilde{W}_t\right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} + \xi_w \left(\tilde{\pi}_{w,t} \left(\mu_{z^*,t}\right)^{\iota_\mu} \left(\mu_{z^*}\right)^{1-\iota_\mu} \ddot{W}_{t-1} \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}}.
\end{aligned}$$

Divide by W_t and make use of the linear homogeneity of the above expression:

$$\frac{\ddot{W}_t}{W_t} = \left[(1-\xi_w) \left(\frac{\tilde{W}_t}{W_t} \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} + \xi_w \left(\frac{\tilde{\pi}_{w,t} \left(\mu_{z^*,t}\right)^{\iota_\mu} \left(\mu_{z^*}\right)^{1-\iota_\mu}}{\pi_{w,t}} \frac{\ddot{W}_{t-1}}{W_{t-1}} \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}}$$

Define

$$\ddot{w}_t = \frac{\ddot{W}_t}{W_t},$$

so that

$$\ddot{w}_t = \left[(1-\xi_w) (w_t)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} + \xi_w \left(\frac{\tilde{\pi}_{w,t} \left(\mu_{z^*,t}\right)^{\iota_\mu} \left(\mu_{z^*}\right)^{1-\iota_\mu}}{\pi_{w,t}} \ddot{w}_{t-1} \right)^{\frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w(1+\sigma_L)}}, \tag{2.28}$$

using (2.26). We conclude

$$\begin{aligned}
\int_0^1 h_{it}^{1+\sigma_L} di &= \left[l_t \left(\ddot{w}_t\right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{(1+\sigma_L)} \qquad (2.29) \\
&= \left[h_t \left(\frac{\ddot{w}_t}{w_t^*} \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{(1+\sigma_L)}.
\end{aligned}$$

using (2.27).

The optimality conditions associated with wage-setting are characterized by:

$$(5) E_t\{\zeta_{c,t}\lambda_{z,t} \frac{\left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w-1}} h_t \left(1-\tau_t^l\right)}{\lambda_w} + \beta\xi_w \left(\mu_{z^*}\right)^{\frac{1-\iota_\mu}{1-\lambda_w}} E_t \left(\mu_{z^*,t+1}\right)^{\frac{\iota_\mu}{1-\lambda_w}-1} \left(\frac{1}{\pi_{w,t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}} \frac{\tilde{\pi}_{w,t+1}^{\frac{1}{1-\lambda_w}}}{\pi_{t+1}} F_{w,t+1} - F_{w,t}\} = 0 \tag{2.30}$$

and

$$(6)\ E_t\{\zeta_{c,t}\zeta_t\left[(w_t^*)^{\frac{\lambda_w}{\lambda_w-1}}h_t\right]^{1+\sigma_L}+\beta\xi_w\left(\frac{\tilde{\pi}_{w,t+1}\left(\mu_{z,t+1}^*\right)^{\iota_\mu}\left(\mu_z^*\right)^{1-\iota_\mu}}{\pi_{wt+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}K_{w,t+1}-K_{w,t}\}=0.$$

$$(7)\ \frac{1}{\psi_L}\left[\frac{1-\xi_w\left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}}\left(\mu_{z^*}\right)^{1-\iota_\mu}\left(\mu_{z^*,t}\right)^{\iota_\mu}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right]^{1-\lambda_w(1+\sigma_L)}\tilde{w}_tF_{w,t}-K_{w,t}=0$$

Optimization by households implies:

$$w_t=\left[\frac{\psi_L}{\tilde{w}_t}\frac{K_{w,t}}{F_{w,t}}\right]^{\frac{1-\lambda_w}{1-\lambda_w(1+\sigma_L)}},$$

so that, using (2.25):

$$w_t^*=\left[(1-\xi_w)\left[\frac{\psi_L}{\tilde{w}_t}\frac{K_{w,t}}{F_{w,t}}\right]^{\frac{\lambda_w}{1-\lambda_w(1+\sigma_L)}}+\xi_w\left(\frac{\tilde{\pi}_{w,t}\left(\mu_{z,t}^*\right)^{\iota_\mu}\left(\mu_z^*\right)^{1-\iota_\mu}}{\pi_{wt}}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}.$$

We can replace $K_{w,t}/F_{w,t}$ with the expression implied by (7) above:

$$(8)\ w_t^*=\left[(1-\xi_w)\left(\frac{1-\xi_w\left(\frac{\tilde{\pi}_{w,t}}{\pi_{w,t}}\left(\mu_{z^*}\right)^{1-\iota_\mu}\left(\mu_{z^*,t}\right)^{\iota_\mu}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right)^{\lambda_w}+\xi_w\left(\frac{\tilde{\pi}_{w,t}\left(\mu_{z,t}^*\right)^{\iota_\mu}\left(\mu_z^*\right)^{1-\iota_\mu}}{\pi_{wt}}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}}$$

B.1.3 Capital Utilization, Marginal Cost, Return on Capital, Investment, Monetary Policy

The first order necessary condition associated with the capital utilization decision is:

$$P_t\frac{1}{\Upsilon^t}\tau_t^oa'\left(u_t\right)\bar{K}_t=P_t\tilde{r}_t^k\bar{K}_t,$$

or,

$$\tau_t^oa'\left(u_t\right)=\Upsilon^t\tilde{r}_t^k=r_t^k,$$

after scaling. Making use of our assumed utilization cost function, this reduces to:

$$(9)\ r_t^k=\tau_t^or^k\exp(\sigma_a\left[u-1\right]).$$

Marginal cost:

$$
\begin{aligned}
(10) r_t^k &= \frac{\alpha \epsilon_t}{\left[1+\psi_{k,t} R_t\right]} \left(\frac{\Upsilon \mu_{z,t}^* L_t \left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w - 1}}}{u_t \bar{k}_t} \right)^{1-\alpha} s_t \\
\tilde{w}_t &= \frac{(1-\alpha) \epsilon_t}{\left[1+\psi_{l,t} R_t\right]} \left(\frac{\Upsilon \mu_{z,t}^* L_t \left(w_t^*\right)^{\frac{\lambda_w}{\lambda_w - 1}}}{u_t \bar{k}_t} \right)^{-\alpha} s_t,
\end{aligned} \tag{2.31}
$$

where $\psi_{k,t}$ and $\psi_{l,t}$ denote the fraction of the capital services and labor bills, respectively, that must be financed in advance. Combining the last two equations, we obtain the familiar expression for marginal cost:

$$
(11)\ s_t = \frac{1}{\epsilon_t} \left(\frac{r_t^k \left[1+\psi_{k,t} R_t\right]}{\alpha} \right)^{\alpha} \left(\frac{\tilde{w}_t \left[1+\psi_{l,t} R_t\right]}{1-\alpha} \right)^{1-\alpha} \tag{2.32}
$$

Resource constraint:

$$
(12) \tau_t^o a(u_t) \frac{\bar{k}_t}{\Upsilon \mu_{z,t}^*} + g_t + c_t + \frac{i_t}{\mu_{\Upsilon,t}} = y_{z,t} \tag{2.33}
$$

where g_t is an exogenous stochastic process and

$$
(13) \bar{k}_{t+1} = (1-\delta) \frac{1}{\mu_{z,t}^* \Upsilon} \bar{k}_t + \left[1 - S\left(\frac{\zeta_{i,t}\, i_t \mu_{z,t}^* \Upsilon}{i_{t-1}} \right)\right] i_t, \tag{2.34}
$$

where i_t is investment scaled by $z_t^* \Upsilon^t$.

Equation defining the nominal non-state contingent rate of interest:

$$
(14) E_t \{ \beta \frac{1}{\pi_{t+1} \mu_{z,t+1}^*} \zeta_{c,t+1} \lambda_{z,t+1} \left(1+R_t\right) - \zeta_{c,t} \lambda_{z,t} \} = 0 \tag{2.35}
$$

The derivative of utility with respect to consumption is,

$$
(15) E_t \left[(1+\tau^C) \zeta_{c,t} \lambda_{z,t} - \frac{\mu_{z,t}^* \zeta_{c,t}}{c_t \mu_{z,t}^* - b c_{t-1}} + b\beta \frac{\zeta_{c,t+1}}{c_{t+1} \mu_{z,t+1}^* - b c_t} \right] = 0, \tag{2.36}
$$

where c_t denotes consumption scaled by z_t^*. The capital first order condition:

$$
(16) E_t \left\{ -\zeta_{c,t} \lambda_{zt} + \frac{\beta}{\pi_{t+1} \mu_{z,t+1}^*} \zeta_{c,t+1} \lambda_{zt+1} \left(1 + R_{t+1}^k\right) \right\} = 0, \tag{2.37}
$$

where R_{t+1}^k denotes the rate of return on capital:

$$
(17)\ 1 + R_t^k = \frac{(1-\tau_{t-1}^k) \left[u_t r_t^k - \tau_t^o a(u_t)\right] + (1-\delta) q_t}{\Upsilon q_{t-1}} \pi_t + \tau_{t-1}^k \delta
$$

where q_t denotes the scaled market price of capital, $Q_{\bar{K}',t}$:

$$q_t = \Upsilon^t \frac{Q_{\bar{K}',t}}{P_t}.$$

The investment first order condition:

$$\begin{aligned}
(18)\ E_t\{\zeta_{c,t}\lambda_{zt}q_t \left[1 - S(\frac{\zeta_{i,t}\mu^*_{z,t}\Upsilon i_t}{i_{t-1}}) - S'(\frac{\zeta_{i,t}\mu^*_{z,t}\Upsilon i_t}{i_{t-1}})\frac{\zeta_{i,t}\mu^*_{z,t}\Upsilon i_t}{i_{t-1}}\right] \qquad (2.38)\\
-\frac{\zeta_{c,t}\lambda_{zt}}{\mu_{\Upsilon,t}} + \frac{\beta\lambda_{zt+1}\zeta_{c,t+1}\zeta_{i,t+1}q_{t+1}}{\mu^*_{z,t+1}\Upsilon}S'(\frac{\zeta_{i,t+1}\mu^*_{z,t+1}\Upsilon i_{t+1}}{i_t})\left(\frac{\mu^*_{z,t+1}\Upsilon i_{t+1}}{i_t}\right)^2\} &= 0,
\end{aligned}$$

where i_t is scaled (by $z_t^*\Upsilon^t$) investment. The scaled representation of aggregagte output is:

$$(19)\ y_{z,t} \equiv \frac{Y_t}{z_t^*} = (p_t^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\epsilon_t \left(\frac{u_t\bar{k}_t}{\mu^*_{z,t}\Upsilon}\right)^\alpha \left((w_t^*)^{\frac{\lambda_w}{\lambda_w-1}} h_t\right)^{1-\alpha} - \phi\right]$$

The monetary policy rule:

$$\begin{aligned}
(20)\ \log(1+R_t) &= (1-\tilde{\rho})\log(1+R) + \tilde{\rho}\log(1+R_{t-1}) \qquad (2.39)\\
&\quad + \frac{1-\tilde{\rho}}{1+R}\left[\tilde{a}_p\pi\log\frac{\pi_{t+1}}{\pi_t^*} + \tilde{a}_y\frac{1}{4}\log\frac{y_t}{y}\right] + x_t^p,
\end{aligned}$$

where x_t^p is an iid monetary policy shock and y_t denotes scaled GDP:

$$(21)\ y_t = g_t + c_t + \frac{i_t}{\mu_{\Upsilon,t}}.$$

Total nonfinancial sector borrowing is an important variable to match with the data.

Borrowing is an important variable in the model. In the CEE model, borrowing by nonfinancial firms is for paying the capital rental bill and the wage bill. In unscaled terms, this is:

$$\psi_{k,t}P_t\tilde{r}_t^kK_t + \psi_{l,t}W_tl_t.$$

We scale this by dividing by $P_tz_t^*$:

$$\begin{aligned}
&\psi_{k,t}\frac{\tilde{r}_t^ku_t\bar{K}_t}{z_t^*} + \psi_{l,t}\frac{W_t}{P_tz_t^*}l_t\\
&= \psi_{k,t}\frac{r_t^ku_tz^*_{t-1}\Upsilon^{t-1}\bar{k}_t}{\Upsilon^tz_t^*} + \psi_{l,t}\tilde{w}_t\\
&= \psi_{k,t}\frac{r_t^ku_t\bar{k}_t}{\Upsilon\mu_{z^*,t}} + \psi_{l,t}\tilde{w}_t. \qquad (2.40)
\end{aligned}$$

B.1.4 Entrepreneurs

cutoff equation we obtain:

$$(16)\ \frac{q_t \bar{k}_{t+1}}{n_{t+1}} \frac{1+R^k_{t+1}}{1+R_t} \left[\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right] - \frac{q_t \bar{k}_{t+1}}{n_{t+1}} + 1 = 0, \tag{2.41}$$

which must hold in each realized $t+1$ state of nature. Here,

$$\begin{aligned} \overbrace{\Gamma_t(\bar{\omega}_{t+1})}^{\text{share of entrepreneurial earnings, } \left(1+R^k_{t+1}\right) q_t \bar{k}_{t+1} \text{, received by bank}} &\equiv \bar{\omega}_{t+1}\left[1 - F_t(\bar{\omega}_{t+1})\right] + G_t(\bar{\omega}_{t+1}) \\ G_t(\bar{\omega}_{t+1}) &\equiv \int_0^{\bar{\omega}_{t+1}} \omega dF_t(\omega). \end{aligned} \tag{2.42}$$

Substituting out for η_{t+1} from the second first order condition into the first, we obtain:

$$(17) E_t \left\{ \left[1 - \Gamma_t(\bar{\omega}_{t+1})\right] \frac{1+R^k_{t+1}}{1+R_t} + \frac{\Gamma'_t(\bar{\omega}_{t+1})}{\Gamma'_t(\bar{\omega}_{t+1}) - \mu G'_t(\bar{\omega}_{t+1})} \left[\frac{1+R^k_{t+1}}{1+R_t} \left(\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right) - 1 \right] \right\} = 0, \tag{2.43}$$

where $\Gamma'_t(\bar{\omega}_{t+1}) = 1 - F_t(\bar{\omega}_{t+1})$. In principle these equations should have been derived separately for entrepreneurs with each different level of possible net worth. It is clear from the first order conditions that had we done so, each one's standard debt contract would have been characterized by the same ϱ_t, $\{\bar{\omega}_{t+1}\}$.

We now derive the law of motion of net worth. After the loan contract received in $t-1$ is settled, but before it is known which entrepreneur exits and which stays, the (scaled by $P_t z^*_t$) net worth in period t of entrepreneurs is

$$V_t = \overbrace{\left[1 - \Gamma_{t-1}(\bar{\omega}_t)\right]}^{\text{share of entrepreneurial earnings received by entrepreneurs}} \times R^k_t \frac{q_{t-1}}{\pi_t \mu^*_{z,t}} \bar{k}_t,$$

where the appearance of $\pi_t \mu^*_{z,t}$ in the denominator reflects that $q_{t-1}\bar{k}_t$ has been scaled by

$P_{t-1}z^*_{t-1}$. The above expression can be written

$$\begin{aligned} V_t &= \overbrace{\left\{1-\bar{\omega}_t\left[1-F_{t-1}(\bar{\omega}_t)\right]-\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)\right\}}^{=1-\Gamma_{t-1}(\bar{\omega}_t)} R^k_t \frac{q_{t-1}}{\pi_t \mu^*_{z,t}}\bar{k}_t \\ &= \left(1+R^k_t\right)\frac{q_{t-1}}{\pi_t \mu^*_{z,t}\Upsilon}\bar{k}_t - \overbrace{\left(\bar{\omega}_t\left[1-F_{t-1}(\bar{\omega}_t)\right]+(1-\mu)\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)\right)R^k_t\frac{q_{t-1}}{\pi_t \mu^*_{z,t}\Upsilon}\bar{k}_t}^{\text{earnings of banks, which must equal } B_t(1+R_{t-1})=(1+R_{t-1})(q_{t-1}\bar{k}_t-n_t)} \\ &\quad -\mu\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)R^k_t\frac{q_{t-1}}{\pi_t \mu^*_{z,t}\Upsilon}\bar{k}_t \\ &= \left[1+R^k_t-(1+R_{t-1})-\mu\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)\left(1+R^k_t\right)\right]\frac{q_{t-1}}{\pi_t \mu^*_{z,t}}\bar{k}_t+\frac{1+R_{t-1}}{\pi_t \mu^*_{z,t}}n_t. \end{aligned}$$

At this point, γ_t entrepreneurs exit and are replaced by γ_t new arrivals. Both surviving entrepreneurs and new arrivals receive a lump sum transfer in the amount, w^e. Thus, $n_{t+1} = \gamma_t V_t + w^e$, or,

$$n_{t+1} = \frac{\gamma_t}{\pi_t \mu^*_{z,t}}\left\{R^k_t - R_{t-1} - \mu\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)\left(1+R^k_t\right)\right\}\bar{k}_t q_{t-1} + w^e + \gamma_t\left(\frac{1+R_{t-1}}{\pi_t \mu^*_{z,t}}\right)n_t. \tag{2.44}$$

The resource constraint becomes:

$$\begin{aligned} & d_t + c_t + g_t + \frac{i_t}{\mu_{\Upsilon,t}} + \Theta\frac{1-\gamma_t}{\gamma_t}\left[n_{t+1}-w^e\right] + \tau^o_t a(u_t)\frac{\bar{k}_t}{\Upsilon\mu^*_{z,t}} \\ = \;& y_{z,t} \end{aligned} \tag{2.45}$$

Here, $\left[n_{t+1}-w^e\right]/\gamma_t$ denotes the assets of entrepreneurs before they have received their real transfer, w^e, and before it is determined which is selected to exit. The assets of the fraction of entrepreneurs that exit is $(1-\gamma_t)$ times this amount, and they consume Θ of their assets, with the other $1-\Theta$ being transferred to households. Also, d_t denotes the resources used up in monitoring:

$$d_t = \frac{\mu G(\bar{\omega}_t)\left(1+R^k_t\right)q_{t-1}\bar{k}_t}{\pi_t \mu^*_{z,t}}. \tag{2.46}$$

In the modified economy, entrepreneurs rather than households accumulate capital. This means that the household intertemporal equation, (2.37), (i.e., (12)) must be deleted. So, we have added three new equations, (2.43), (2.11) and (2.44) and deleted one. The net increase in the number of equations is two. We increase the number of endogenous variables by two: $\bar{\omega}_{t+1}$ and n_{t+1} (the first variable is a function of the period $t+1$ state of nature, while the second is a function of the period t state of nature).

B.1.5 Social Welfare Function

We now turn to developing an expression for the representative household's utility function

$$\begin{aligned} Util_t &= \zeta_{c,t} \log(z_t^+ c_t - b z_{t-1}^+ c_{t-1}) - \psi_L \int_0^1 \frac{h_{it}^{1+\sigma_L}}{1+\sigma_L} di \\ &= \zeta_{c,t} \left\{ \log \left[z_t^+ (c_t - b \frac{z_{t-1}^+}{z_t^+} c_{t-1}) \right] - \psi_L \int_0^1 \frac{h_{it}^{1+\sigma_L}}{1+\sigma_L} di \right\} \\ &= \zeta_{c,t} \left\{ \log(c_t - \frac{b}{\mu_{z,t}^*} c_{t-1}) - \frac{\psi_L}{1+\sigma_L} \int_0^1 h_{it}^{1+\sigma_L} di \right\}, \end{aligned}$$

apart from a constant term. Using (2.29):

$$\frac{\psi_L}{1+\sigma_L} \int_0^1 h_{it}^{1+\sigma_L} di = \frac{\psi_L}{1+\sigma_L} \left[h_t \left(\frac{\ddot{w}_t}{w_t^*} \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{(1+\sigma_L)},$$

so that

$$Util_t = \zeta_{c,t} \left\{ \log(c_t - \frac{b}{\mu_{z,t}^*} c_{t-1}) - \frac{\psi_L}{1+\sigma_L} \left[h_t \left(\frac{\ddot{w}_t}{w_t^*} \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{(1+\sigma_L)} \right\},$$

where $\ddot{w}_t$ is defined in (2.28) and w_t^* is defined in (8). Both these variables are unity in steady state.

C Appendix C: Understanding the Effects of Financial Friction Shocks

Our key empirical finding is that shocks to risk, σ, can account for a large portion of business cycle fluctuations. To clarify the economics of this result, we discuss the impact of an increase in σ on equilibrium loan contracts. In addition, it is standard in analyses of the 2008 crisis, to suppose that it was triggered by a shock to net worth. In our environment, this is captured by a perturbation in γ.[57] One of our empirical findings is that, from the perspective of our model, such a shock is an unlikely candidate as business cycle shock because it implies, counterfactually, that credit is countercyclical. To build intuition, this section performs a microeconomic analysis of the credit market. Our results are summarized in the two comparative statics exercises summarized in Figures 1a and 1b.

To simplify notation and because we are concerned with only one period of time, we delete

[57] See Christiano and Ikeda (2011) and the references they cite.

time subscripts. We highlight a partial equilibrium and a general equilibrium effect on the loan contract of an increase in σ. The former effect refers to what happens to the loan contract, holding fixed the key market variable, R^k/R, taken as given by participants in the market for entrepreneurial credit. Recall, R^k is the across-entrepreneur average return on capital and R is the interest rate on the mutual funds' source of funds. The general equilibrium effect refers to the additional changes to the loan contract that occur when R^k/R also adjusts in response to a change in risk. The general equilibrium effects of a change in risk are important for understanding our empirical results.

Entrepreneurs have access to a constant returns to scale project with return, $R^k\omega$, where R^k is common knowledge and ω has a log-normal distribution with $E\omega = 1$ and $\log\omega$ has standard deviation, σ. Denote the total assets acquired by entrepreneurs by $A = N + B$, where B is the size of the loans received from mutual funds. Denote leverage by $L = A/N$. We characterize the standard debt contract received by entrepreneurs in terms of a value for L and a interest rate spread, Z/R, where Z is the interest rate on the entrepreneurial loan. As in (2.10),

$$\bar{\omega} = \frac{Z}{R}\frac{R}{R^k}\frac{L-1}{L}, \tag{3.47}$$

represents the value of ω that separates bankrupt and non-bankrupt entrepreneurs. The objective of entrepreneurs is proportional to:

$$\left[1 - \Gamma\left(\bar{\omega}\right)\right] L. \tag{3.48}$$

The menu of standard debt contracts available to entrepreneurs is given by:

$$L = \frac{1}{1 - \left[\Gamma\left(\bar{\omega}\right) - \mu G\left(\bar{\omega}\right)\right]\frac{R^k}{R}}. \tag{3.49}$$

In our numerical example, we use the steady state values of the variables used in our empirical analysis:

$$\mu = 0.21,\ \frac{R^k}{R} = 1.0073,\ \sigma = 0.26.$$

Our partial equilibrium experiment increases σ by 5 percent and holds R^k/R fixed. The equilibria corresponding to the two values of σ are exhibited in Figure 1a, which displays the interest rate spread, Z/R, on the vertical axis and leverage, L, on the horizontal. The graphs of (3.49) corresponding to the two values of σ are indicated in the figure. Both are upward-sloping, so that an entrepreneur can obtain a loan contract with higher leverage but this requires paying a higher interest rate spread. This is because, with higher leverage the entrepreneur imposes a greater cost on its mutual fund in the event of default. In both cases, the menu of contracts implied by (3.49) is bowed towards the southeast.[58]

Expression (3.48) can be used to construct an indifference map for entrepreneurs, though we only display the indifference curves that are tangent to the relevant menu of contracts. Indifference curves have a positive slope. This is because, holding the interest rate fixed, (3.48) is increasing in L and holding L fixed (3.48) is decreasing in Z/R.[59] The indifference curves are bowed towards the northwest and entrepreneurial utility is decreasing in that direction. The equilibrium loan contract occurs at a point of tangency of the entrepreneur's indifference curve and the menu of contracts.

The equilibrium for the lower of the two values of σ is associated with a level of leverage, $L = 2.02$, and an interest rate spread of 0.616 in annual, percent terms. With the jump in σ, the indifference curves change shape and the menu of contracts shifts. Not surprisingly, the menu shifts up. That is, entrepreneurs may still obtain the same leverage as before the rise in σ, but in this case they must pay a higher interest rate spread. The higher interest rate spread is required because the rise in σ increases the probability of default, and so raises the cost of lending to banks. If they chose to do so, entrepreneurs could even select a higher level of leverage in response to the increase in σ. As it happens, the new point of tangency involves a 3 percent jump in the interest rate premium, to 0.635 percent, and a slightly larger percent drop in leverage, to 1.95.

In the general equilibrium of our model, there is another effect associated with a temporary increase in risk. The fall in credit associated with the reduction in leverage leads to a reduction in entrepreneurial purchases of physical capital. This in turn leads to a fall in the production of capital by households which results in a fall in its price, $Q_{\bar{K}}$. The anticipated capital gains associated with the expectation that the effects on $Q_{\bar{K}}$ will be undone raises R^k. Figure 1b shows the impact of an increase in R^k/R by 1 percent. This corresponds roughly to a 1 percentage point increase in in the *net* return, $R^k - 1$, expressed in the time units of the model (i.e., one quarter). Given the large rise in the return on capital it is not surprising that the equilibrium involves a substantial increase in leverage. Thus, we can expect this general equilibrium effect to mute the negative impact on leverage of a jump in σ. In our numerical

[58] For a thorough discussion of the menu of contracts, see http://faculty.wcas.northwestern.edu/~lchrist/research/ECB/risk_shocks/risk_shocks.html

[59] Expression (3.48) may not be increasing in L for small values of L. This is because an increase in L has two countervailing effects on entrepreneurial utility. For each fixed and finite value of $\bar{\omega}$ fixed, (3.48) indicates that utility is strictly increasing in L (it is easy to show that $0 < \Gamma < 1$ when $F(\bar{\omega}) < 1$ for each finite $\bar{\omega}$, an assumption that is satisfied when F corresponds to the log-normal distribution). At the same time, an increase in L leads to a rise in $\bar{\omega}$ and this makes $1 - \Gamma$ fall, as the probability that the entrepreneur makes positive profits falls. This latter effect vanishes for sufficiently large L because in that case $\bar{\omega}$ ceases to vary with L.For additional discussion, see http://faculty.wcas.northwestern.edu/~lchrist/research/ECB/risk_shocks/risk_shocks.html

experiments, we have never found examples where this general equilibrium effect on leverage was actually larger than the partial equilibrium effect.[60]

We did find that general equilbrium effects tend to dominate partial equilibrium effects in the case of shocks to equity. Thus, suppose there is a drop in γ, captured in our numerical example by a drop in N. The impact on leverage in partial equilbirium is nil, since N does not separately enter the analysis. Thus, the partial equilibrium impact of a drop in N is an equiproportionate cut in credit, i.e., B. In general equilibrium the consequent drop in $A = N + B$ produces a drop in $Q_{\bar{K}}$ and a rise in R^k as discussed above. This in turn leads to a rise in B, as indicated in Figure 1b. We found that there is a tendency for the general equilibrium rise in B to dominate the partial equilibrium fall in B. That is, in numerical simulations of our dynamic model, a drop in γ tends to produce a rise in B. Because this rise in B in practice is smaller than the initial drop in N, $N + B$ still drops when both partial and general equilibrum effects are accounted for.

D Appendix D: The Fisherian Debt-Deflation Hypothesis

We wish to diagnose the role of the assumption that payments to households are non-state contingent in nominal terms. We do this by exploring the BGG version of the model in which the payment on households' bank deposits is non-state contingent in real terms. Thus, suppose that instead of earning gross nominal return, $1 + R_t$, from t to $t+1$ households instead earn gross nominal return,

$$F_t \pi_{t+1},$$

from t to $t+1$. Here, F_t denotes the real return from t to $t+1$, which is non-state contingent in real terms. With two exceptions, we substitute $1 + R_t$ with $F_t \pi_{t+1}$ everywhere. The two exceptions are the Taylor rule, where we continue to assume a non-state contingent nominal rate of interest is 'controlled'. To ensure that that rate of interest is well defined, we keep equation (10). We add an equation for household deposits:

$$(10)' E_t\{\beta \frac{1}{\mu_z} \lambda_{z,t+1} F_t - \lambda_{z,t}\} = 0.$$

[60] We suspect such an example may be impossible. If the general equilibrium effect dominated, then credit flows would increase after a positive shock to σ, and these would give rise to a fall in R^k, contradicting the rise needed to get the general equilibrium effect to be operative in the first place.

We must change the relevant equations associated with the entrepreneur. The zero profit condition becomes:

$$(16)' \; \Gamma_{t-1}(\bar{\omega}_t) - \mu G_{t-1}(\bar{\omega}_t) = \frac{F_{t-1}\pi_t}{1+R_t^k}\left(1 - \frac{n_t}{q_{t-1}k_t}\right).$$

The optimality condition becomes:

$$(17)' E_t\left\{[1-\Gamma_t(\bar{\omega}_{t+1})]\frac{1+R_{t+1}^k}{F_t\pi_{t+1}} + \frac{\Gamma_t'(\bar{\omega}_{t+1})}{\Gamma_t'(\bar{\omega}_{t+1}) - \mu G_t'(\bar{\omega}_{t+1})}\left[\frac{1+R_{t+1}^k}{F_t\pi_{t+1}}\left(\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right) - 1\right]\right\} = 0$$

and the law of motion of net worth becomes:

$$(18)' n_{t+1} = \frac{\gamma_t}{\pi_t\mu_z^*}\left\{1 + R_t^k - F_{t-1}\pi_t - \mu\int_0^{\bar{\omega}_t}\omega dF_{t-1}(\omega)\left(1+R_t^k\right)\right\}k_t q_{t-1} + w^e + \gamma_t\frac{F_{t-1}}{\mu_z}n_t$$

E Appendix E: Steady State

Here, we discuss an algorithm for computing the steady state of the model. In our analysis, we distinguish between steady state inflation, π, and the quantity appearing in the price and wage updating equations, $\bar{\pi}$. Equation (2.21) in steady state, is:

$$p^* = \left[\frac{\left(1-\xi_p\right)\left(\frac{1-\xi_p\left(\frac{\bar{\pi}}{\pi}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right)^{\lambda_f}}{1-\xi_p\left(\frac{\bar{\pi}}{\pi}\right)^{\frac{\lambda_f}{1-\lambda_f}}}\right]^{\frac{1-\lambda_f}{\lambda_f}}.$$

Note that, if $\pi = \bar{\pi}$ then $p^* = 1$. Equation (2.22):

$$F_p = \frac{\lambda_z\left(p^*\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\left(\frac{k}{\mu_z^*\Upsilon}\right)^{\alpha}\left(\left(w^*\right)^{\frac{\lambda_w}{\lambda_w-1}}h\right)^{1-\alpha} - \phi\right]}{1-\left(\frac{\bar{\pi}}{\pi}\right)^{\frac{1}{1-\lambda_f}}\beta\xi_p},$$

assuming

$$\left(\frac{\tilde{\pi}}{\pi}\right)^{\frac{1}{1-\lambda_f}}\beta\xi_p < 1.$$

Equation (2.23) in steady state is:

$$F_p = \frac{\lambda_z\lambda_f\left(p^*\right)^{\frac{\lambda_f}{\lambda_f-1}}\left[\left(\frac{k}{\mu_z}\right)^{\alpha}\left(\left(w^*\right)^{\frac{\lambda_w}{\lambda_w-1}}h\right)^{1-\alpha} - \phi\right]s}{\left[\frac{1-\xi_p\left(\frac{\bar{\pi}}{\pi}\right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p}\right]^{1-\lambda_f}\left[1-\beta\xi_p\left(\frac{\bar{\pi}}{\pi}\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]}$$

Equating the preceding two equations:

$$s = \frac{1}{\lambda_f} \frac{\left[\frac{1-\xi_p \left(\frac{\bar{\pi}}{\pi} \right)^{\frac{1}{1-\lambda_f}}}{1-\xi_p} \right]^{1-\lambda_f} \left[1 - \beta \xi_p \left(\frac{\bar{\pi}}{\pi} \right)^{\frac{\lambda_f}{1-\lambda_f}} \right]}{1 - \left(\frac{\bar{\pi}}{\pi} \right)^{\frac{1}{1-\lambda_f}} \beta \xi_p}. \tag{5.50}$$

In the case, $\pi = \bar{\pi}$, $s = 1/\lambda_f$. Equation (2.30) in steady state is:

$$F_w = \frac{\lambda_z \frac{(w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h \left(1-\tau^l\right)}{\lambda_w}}{1 - \beta \xi_w \tilde{\pi}_w^{\frac{1}{1-\lambda_w}} \frac{\left(\frac{1}{\pi} \right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi}},$$

as long as the condition,

$$\beta \xi_w \tilde{\pi}_w^{\frac{1}{1-\lambda_w}} \frac{\left(\frac{1}{\pi} \right)^{\frac{\lambda_w}{1-\lambda_w}}}{\pi} < 1,$$

is satisfied. Also

$$\tilde{\pi}_w = (\pi)^{\iota_{w,2}} \bar{\pi}^{1-\iota_{w,2}}.$$

Equation (**??**) is

$$F_w = \frac{\left[(w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h \right]^{1+\sigma_L}}{\frac{1}{\psi_L} \left[\frac{1-\xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w} \right]^{1-\lambda_w(1+\sigma_L)} \tilde{w} \left[1 - \beta \xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \right]},$$

as long as

$$\beta \xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} < 1.$$

Equating the two expressions for F_w, we obtain:

$$\tilde{w} = W \lambda_w \frac{\psi_L h^{\sigma_L}}{\left(1-\tau^l\right) \lambda_z}, \tag{5.51}$$

where

$$W = (w^*)^{\frac{\lambda_w}{\lambda_w - 1} \sigma_L} \left[\frac{1 - \xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w} \right]^{\lambda_w(1+\sigma_L)-1} \frac{1 - \beta \xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{1}{1-\lambda_w}}}{1 - \beta \xi_w \left(\frac{\tilde{\pi}_w}{\pi} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}}. \tag{5.52}$$

In steady state, (**??**) reduces to:

$$w^* = \left[\frac{(1-\xi_w)\left(\frac{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{1}{1-\lambda_w}}}{1-\xi_w}\right)^{\lambda_w}}{1-\xi_w\left(\frac{\tilde{\pi}_w}{\pi}\right)^{\frac{\lambda_w}{1-\lambda_w}}} \right]^{\frac{1-\lambda_w}{\lambda_w}} \tag{5.53}$$

According to the wage equation, the wage is a markup, $W\lambda_w$, over the household's marginal cost. Note that the magnitude of the markup depends on the degree of wage distortions in the steady state. These will be important to the extent that $\tilde{\pi}_w \neq \pi_w$.

The marginal cost equation, (2.31) implies:

$$r^k = \frac{\alpha\epsilon}{[1+\psi_k R]} \left(\frac{\Upsilon\mu_z^* L\,(w^*)^{\frac{\lambda_w}{\lambda_w-1}}}{\bar{k}} \right)^{1-\alpha} s, \tag{5.54}$$

where w^* is determined by (5.53). In steady state, the capital accumulation equation, (2.34), is

$$\bar{k}\left[1-\frac{1-\delta}{\mu_z^*\Upsilon}\right] = i.$$

In steady state, the equation for the nominal rate of interest, (2.35), reduces to:

$$1+R = \frac{\pi\mu_z^*}{\beta}. \tag{5.55}$$

In steady state, the marginal utility of consumption, (2.36), is

$$\lambda_z = \frac{1}{(1+\tau^C)c}\frac{\mu_z^*-b\beta}{\mu_z^*-b}. \tag{5.56}$$

Finally, the euler equation for investment, (2.38), reduces to

$$q = 1.$$

We proceed as follows. First, fix the nominal rate of interest according to (5.55). Now, fix a value for r^k. Solve (5.54) for

$$(1)\frac{h}{\bar{k}} = \frac{(w^*)^{\frac{\lambda_w}{1-\lambda_w}}}{\Upsilon\mu_z^*}\left(\frac{[1+\psi_k R]\,r^k}{s\alpha\epsilon}\right)^{\frac{1}{1-\alpha}}, \tag{5.57}$$

where s is determined by (5.50). Then,

$$(2) R^k = \frac{(1-\tau^k)r^k + 1 - \delta}{\Upsilon}\pi + \tau^k\delta - 1. \tag{5.58}$$

Then, solve

$$(3)\ [1-\Gamma(\bar{\omega})]\frac{1+R^k}{1+R} + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})}\left[\frac{1+R^k}{1+R}\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right) - 1\right] = 0. \tag{5.59}$$

for $\bar{\omega}$. When we estimate the model, for each $\bar{\omega}$, we impose that $F(\bar{\omega})$ is equal to a specified calibrated value. Since F is cdf of the log normal distribution, with $E\omega = 1$, then F has one free parameter, a variance. For each $\bar{\omega}$, this variance is computed to ensure that $F(\bar{\omega})$ is the value required. When we compute the Ramsey equilibrium, then we take the variance of the model in the posterior mode as fixed. To evaluate (5.59) it is useful to have a formula:

$$G(\bar{\omega}) = \int_0^{\bar{\omega}} \omega dF(\omega).$$

Making the following change of variables: $\omega = e^x$, $d\omega = e^x dx$, $x = \log\omega$, $dx = d\omega/\omega$, we obtain:

$$\int_0^{\bar{\omega}} \omega dF(\omega) = \int_{-\infty}^{\log\bar{\omega}} e^x f(x)\, dx.$$

Here, $x = \log(\omega)$ and f is the Normal density function. Writing this explicitly:

$$\begin{aligned}
\int_0^{\bar{\omega}} \omega dF(\omega) &= \int_{-\infty}^{\log\bar{\omega}} e^x f(x)\, dx \\
&= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}} e^x \exp^{\frac{-(x-Ex)^2}{2\sigma_x^2}}\, dx,
\end{aligned}$$

where σ_x^2 is the variance of x. Now, $E\omega = 1$ implies $Ex = -(1/2)\,\sigma_x^2$, so that

$$\begin{aligned}
\int_0^{\bar{\omega}} \omega dF(\omega) &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}} e^x \exp^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}\, dx \\
&= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}_t} \exp^{\frac{x2\sigma_x^2-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}\, dx \\
&= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}_t} \exp^{\frac{-\left(x-\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}\, dx.
\end{aligned}$$

Now, make the change of variable,

$$
\begin{aligned}
v &= \frac{x-\frac{1}{2}\sigma_x^2}{\sigma_x} = \frac{x+\frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x \\
\bar{v} &= \frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x \\
dv &= \frac{1}{\sigma_x} dx
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_0^{\bar{\omega}} \omega dF(\omega) &= \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}-\sigma_x} \exp^{\frac{-v^2}{2}} \sigma_x dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}-\sigma_x} \exp^{\frac{-v^2}{2}} dv \\
&= prob\left[x < \frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x\right].
\end{aligned}
$$

where

$$
\begin{aligned}
E\omega &= Ee^x = e^{\left[Ex+\frac{1}{2}\sigma_x^2\right]} = 1 \\
Ex &= -\frac{1}{2}\sigma_x^2.
\end{aligned}
$$

Next, find n/k which solves (2.11):

$$
(4)\frac{n}{\bar{k}} = 1 - \frac{1+R^k}{1+R}\left[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right] \tag{5.60}
$$

In steady state, (2.44) is

$$
n = \frac{\gamma}{\pi\mu_z^*}\left\{R^k - R - \mu\int_0^{\bar{\omega}} \omega dF(\omega)\left(1+R^k\right)\right\}\left(\frac{\bar{k}}{n}\right)n + w^e + \gamma\left(\frac{1+R}{\pi\mu_z^*}\right)n,
$$

so that

$$
(5)n = \frac{w^e}{1 - \frac{\gamma}{\pi\mu_z^*}\left\{R^k - R - \mu G(\bar{\omega})(1+R^k)\right\}\left(\frac{\bar{k}}{n}\right) - \gamma\left(\frac{1+R}{\pi\mu_z^*}\right)}, \tag{5.61}
$$

$$
\begin{aligned}
\bar{k} &= \left(\frac{\bar{k}}{n}\right)n \\
h &= \left(\frac{h}{\bar{k}}\right)\bar{k}
\end{aligned}
$$

$$
(6)i = \left[1 - (1-\delta)\frac{1}{\mu_z^*\Upsilon}\right]\bar{k}, \tag{5.62}
$$

where $G(\bar{\omega})$ is obtained from (2.17).

We now need to solve the resource constraint for consumption. But, first we require ϕ. Normally, this parameter is set so that steady state profits of the intermediate good producer are zero. However, those profits are not constant in the version of the model in which prices are distorted along a steady state growth path. Instead, we choose ϕ so that profits are zero in the version of the model in which there are no distortions in the steady state. We suppose that this way of setting ϕ or other ways will make little difference. Thus, we compute ϕ to guarantee that firm profits are zero in a steady state where $\pi = \bar{\pi}$. Let h and $\bar{k}$ denote hours worked and capital in such a steady state. Also, let F denote gross output of the final good in that steady state. Write sales of final good firm as $F - \phi$. Real marginal cost in this steady state is $s = 1/\lambda_f$. Since this is a constant, the total costs of the firm are sF. Zero profits requires $sF = F - \phi$. Thus, $\phi = (1-s)F = F(1 - 1/\lambda_f)$, or,

$$(7)\phi = \left(\frac{\bar{k}}{\mu_z^* \Upsilon}\right)^{\alpha} (h)^{1-\alpha} \left(1 - \frac{1}{\lambda_f}\right). \tag{5.63}$$

Solve the steady state version of the resource constraint, (2.45), for c :

$$(8) d + c + g + \frac{i}{\mu_\Upsilon} + \Theta \frac{1-\gamma}{\gamma}\left[n - w^e\right] = (p^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left(\frac{\bar{k}}{\mu_z^* \Upsilon}\right)^{\alpha} \left[(w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h\right]^{1-\alpha} - \phi, \tag{5.64}$$

where d is determined by the steady state version of (2.46). Compute the steady state real wage using (2.31):

$$(9)\tilde{w} = s(1-\alpha)\left[\frac{\Upsilon \mu_{z^*} (w^*)^{\frac{\lambda_w}{\lambda_w - 1}} h}{\bar{k}}\right]^{-\alpha}. \tag{5.65}$$

Then, solve the labor supply equation, (5.51), for h :

$$(10) h = \left[\frac{\left(1 - \tau^l\right)\lambda_z}{W \lambda_w \psi_L} \tilde{w}\right]^{\frac{1}{\sigma_L}}, \tag{5.66}$$

where λ_z is obtained using (5.56) and W is obtained from (5.52). These calculations began by fixing a value for r^k. Adjust r^k until the value of h obtained from (5.66) coincides with the value implied by multiplying $h/\bar{k}$ in (5.57) by $\bar{k}$.

It is of interest to understand what happens when $\mu = 0$. In this case, (5.59) implies $R = R^k$. So, one chooses r^k so that $R = \left[r^k + (1-\delta)\right]\pi - 1$. Then, (5.57) implies a value for h/k. From (5.66),

$$n = \frac{w^e}{1 - \frac{\gamma}{\beta}}.$$

In the case, $\bar{\pi} = \pi$, $\mu = 0$ implies:

$$\begin{aligned} c + I + \Theta\frac{1-\gamma}{\gamma}\left[n - w^e\right] &= \left(\frac{k}{\mu_z^*}\right)^{\alpha} h^{1-\alpha} - \left(\frac{k}{\mu_z^*}\right)^{\alpha} h^{1-\alpha}\left(1 - \frac{1}{\lambda_f}\right) \\ &= \frac{1}{\lambda_f}\left(\frac{1}{\mu_z^*}\right)^{\alpha}\left(\frac{h}{k}\right)^{1-\alpha} k, \end{aligned}$$

or,

$$\frac{c}{k} + \left[1 - (1-\delta)\mu_z^{-1}\right] + \Theta\frac{1-\gamma}{\gamma}\frac{\left[n - w^e\right]}{k} = \frac{1}{\lambda_f}\left(\frac{1}{\mu_z^*}\right)^{\alpha}\left(\frac{h}{k}\right)^{1-\alpha}$$

The labor-leisure choice implies:

$$c = \frac{\frac{\mu_z - b\beta}{\mu_z - b}}{W\lambda_w\psi_L}\tilde{w}h^{-\sigma_L},$$

where $\tilde{w}$ can be computed from (5.65) and $W = 1$ according to (5.52). Substituting this into the resource constraint, we obtain:

$$\frac{\frac{\mu_z - b\beta}{\mu_z - b}}{W\lambda_w\psi_L}\tilde{w}\frac{1}{h^{1+\sigma_L}} + \Theta\frac{(1-\gamma)\,w^e}{\beta-\gamma}\frac{1}{h} = \frac{\frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\right)^{\alpha}\left(\frac{h}{k}\right)^{1-\alpha} - \left(1 - \frac{1-\delta}{\mu_z}\right)}{\frac{h}{k}},$$

which is a single equation in one unknown, h. Note that the right side must be positive for consumption to be positive. Also, the left side goes from 0 to ∞ as h goes from ∞ to 0. Thus, there is a unique solution, as long as the model implies positive steady state consumption. Once this is solved for h, then we have k. Then, given k we can compute $\bar{\omega}$ from (5.60):

$$\frac{n}{k} = 1 - \Gamma(\bar{\omega})$$

$$\Gamma(\bar{\omega}) = 1 - \frac{n}{k}$$

This gives the same solution as the model without financial frictions, except for the fact that entrepreneurs consume resources.

F Appendix F: Laplace-type Approximation for Bimodal Posterior Distribution

When we estimate our model on the standard data set, we find two isolated local modes for the posterior distribution. The difference of the log posterior distribution, L, is only about 4 points across these two modes. The local curvature about the two modes makes locally computed probability intervals seem narrow, yet the properties of the model differs sharply across the two modes. In this sense, correctly computed probability intervals encompass sharply different behavior and are not convex sets. In this appendix, we describe a Laplace approximation procedure computing the posterior distribution under circumstances when the posterior distribution is bimodal. We use it, among other things, to create a visual representation of the posterior distribution in terms of the model property of interest. In particular, at one mode the fraction of variance in output due to the risk shock is high and the fraction due to the marginal efficiency of investment is low. The reverse is true at the other mode. The procedure developed here approximates the posterior distribution of the fraction of variance due to risk under these circumstances. When represented visually in a diagram with the fraction of variance in output due to risk on the horizontal axis and the associated posterior density on the vertical axis, we obtain the following. The density has two humps, one above a high value for the fraction of variance and the other over a low value of that fraction. One of the humps is slightly higher than the other one. The local curvature at each hump greatly exagerates the precision, according to the posterior distribution, assigned to that value of the fraction. The small difference in the height of the posteriors over the two humps provides a correct assessment of the precision with which the two fraction of variances are differentiated according to the posterior distribution.

Our approximation of the posterior distribution is that it is a mixture of two normals, with mixture probability, p. The approximation is valid as long as the posterior probability of each mode is nearly zero under the local approximation about the other mode. We develop this approximation for two reasons. First, the exact procedure based on the MCMC algorithm is impractical, because of the great amount of computer time it would require. Second, we wish to develop an alternative measure of the distance between two posterior modes that is not based on the posterior odds computed by exponentiating L. In practice, one often has to give an interpretation to differences in the log criterion on the order of 4 or 10. Such differences seem small and yet the posterior odds at these points are, $\exp(4)$ and $\exp(10)$, respectively. This gives rise to enormous posterior odds, which seem to overstate the significance of such small differences in the log criterion. We propose, as an alternative to the posterior odds, the mixture probability parameter, p.

Our approximation procedure simply requires the hessians at the two modes, in addition to L. With our mixture of normals approximation of the posterior, we can draw the model parameters, θ, many (say, N) times, $\theta_1, ..., \theta_N$. For any statistic of interest, $s(\theta)$, we then obtain the posterior distribution for that statistic from $s(\theta_1), ..., s(\theta_N)$.

Consider first the standard Laplace approximation to a unimodal distribution. Let $f(\theta)$

denote the product of the likelihood and the prior, so that $f(\theta)$ is proportional to the posterior distribution, where the factor of proportionality is independent of θ. Let $g(\theta) \equiv \log f(\theta)$. Define

$$g_{\theta\theta} = -\frac{\partial^2 g(\theta)}{\partial\theta\partial\theta'}|_{\theta=\theta^*},$$

where θ^* is the mode. The second order Taylor series expansion of g about $\theta = \theta^*$ is:

$$g(\theta) = g(\theta^*) - \frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*),$$

where the slope term is zero because of our assumption θ^* is a local maximum of g. Then,

$$f(\theta) \approx f(\theta^*) \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right].$$

Note that

$$\frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right]$$

is a multivariate normal distribution, so that

$$\int \frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right] d\theta = 1.$$

Bringing together the previous results, we obtain:

$$\begin{aligned}
& \int f(\theta)\, d\theta \\
\approx\ & \int f(\theta^*) \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right] d\theta \\
=\ & \frac{f(\theta^*)}{\frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}}} \int \frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right] d\theta \\
=\ & \frac{f(\theta^*)}{\frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}}},
\end{aligned}$$

by the integral property of the normal distribution. Thus, the posterior distribution is, approximately,

$$\frac{f(\theta)}{\frac{f(\theta^*)}{\frac{1}{(2\pi)^{\frac{n}{2}}}|g_{\theta\theta}|^{\frac{1}{2}}}} \approx \frac{f(\theta^*) \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right]}{\frac{f(\theta^*)}{\frac{1}{(2\pi)^{\frac{n}{2}}}|g_{\theta\theta}|^{\frac{1}{2}}}} = \frac{1}{(2\pi)^{\frac{n}{2}}} |g_{\theta\theta}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\theta - \theta^*)' g_{\theta\theta}(\theta - \theta^*)\right].$$

This covers the unimodal case.

Suppose now that we have two local maxima of g : θ_1^* and θ_2^*. Denote the analogs of $g_{\theta\theta}$ by

$g^1_{\theta\theta}$ and $g^2_{\theta\theta}$. Suppose we approximate the posterior distribution by a mixture of normals:

$$\begin{aligned} F(\theta) &= p\frac{1}{(2\pi)^{\frac{n}{2}}}\left|g^1_{\theta\theta}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}(\theta-\theta_1^*)' g^1_{\theta\theta}(\theta-\theta_1^*)\right] \\ &\quad +(1-p)\frac{1}{(2\pi)^{\frac{n}{2}}}\left|g^2_{\theta\theta}\right|^{\frac{1}{2}}\exp\left[-\frac{1}{2}(\theta-\theta_2^*)' g^2_{\theta\theta}(\theta-\theta_2^*)\right] \\ &= p\exp\left[G_1(\theta)\right]+(1-p)\exp\left[G_2(\theta)\right], \end{aligned}$$

where $0 \le p \le 1$ and $G_1(\theta)$ is the second order approximation of $g(\theta)$ about $\theta = \theta_i^*$:

$$G_i(\theta) = -\frac{n}{2}\log(2\pi) + \frac{1}{2}\log\left|g^i_{\theta\theta}\right| - \frac{1}{2}(\theta-\theta_i^*)' g^i_{\theta\theta}(\theta-\theta_i^*), \tag{6.67}$$

for $i = 1, 2$. Note that

$$G_i'(\theta_i^*) = \underbrace{0}_{N\times 1},\ G_i''(\theta) = \underbrace{g^i_{\theta\theta}}_{N\times N}.$$

Let $\mathcal{F}(\theta)$ denote $\log F(\theta)$. Then,

$$\begin{aligned} \mathcal{F}'(\theta) &= \frac{1}{F(\theta)}\left\{p\exp\left[G_1(\theta)\right]G_1'(\theta)+(1-p)\exp\left[G_2(\theta)\right]G_2'(\theta)\right\} \\ \mathcal{F}''(\theta) &= -\frac{1}{F(\theta)}g'(\theta)\left\{p\exp\left[G_1(\theta)\right]\left[G_1'(\theta)\right]^T+(1-p)\exp\left[G_2(\theta)\right]\left[G_2'(\theta)\right]^T\right\} \\ &\quad +\frac{1}{F(\theta)}\left\{p\exp\left[G_1(\theta)\right]G_1''(\theta)+(1-p)\exp\left[G_2(\theta)\right]G_2''(\theta)\right\} \end{aligned}$$

Evaluate the latter at θ_1^* :

$$\begin{aligned} \mathcal{F}''(\theta_1^*) &= -\frac{1}{F(\theta_1^*)}\mathcal{F}'(\theta_1^*)\left\{p\exp\left[G_1(\theta_1^*)\right]\overbrace{\left[G_1'(\theta_1^*)\right]^T}^{=0}+(1-p)\overbrace{\exp\left[G_2(\theta_1^*)\right]}^{\simeq 0}\left[G_2'(\theta_1^*)\right]^T\right\} \\ &\quad +\frac{1}{F(\theta_1^*)}\{p\exp\left[G_1(\theta_1^*)\right]G_1''(\theta_1^*)+(1-p)\overbrace{\exp\left[G_2(\theta_1^*)\right]}^{\simeq 0}G_2''(\theta_1^*)\}, \end{aligned}$$

where the terms with $\simeq 0$ reflect our assumption that θ_1^* is very unlikely under the Laplace approximation about θ_2^* similarly for θ_2^* :

$$\exp\left[G_2(\theta_1^*)\right] \simeq 0,\ \ \exp\left[G_1(\theta_2^*)\right] \simeq 0. \tag{6.68}$$

Then,

$$\begin{aligned}
\mathcal{F}''\left(\theta_1^*\right) &= \frac{1}{F\left(\theta_1^*\right)} p \exp\left[G_1\left(\theta_1^*\right)\right] g_{\theta\theta}^1 \\
&= \frac{p \exp\left[G_1\left(\theta_1^*\right)\right] g_{\theta\theta}^1}{p \exp\left[G_1\left(\theta_1^*\right)\right] + (1-p)\exp\left[G_2\left(\theta_1^*\right)\right]} \\
&= \frac{p \exp\left[G_1\left(\theta_1^*\right)\right] g_{\theta\theta}^1}{p \exp\left[G_1\left(\theta_1^*\right)\right]} = g_{\theta\theta}^1.
\end{aligned}$$

Thus, under (6.68), the curvature of our mixted Normal approximation about $\theta = \theta_1^*$ coincides with the curvature of the actual posterior distribution. This is a basic requirement of consistency. Of course, in practice it is necessary to verify (6.68). We have an analogous result for $\mathcal{F}''\left(\theta_2^*\right)$.

It remains to compute p, the Normal mixture probability. We obtain this as follows. Let L denote the difference in the log posterior between the two modes. Thus,

$$L = g\left(\theta_1^*\right) - g\left(\theta_2^*\right) > 0,$$

so that θ_1^* is the global maximum of g. We can use L to pin down the value of p in the mixture distribution. According to our mixture approximation to the posterior distribution,

$$\begin{aligned}
L &= \log \frac{p \exp\left[G_1\left(\theta_1^*\right)\right] + (1-p)\exp\left[G_2\left(\theta_1^*\right)\right]}{p \exp\left[G_1\left(\theta_2^*\right)\right] + (1-p)\exp\left[G_2\left(\theta_2^*\right)\right]} \\
&= \log \frac{p \exp\left[G_1\left(\theta_1^*\right)\right]}{(1-p)\exp\left[G_2\left(\theta_2^*\right)\right]} \\
&= \log \frac{p}{(1-p)} + G_1\left(\theta_1^*\right) - G_2\left(\theta_2^*\right) \\
&= \log \frac{p}{1-p} + \frac{1}{2}\log \frac{\left|g_{\theta\theta}^1\right|}{\left|g_{\theta\theta}^2\right|}.
\end{aligned}$$

The second equality reflects the assumption, (6.68), that under the local approximation, the alternative mode is highly improbable. The fourth equality uses (6.67). Thus,

$$\frac{p}{1-p} = \exp\left[L - \frac{1}{2}\log \frac{\left|g_{\theta\theta}^1\right|}{\left|g_{\theta\theta}^2\right|}\right] = d,$$

say, which can be used to solve for p :

$$p = \frac{d}{1+d}$$

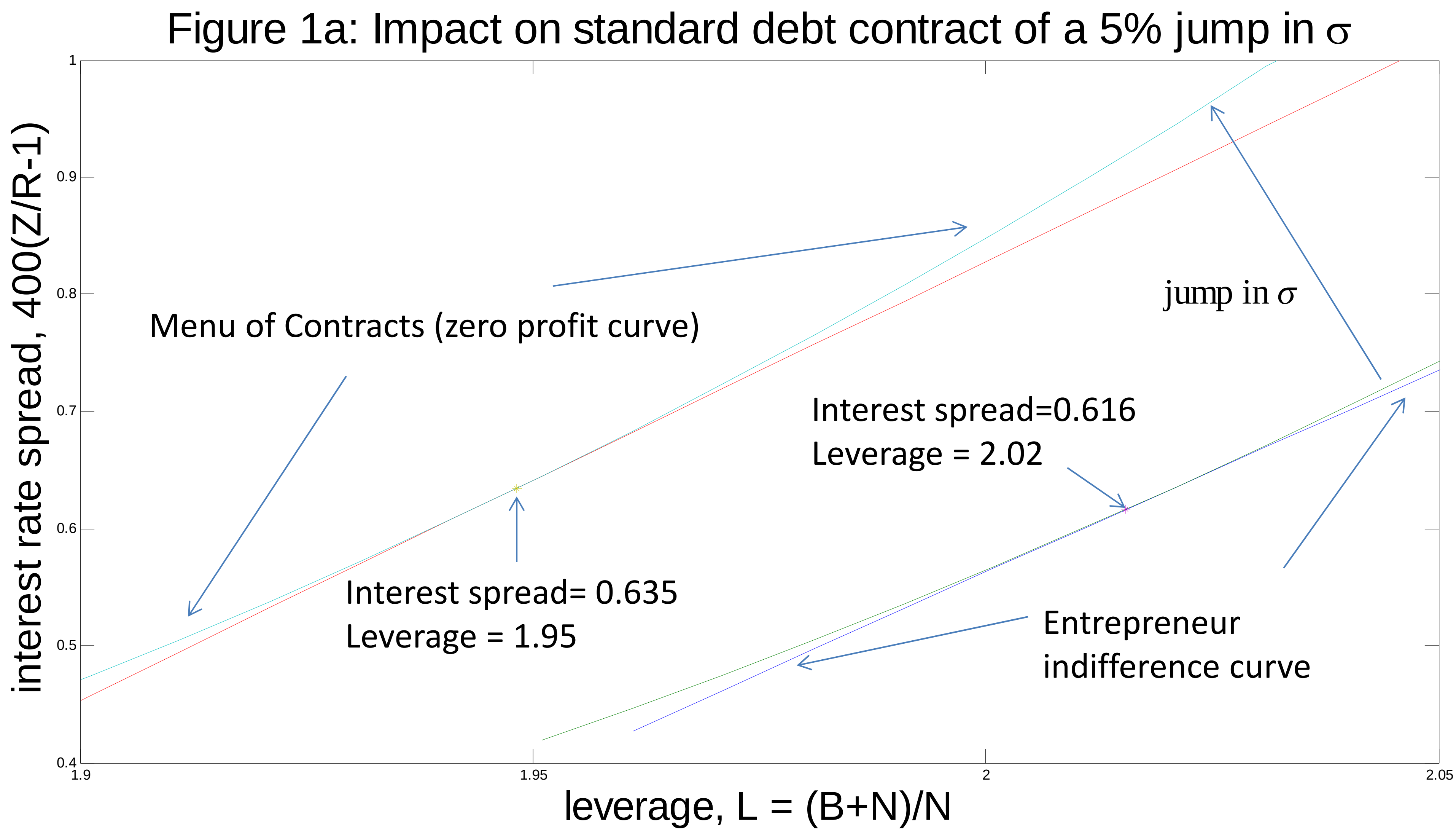
Figure 1a: Impact on standard debt contract of a 5% jump in σ
interest rate spread, 400(Z/R-1)
leverage, L = (B+N)/N
1
0.9
0.8
0.7
0.6
0.5
0.4
1.9
1.95
2
2.05
Menu of Contracts (zero profit curve)
jump in σ
Interest spread=0.616
Leverage = 2.02
Interest spread= 0.635
Leverage = 1.95
Entrepreneur
indifference curve

Figure 1b: Impact on standard debt contract of a 1% jump in R^k/R