Technical Notes for Discussion of Eggertsson, What Fiscal Policy Is Effective at Zero Interest Rates?

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The model that was simulated for the discussion is presented in the first section. The simulation strategy is described in the next section.

# 1. Household

The utility function of the $j^{th}$ household is:

$$U\left(C_t, h_{j,t}\right) = \log C_t - A\frac{h_{j,t}^{1+\phi}}{1+\phi}$$

The budget constraint is:

$$P_t C_t + B_{t+1} \leq R_{t-1} B_t + W_{j,t} h_{j,t} + \Pi_{j,t},$$

where $\Pi_{j,t}$ denotes lump sum profits and lump sum taxes.

## 1.1. Intertemporal Condition

The discount rate from $t$ to $t+1$ has the following representation:

$$\begin{aligned} \beta_t &= \frac{1}{1+r_t} \\ \beta\hat{\beta}_t &= -\left(\frac{1}{1+r}\right)^2 dr_t, \end{aligned}$$

or

$$\hat{\beta}_t = -\beta dr_t$$

The first order condition associated with bonds is:

$$\frac{1}{C_t} = \beta_t \frac{1}{C_{t+1}}\left(1+R_t\right)\frac{1}{\pi_{t+1}}.$$

Linearizing around a zero inflation steady state:

$$-\hat{C}_t = -C_{t+1} - \beta dr_t + \widehat{(1+R_t)} - \hat{\pi}_{t+1},$$

or

$$\hat{C}_t = C_{t+1} + \beta dr_t - \beta dR_t + \hat{\pi}_{t+1},$$

or,

$$\hat{C}_t = \hat{C}_{t+1} - \beta\left(R_t - r_t\right) + \hat{\pi}_{t+1},$$

### 1.2. Household Wage/Employment Decision

The household selects the wage rate to optimize:

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \left[ \upsilon_{t+i} W_{j,t+i} h_{j,t+i} - A \frac{h_{j,t+i}^{1+\phi}}{1+\phi} \right],$$

where $\upsilon_{t+i}$ denotes the multiplier on the household budget constraint in the Lagrangian representation of its problem. The household treats this object as an exogenous constant. Each household that optimizes its wage, chooses the same wage rate, $\tilde{W}_t$ :

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t h_{t+i}^t - A \frac{\left(h_{t+i}^t\right)^{1+\phi}}{(1+\phi)\, \upsilon_{t+i}} \right],$$

where $h_{t+i}^t$ denotes the level of employment in period $t+i$ of a household that optimizes its wage in period $t$ :

$$h_{t+i}^t = \left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i},$$

and $H_{t+i}$ denotes aggregate employment. Also, $W_t$ denotes the aggregate wage rate. Imposing the requirement that the household is always on the labor demand curve implies:

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t \left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - A \frac{\left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{(1+\phi)\, \upsilon_{t+i}} \right].$$

Differentiate with respect to $\tilde{W}_t$ :

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \upsilon_{t+i} \left[ \left(1 + \frac{\lambda_w}{1-\lambda_w}\right) \tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}} \left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - \frac{\lambda_w}{1-\lambda_w} A \frac{\tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)-1} \left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{\upsilon_{t+i}} \right.$$

or,

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \upsilon_{t+i} \left[ \left(1 + \frac{\lambda_w}{1-\lambda_w}\right) \tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}+1-\frac{\lambda_w}{1-\lambda_w}(1+\phi)} \left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - \frac{\lambda_w}{1-\lambda_w} A \frac{\left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{\upsilon_{t+i}} \right.$$

or,

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t^{1-\frac{\lambda_w}{1-\lambda_w}\phi} \left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - \lambda_w A \frac{\left( \frac{1}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{\upsilon_{t+i}} \right],$$

or,

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t^{1-\frac{\lambda_w}{1-\lambda_w}\phi} \tilde{W}_t^{-\frac{\lambda_w}{1-\lambda_w}} \left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - \lambda_w A \frac{\tilde{W}_t^{-\frac{\lambda_w}{1-\lambda_w}(1+\phi)} \left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{\upsilon_{t+i}} \right],$$

or,

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t \left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} - \lambda_w A \frac{\left( \frac{\tilde{W}_t}{W_{t+i}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\phi)} H_{t+i}^{1+\phi}}{\upsilon_{t+i}} \right],$$

or,

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \upsilon_{t+i} \left[ \tilde{W}_t h_{t+i}^t - \lambda_w A \frac{\left( h_{t+i}^t \right)^{1+\phi}}{\upsilon_{t+i}} \right],$$

or

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \upsilon_{t+i} h_{t+i}^t P_{t+i} \left[ \frac{\tilde{W}_t}{P_{t+i}} - \lambda_w A \frac{\left( h_{t+i}^t \right)^{\phi}}{P_{t+i}\upsilon_{t+i}} \right],$$

or

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \upsilon_{t+i} h_{t+i}^t P_{t+i} \left[ \frac{\tilde{W}_t}{P_{t+i}} - \lambda_w A \frac{\left( h_{t+i}^t \right)^{\phi}}{P_{t+i}\upsilon_{t+i}} \right],$$

Given our utility function, we have

$$\upsilon_{t+i} = \frac{U_{c,t+i}}{P_{t+i}} = \frac{1}{P_{t+i}C_{t+i}}.$$

Note that

$$MRS_{t+i}^t \equiv A \frac{\left( h_{t+i}^t \right)^{\phi}}{P_{t+i}\upsilon_{t+i}} = AC_{t+i} \left( h_{t+i}^t \right)^{\phi},$$

where $MRS_{t+i}^t$ denotes the marginal rate of substitution between consumption and leisure in period $t+i$ for a person that reoptimizes in period $t$ and does not reoptimize between $t+1$ and $t+i$. Substitute this into the first order condition:

$$E_t \sum_{i=0}^{\infty} (\beta\xi_w)^i \frac{h_{t+i}^t}{C_{t+i}} \left[ \frac{\tilde{W}_t}{P_{t+i}} - \lambda_w MRS_{t+i}^t \right].$$

Now, consider the following scaling:

$$w_t = \frac{\tilde{W}_t}{W_t},\ \bar{w}_t = \frac{W_t}{P_t},\ X_{t,t+i} = \begin{cases} \frac{1}{\pi_{t+1}\cdots\pi_{t+i}} & i \geq 1 \\ 1 & i = 0 \end{cases}.$$

Note that with this definition,

$$X_{t,t+i}\frac{1}{P_t} = \frac{1}{P_{t+i}},\ \text{all } i \geq 0.$$

Then, the first order condition reduces to:

$$E_t \sum_{i=0}^{\infty} (\beta \xi_w)^i \frac{h_{t+i}^t}{C_{t+i}} \left[ \bar{w}_t w_t X_{t,t+i} - \lambda_w MRS_{t+i}^t \right] = 0.$$

Note,

$$\hat{X}_{t,t+i} = -\hat{\pi}_{t+1} - \cdots - \hat{\pi}_{t+i}.$$

We are interested in the case where $\beta$ is time varying. Although in the end the time varying $\beta$ has no impact on the reduced form wage equation, it is useful to establish this. Thus,

$$\begin{aligned}
\beta_{t+1} &= \frac{1}{1+r_{t+1}} \\
\beta \hat{\beta}_{t+1} &= -\left( \frac{1}{1+r} \right)^2 dr_{t+1} \\
&= -\beta^2 dr_{t+1} \\
\hat{\beta}_{t+1} &= -\beta dr_{t+1}
\end{aligned}$$

We write out the first order condition like this:

$$\begin{aligned}
& \beta_t \frac{h_t^t}{C_t} \left[ \bar{w}_t w_t - \lambda_w MRS_t^t \right] \\
& + \beta_t \beta_{t+1} \xi_w \frac{h_{t+1}^t}{C_{t+1}} \left[ \bar{w}_t w_t X_{t,t+1} - \lambda_w MRS_{t+1}^t \right] \\
& + \beta_t \beta_{t+1} \beta_{t+2} \xi_w^2 \frac{h_{t+2}^t}{C_{t+2}} \left[ \bar{w}_t w_t X_{t,t+2} - \lambda_w MRS_{t+2}^t \right] \\
& + \beta_t \beta_{t+1} \beta_{t+2} \beta_{t+3} \xi_w^3 \frac{h_{t+3}^t}{C_{t+3}} \left[ \bar{w}_t w_t X_{t,t+3} - \lambda_w MRS_{t+3}^t \right] \\
+ ... &= 0
\end{aligned}$$

Log-linearly expanding this expression, and taking into account that the object in square brackets is zero in steady state:

$$\begin{aligned}
& \beta \frac{h}{C} \left[ \bar{w} \left( \widehat{\bar{w}}_t + \hat{w}_t \right) - (\lambda_w MRS) \widehat{MRS}_t^t \right] \\
& + \beta^2 \xi_w \frac{h}{C} \left[ \bar{w} \left( \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+1} \right) - \lambda_w MRS \left( \widehat{MRS}_{t+1}^t \right) \right] \\
& + \beta^3 \xi_w^2 \frac{h}{C} \left[ \bar{w} \left( \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+2} \right) - \lambda_w MRS \left( \widehat{MRS}_{t+2}^t \right) \right] \\
& + \beta^4 \xi_w^3 \frac{h}{C} \left[ \bar{w} \left( \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+3} \right) - \lambda_w MRS \left( \widehat{MRS}_{t+3}^t \right) \right] \\
& + ... = 0
\end{aligned}$$

(note how the time varying $\beta$ disappeared) or, noting $\bar{w} = \lambda_w MRS$

$$\begin{aligned}
&\beta \frac{h}{C} \bar{w} \left[ \left( \widehat{\bar{w}}_t + \hat{w}_t \right) - \widehat{MRS}_t^t \right] \\
&+\beta^2 \xi_w \frac{h}{C} \bar{w} \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+1} - \widehat{MRS}_{t+1}^t \right] \\
&+\beta^3 \xi_w^2 \frac{h}{C} \bar{w} \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+2} - \widehat{MRS}_{t+2}^t \right] \\
&+\beta^4 \xi_w^3 \frac{h}{C} \bar{w} \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+3} - \widehat{MRS}_{t+3}^t \right] \\
&+... = 0
\end{aligned}$$

dividing through by $\beta \frac{h}{C} \bar{w}$ :

$$\begin{aligned}
&\widehat{\bar{w}}_t + \hat{w}_t - \widehat{MRS}_t^t \\
&+\beta \xi_w \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+1} - \widehat{MRS}_{t+1}^t \right] \\
&+\beta^2 \xi_w^2 \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+2} - \widehat{MRS}_{t+2}^t \right] \\
&+\beta^3 \xi_w^3 \left[ \widehat{\bar{w}}_t + \hat{w}_t + \hat{X}_{t,t+3} - \widehat{MRS}_{t+3}^t \right] \\
&+... = 0
\end{aligned}$$

Then,

$$\begin{aligned}
&\widehat{\bar{w}}_t + \hat{w}_t - \widehat{MRS}_t^t \\
&+\beta \xi_w \left[ \widehat{\bar{w}}_t + \hat{w}_t - \hat{\pi}_{t+1} - \widehat{MRS}_{t+1}^t \right] \\
&+\beta^2 \xi_w^2 \left[ \widehat{\bar{w}}_t + \hat{w}_t - \hat{\pi}_{t+1} - \hat{\pi}_{t+2} - \widehat{MRS}_{t+2}^t \right] \\
&+\beta^3 \xi_w^3 \left[ \widehat{\bar{w}}_t + \hat{w}_t - \hat{\pi}_{t+1} - \hat{\pi}_{t+2} - \hat{\pi}_{t+3} - \widehat{MRS}_{t+3}^t \right] \\
&+... = 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{1-\beta \xi_w} \left( \widehat{\bar{w}}_t + \hat{w}_t \right) - \widehat{MRS}_t^t \\
&+\beta \xi_w \left[ -\hat{\pi}_{t+1} - \widehat{MRS}_{t+1}^t \right] \\
&+\beta^2 \xi_w^2 \left[ -\hat{\pi}_{t+1} - \hat{\pi}_{t+2} - \widehat{MRS}_{t+2}^t \right] \\
&+\beta^3 \xi_w^3 \left[ -\hat{\pi}_{t+1} - \hat{\pi}_{t+2} - \hat{\pi}_{t+3} - \widehat{MRS}_{t+3}^t \right] \\
&+... = 0
\end{aligned}$$

or,

$$\begin{aligned} & \frac{1}{1-\beta\xi_w}\left(\widehat{\overline{w}}_t+\hat{w}_t\right)-\frac{\beta\xi_w}{1-\beta\xi_w}\hat{\pi}_{t+1}-\frac{(\beta\xi_w)^2}{1-\beta\xi_w}\hat{\pi}_{t+2}-... \\ & -\left[\widehat{MRS}_t^t+\beta\xi_w\widehat{MRS}_{t+1}^t+(\beta\xi_w)^2\,\widehat{MRS}_{t+2}^t+...\right] \\ = & \ 0 \end{aligned} \tag{1.1}$$

Note,

$$\begin{aligned} \widehat{MRS}_{t+i}^t &= \hat{C}_{t+i}+\phi\hat{h}_{t+i}^t \\ &= \hat{C}_{t+i}+\phi\hat{H}_{t+i}+\phi\left(\hat{h}_{t+i}^t-\hat{H}_{t+i}\right). \end{aligned}$$

Recall,

$$\begin{aligned} h_{t+i}^t &= \left(\frac{\tilde{W}_t}{W_{t+i}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i}, \\ &= \left(w_t\frac{W_t}{W_{t+i}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i}, \\ &= \left(w_t\frac{W_t}{W_{t+1}}\frac{W_{t+1}}{W_{t+2}}\cdots\frac{W_{t+i-1}}{W_{t+i}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} \\ &= \left(\frac{w_t}{\pi_{w,t+1}\cdots\pi_{w,t+i}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+i} \end{aligned}$$

for $i=0$ :

$$h_t^t=\left(\frac{\tilde{W}_t}{W_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_t=(w_t)^{\frac{\lambda_w}{1-\lambda_w}}\,H_t$$

so that

$$\frac{h_{t+i}^t}{H_{t+i}}=\begin{cases} \left(\frac{w_t}{\pi_{w,t+1}\cdots\pi_{w,t+i}}\right)^{\frac{\lambda_w}{1-\lambda_w}} & i>0 \\ (w_t)^{\frac{\lambda_w}{1-\lambda_w}} & i=0 \end{cases}.$$

Then,

$$\hat{h}_{t+i}^t-\hat{H}_{t+i}=\begin{cases} \frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t-\hat{\pi}_{w,t+1}\cdots-\hat{\pi}_{w,t+i}\right) & i>0 \\ \frac{\lambda_w}{1-\lambda_w}\hat{w}_t & i=0 \end{cases}.$$

Substituting,

$$\begin{aligned} \widehat{MRS}_{t+i}^t &= \hat{C}_{t+i}+\phi\hat{H}_{t+i}+\phi\left(\hat{h}_{t+i}^t-\hat{H}_{t+i}\right) \\ &= \hat{C}_{t+i}+\phi\hat{H}_{t+i}+\phi\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t-\hat{\pi}_{w,t+1}\cdots-\hat{\pi}_{w,t+i}\right), \end{aligned}$$

for $i > 0$. Then,

$$\begin{aligned}
& \widehat{MRS}_t^t + \beta\xi_w \widehat{MRS}_{t+1}^t + (\beta\xi_w)^2 \widehat{MRS}_{t+2}^t + ... \\
= \; & \hat{C}_t + \phi\hat{H}_t + \phi\frac{\lambda_w}{1-\lambda_w}\hat{w}_t \\
& +\beta\xi_w \left[\hat{C}_{t+1} + \phi\hat{H}_{t+1} + \phi\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t - \hat{\pi}_{w,t+1}\right)\right] \\
& + (\beta\xi_w)^2 \left[\hat{C}_{t+2} + \phi\hat{H}_{t+2} + \phi\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t - \hat{\pi}_{w,t+1} - \hat{\pi}_{w,t+2}\right)\right] \\
& + (\beta\xi_w)^3 \left[\hat{C}_{t+3} + \phi\hat{H}_{t+3} + \phi\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t - \hat{\pi}_{w,t+1} - \hat{\pi}_{w,t+2} - \hat{\pi}_{w,t+3}\right)\right] \\
& +...
\end{aligned}$$

or,

$$\begin{aligned}
& \widehat{MRS}_t^t + \beta\xi_w \widehat{MRS}_{t+1}^t + (\beta\xi_w)^2 \widehat{MRS}_{t+2}^t + ... \\
= \; & \sum_{i=0}^{\infty} (\beta\xi_w)^i \left[\hat{C}_{t+i} + \phi\hat{H}_{t+i}\right] + \phi\frac{\lambda_w}{1-\lambda_w}\frac{1}{1-\beta\xi_w}\hat{w}_t - \frac{1}{1-\beta\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i \hat{\pi}_{w,t+i}.
\end{aligned}$$

Substituting the previous expression into (1.1), we conclude that the first order condition for wages looks as follows:

$$\begin{aligned}
& \frac{1}{1-\beta\xi_w}\left(\widehat{\tilde{w}}_t + \hat{w}_t\right) - \frac{1}{1-\beta\xi_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i \hat{\pi}_{t+i} \qquad (1.2) \\
& = \sum_{i=0}^{\infty} (\beta\xi_w)^i \left(\hat{C}_{t+i} + \phi\hat{H}_{t+i}\right) + \phi\frac{\lambda_w}{1-\lambda_w}\frac{1}{1-\beta\xi_w}\hat{w}_t - \frac{1}{1-\beta\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i \hat{\pi}_{w,t+i}.
\end{aligned}$$

We now deduce the implications of the aggregate restrictions across wages:

$$W_t = \left[(1-\xi_w)\left(\tilde{W}_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(W_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.$$

Divide by $W_t$ and use the scaling notation:

$$1 = (1-\xi_w)\left(w_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\frac{1}{\pi_{w,t}}\right)^{\frac{1}{1-\lambda_w}}.$$

Log-linearize this about steady state:

$$0 = (1-\xi_w)\frac{1}{1-\lambda_w}(w)^{\frac{1}{1-\lambda_w}-1} w\hat{w}_t - \frac{1}{1-\lambda_w}\xi_w\left(\pi_w\right)^{-\frac{1}{1-\lambda_w}-1}\pi_w\hat{\pi}_{w,t},$$

or, taking into account $w = \pi_w = 1$ :

$$\hat{w}_t = \frac{\xi_w}{1-\xi_w}\hat{\pi}_{w,t}.$$

Substituting this into (1.2):

$$\begin{aligned}
&\frac{1}{1-\beta\xi_w}\widehat{\overline{w}}_t + \frac{1}{1-\beta\xi_w}\frac{\xi_w}{1-\xi_w}\hat{\pi}_{w,t} - \frac{1}{1-\beta\xi_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{t+i} \\
= &\sum_{i=0}^{\infty}(\beta\xi_w)^i\left(\hat{C}_{t+i}+\phi\hat{H}_{t+i}\right) + \phi\frac{\lambda_w}{1-\lambda_w}\frac{1}{1-\beta\xi_w}\frac{\xi_w}{1-\xi_w}\hat{\pi}_{w,t} - \frac{1}{1-\beta\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{w,t+i}.
\end{aligned}$$

Multiply by

$$\kappa = \frac{(1-\beta\xi_w)(1-\xi_w)}{\xi_w}$$

$$\begin{aligned}
&\frac{1-\xi_w}{\xi_w}\widehat{\overline{w}}_t + \hat{\pi}_{w,t} - \frac{1-\xi_w}{\xi_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{t+i} \\
= &\kappa\sum_{i=0}^{\infty}(\beta\xi_w)^i\left(\hat{C}_{t+i}+\phi\hat{H}_{t+i}\right) + \phi\frac{\lambda_w}{1-\lambda_w}\hat{\pi}_{w,t} - \frac{1-\xi_w}{\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{w,t+i}.
\end{aligned}$$

Note

$$\begin{aligned}
S_t &= \sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{t+i} \qquad (1.3)\\
&= \beta\xi_w\hat{\pi}_{t+1} + (\beta\xi_w)^2\hat{\pi}_{t+2} + (\beta\xi_w)^3\hat{\pi}_{t+3} + ... \\
&= \beta\xi_w\hat{\pi}_{t+1} + \beta\xi_w\left[\beta\xi_w\hat{\pi}_{t+2} + (\beta\xi_w)^2\hat{\pi}_{t+3} + ...\right] \\
&= \beta\xi_w\hat{\pi}_{t+1} + \beta\xi_w S_{t+1} \\
S_{w,t} &= \sum_{i=1}^{\infty}(\beta\xi_w)^i\hat{\pi}_{w,t+i} = \beta\xi_w\hat{\pi}_{w,t+1} + \beta\xi_w S_{w,t+1}. \\
S_{o,t} &= \sum_{i=0}^{\infty}(\beta\xi_w)^i\left(\hat{C}_{t+i}+\phi\hat{H}_{t+i}\right) = \hat{C}_t + \phi\hat{H}_t + \beta\xi_w S_{o,t+1}
\end{aligned}$$

So, the expression for the wage can be written,

$$\begin{aligned}
&\frac{1-\xi_w}{\xi_w}\widehat{\overline{w}}_t + \hat{\pi}_{w,t} - \frac{1-\xi_w}{\xi_w}S_t \\
= &\kappa S_{o,t} + \phi\frac{\lambda_w}{1-\lambda_w}\hat{\pi}_{w,t} - \frac{1-\xi_w}{\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}S_{w,t}
\end{aligned}$$

Lead and multiply by $\beta\xi_w$ :

$$\begin{aligned}
&\frac{1-\xi_w}{\xi_w}\beta\xi_w\widehat{\overline{w}}_{t+1} + \beta\xi_w\hat{\pi}_{w,t+1} - \frac{1-\xi_w}{\xi_w}\beta\xi_w S_{t+1} \\
= &\kappa\beta\xi_w S_{o,t+1} + \phi\frac{\lambda_w}{1-\lambda_w}\beta\xi_w\hat{\pi}_{w,t+1} - \frac{1-\xi_w}{\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\beta\xi_w S_{w,t+1}.
\end{aligned}$$

Subtract the second from the first and make use of (1.3)

$$
\begin{aligned}
& \frac{1-\xi_w}{\xi_w}\widehat{\bar{w}}_t - \frac{1-\xi_w}{\xi_w}\beta\xi_w\widehat{\bar{w}}_{t+1} + \hat{\pi}_{w,t} - \beta\xi_w\hat{\pi}_{w,t+1} \\
& -\frac{1-\xi_w}{\xi_w}\beta\xi_w\hat{\pi}_{t+1} \\
= \;& \kappa\left(\hat{C}_t + \phi\hat{H}_t\right) + \phi\frac{\lambda_w}{1-\lambda_w}\hat{\pi}_{w,t} - \phi\frac{\lambda_w}{1-\lambda_w}\beta\xi_w\hat{\pi}_{w,t+1} \\
& -\frac{1-\xi_w}{\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1-\xi_w}{\xi_w}\widehat{\bar{w}}_t - \frac{1-\xi_w}{\xi_w}\beta\xi_w\widehat{\bar{w}}_{t+1} + \hat{\pi}_{w,t} - \beta\xi_w\hat{\pi}_{w,t+1} \\
& -\frac{1-\xi_w}{\xi_w}\beta\xi_w\hat{\pi}_{t+1} \\
= \;& \kappa\left(\hat{C}_t + \phi\hat{H}_t\right) + \phi\frac{\lambda_w}{1-\lambda_w}\hat{\pi}_{w,t} - \frac{1}{\xi_w}\phi\frac{\lambda_w}{1-\lambda_w}\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

Collecting terms:

$$
\begin{aligned}
& \frac{1-\xi_w}{\xi_w}\widehat{\bar{w}}_t + \left[1-\phi\frac{\lambda_w}{1-\lambda_w}\right]\hat{\pi}_{w,t} \\
= \;& \kappa\left(\hat{C}_t + \phi\hat{H}_t\right) + \frac{1-\xi_w}{\xi_w}\beta\xi_w\left[\hat{\pi}_{t+1} + \widehat{\bar{w}}_{t+1}\right] + \left[1-\frac{\lambda_w}{1-\lambda_w}\frac{\phi}{\xi_w}\right]\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

Note,

$$
\begin{aligned}
\widehat{\bar{w}}_t &= \widehat{\bar{w}}_{t-1} + \hat{\pi}_{w,t} - \hat{\pi}_t \\
\widehat{\bar{w}}_{t+1} &= \widehat{\bar{w}}_t + \hat{\pi}_{w,t+1} - \hat{\pi}_{t+1}
\end{aligned}
$$

so that

$$
\begin{aligned}
& \frac{1-\xi_w}{\xi_w}\widehat{\bar{w}}_t + \left[1-\phi\frac{\lambda_w}{1-\lambda_w}\right]\hat{\pi}_{w,t} \\
= \;& \kappa\left(\hat{C}_t + \phi\hat{H}_t\right) + \frac{1-\xi_w}{\xi_w}\beta\xi_w\left[\widehat{\bar{w}}_t + \hat{\pi}_{w,t+1}\right] + \left[1-\frac{\lambda_w}{1-\lambda_w}\frac{\phi}{\xi_w}\right]\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

then

$$
\begin{aligned}
& \kappa\widehat{\bar{w}}_t + \left[1-\phi\frac{\lambda_w}{1-\lambda_w}\right]\hat{\pi}_{w,t} \\
= \;& \kappa\left(\hat{C}_t + \phi\hat{H}_t\right) + \frac{1-\xi_w}{\xi_w}\beta\xi_w\hat{\pi}_{w,t+1} + \left[1-\frac{\lambda_w}{1-\lambda_w}\frac{\phi}{\xi_w}\right]\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

or,

$$
\begin{aligned}
& \left[1+\phi\frac{\lambda_w}{\lambda_w-1}\right]\hat{\pi}_{w,t} \\
= \;& -\kappa\left(\widehat{\bar{w}}_t - \hat{C}_t - \phi\hat{H}_t\right) + \frac{1-\xi_w}{\xi_w}\beta\xi_w\hat{\pi}_{w,t+1} + \left[1+\phi\frac{\lambda_w}{\lambda_w-1}\frac{1}{\xi_w}\right]\beta\xi_w\hat{\pi}_{w,t+1}
\end{aligned}
$$

or,

$$\begin{aligned}&\left[1+\phi\frac{\lambda_w}{\lambda_w-1}\right]\hat{\pi}_{w,t}\\ = \;& -\kappa\left(\hat{\bar{w}}_t-\hat{C}_t-\phi\hat{H}_t\right)+\left(\frac{1}{\xi_w}+\phi\frac{\lambda_w}{\lambda_w-1}\frac{1}{\xi_w}\right)\beta\xi_w\hat{\pi}_{w,t+1}\end{aligned}$$

or

$$\begin{aligned}&\left[1+\phi\frac{\lambda_w}{\lambda_w-1}\right]\hat{\pi}_{w,t}\\ = \;& -\kappa\left(\hat{\bar{w}}_t-\hat{C}_t-\phi\hat{H}_t\right)+\left(1+\phi\frac{\lambda_w}{\lambda_w-1}\right)\beta\hat{\pi}_{w,t+1}\end{aligned}$$

Now, divide by the term on $\hat{\pi}_{w,t}$ :

$$\hat{\pi}_{w,t}=-\frac{\kappa}{1+\phi\frac{\lambda_w}{\lambda_w-1}}\left(\hat{\bar{w}}_t-\hat{C}_t-\phi\hat{H}_t\right)+\beta\hat{\pi}_{w,t+1}.$$

### 1.3. Goods Production and Price Setting

Suppose that a final good, $Y_t$, is produced using a continuum of inputs as follows:

$$Y_t=\left[\int_0^1 Y_{i,t}^{\frac{1}{\lambda_f}}\,di\right]^{\lambda_f},\ 1\leq\lambda_f<\infty. \tag{1.4}$$

The good is produced by a competitive, representative firm which takes the price of output, $P_t$, and the price of inputs, $P_{i,t}$, as given. The first order necessary condition associated with optimization is:

$$\left(\frac{P_t}{P_{i,t}}\right)^{\frac{\lambda_f}{\lambda_f-1}}Y_t=Y_{i,t}. \tag{1.5}$$

A useful result is obtained by substituting out for $Y_{it}$ in (1.4) from (1.5):

$$P_t=\left[\int_0^1 \left(P_{i,t}\right)^{\frac{-1}{\lambda_f-1}}\,di\right]^{-\left(\lambda_f-1\right)}. \tag{1.6}$$

Each intermediate good is produced by a monopolist using the following production function:

$$Y_{i,t}=A_t h_{i,t}.$$

The equilibrium condition associated with price setting is, after log-linearizing about steady state:

$$\hat{\pi}_t=\beta\hat{\pi}_{t+1}+\frac{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}{\xi_p}\hat{s}_t.$$

Marginal cost in this model is

$$s_t=\frac{W_t}{P_t}=\bar{w}_t,$$

so that

$$\hat{s}_t = \hat{\bar{w}}_t.$$

The resource constraint is:

$$C_t = p_t^* H_t,$$

where $p_t^*$ denotes the Tak Yun distortion, which is unity to a first order approximation.

### 1.4. Equilibrium Conditions

The system has 6 unknowns:

$$\hat{\pi}_t, \hat{\bar{w}}_t, \hat{H}_t, \hat{\pi}_{w,t}, R_t, Z_t,$$

and the following equations. The equations that characterize the private economy are:

$$\begin{aligned}
\hat{\bar{w}}_t &= \hat{\bar{w}}_{t-1} + \hat{\pi}_{w,t} - \hat{\pi}_t \\
\hat{\pi}_t &= \beta \hat{\pi}_{t+1} + \frac{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}{\xi_p} \hat{\bar{w}}_t \\
\hat{\pi}_{w,t} &= -\frac{\kappa}{1+\phi\frac{\lambda_w}{\lambda_w-1}} \left[\hat{\bar{w}}_t - \frac{d\tau_t}{1-\tau} - (1+\phi)\hat{H}_t\right] + \beta\hat{\pi}_{w,t+1} \\
\hat{H}_t &= \hat{H}_{t+1} - \beta\left(dZ_t - dr_t\right) + \hat{\pi}_{t+1},
\end{aligned}$$

and monetary policy:

$$\begin{aligned}
dZ_t &= \rho_R dR_{t-1} + (1-\rho_R)\frac{1}{\beta}\left[r_\pi \hat{\pi}_t + r_y \hat{H}_t\right], \\
dR_t &= \left\{ \begin{array}{ccc} dZ_t & dZ_t \geq -\left(\frac{1}{\beta}-1\right) & \text{`zero bound not binding'} \\ -\left(\frac{1}{\beta}-1\right) & \text{otherwise} & \text{`zero bound binding'} \end{array} \right. .
\end{aligned}$$

The latter captures the fact that $R_t \geq 0$, which means $R_t \geq R = 1/\beta - 1$.

We write this in matrix form as follows. Supppose the zero bound is not binding, so that

$$dR_t = dZ_t. \tag{1.7}$$

This gives us six equations in the six unknowns:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} dR_{t+1} \\ \hat{\pi}_{w,t+1} \\ \hat{H}_{t+1} \\ \hat{\overline{w}}_{t+1} \\ \hat{\pi}_{t+1} \\ dZ_{t+1} \end{pmatrix} + \begin{bmatrix} 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}{\xi_p} & 1 & 0 \\ 0 & 1 & -\frac{\kappa(1+\phi)}{1+\phi\frac{\lambda_w}{\lambda_w-1}} & \frac{\kappa}{1+\phi\frac{\lambda_w}{\lambda_w-1}} & 0 & 0 \\ \beta & 0 & 1 & 0 & 0 & \beta \\ 0 & 0 & -\left(1-\rho_R\right)\frac{1}{\beta}r_y & 0 & -\left(1-\rho_R\right)\frac{1}{\beta}r_\pi & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho_R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} dR_{t-1} \\ \hat{\pi}_{w,t-1} \\ \hat{H}_{t-1} \\ \hat{\overline{w}}_{t-1} \\ \hat{\pi}_{t-1} \\ dZ_{t-1} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} dr_{t+1} \\ d\tau_{t+1} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\frac{\kappa}{1+\phi\frac{\lambda_w}{\lambda_w-1}}\frac{1}{1-\tau} \\ -\beta & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or,

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t = 0,$$

where the definitions of the matrices are obvious and Let,

$$z_t = \begin{pmatrix} dR_t \\ \hat{\pi}_{w,t} \\ \hat{H}_t \\ \hat{\overline{w}}_t \\ \hat{\pi}_t \\ dZ_t \end{pmatrix}, \; s_t = \begin{pmatrix} dr_t \\ d\tau_t \end{pmatrix}.$$

The linearized system when the zero bound is binding is as follows:

$$d + \alpha_0 z_{t+1} + \tilde{\alpha}_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t = 0,$$

where

$$d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\beta} - 1 \end{pmatrix},$$

and $\tilde{\alpha}_1$ is $\alpha_1$ with its $6,6$ element replaced by zero. This system is simply the previous one with (1.7) replaced by:

$$dR_t = -\left(\frac{1}{\beta} - 1\right).$$

## 2. Simulating the Model

The general algorithm appears in the third subsection below. It captures the feature of our setting, that the equations that characterize equilibrium change during the simulation. Before we discuss the algorithm in its full generality, we provide two examples to illustrate features of the algorithm not related to the equation switching.

### 2.0.1. Simple Example

We use a slight perturbation on a standard shooting algorithm. The perturbation is designed so that the algorithm is required to 'hit' a specific target at a specific date, as opposed to the usual shooting in which a target is reached asymptotically. Because the general algorithm involves other complications, it is useful to point out the perturbation that we use, in isolation from the other complications. Suppose that the system obeys the following scalar difference equation:

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = d_t,$$

for $t = 1, 2, ..., T$. Here, $z_0$ is given. For $t \geq T + 1$,

$$z_t = A z_{t-1}.$$

Writing the equations out explicitly:

$$\begin{aligned}
\alpha_0 z_2 + \alpha_1 z_1 + \alpha_2 z_0 &= d_1 \\
\\
\alpha_0 z_{T+1} + \alpha_1 z_T + \alpha_2 z_{T-1} &= d_T.
\end{aligned}$$

Given $\alpha_0 \neq 0$, and given an arbitrary $z_1 \in R^1$, we can use these equations to compute $z_2, ..., z_{T+1}$. But, it has to be the case that

$$z_{T+1} - A z_T = 0.$$

So, the algorithm is to adjust $z_1$ until the above equation is satisfied.

### 2.0.2. Algorithm Based on QZ Decomposition

The problem we have to confront in applying the simple algorithm in the previous section is that $\alpha_0$ is not invertible. One way to adapt the algorithm applies the $QZ$ decomposition. Thus, let

$$\begin{aligned}
Q\alpha_0 Z &= H_0,\ Q\tilde{\alpha}_1 Z = H_1, \\
Z' z_t &= \gamma_t.
\end{aligned}$$

Multiply (**??**) by $Q$ :

$$Q\alpha_0 Z Z' z_{t+1} + Q\tilde{\alpha}_1 Z Z' z_t + Q\alpha_2 Z Z' z_{t-1} + Qd + Q\beta_0 s_{t+1} + Q\beta_1 s_t = 0,$$

or,

$$H_0\gamma_{t+1}+H_1\gamma_t+Q\alpha_2 Z\gamma_{t-1}+Q\overbrace{[d+\beta_0 s_{t+1}+\beta_1 s_t]}^{d_t}=0.$$

Summarizing,

$$H_0\gamma_{t+1}+H_1\gamma_t=D_t,$$

for $t=1,2,...,T-1$. Write

$$\begin{aligned}H_0 &= \begin{bmatrix} G_0 & H_0^{12} \\ 0 & H_0^{22}\end{bmatrix},\ H_1=\begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22}\end{bmatrix},\gamma_t=\begin{pmatrix}\gamma_t^1 \\ \gamma_t^2\end{pmatrix}=Z'z_t=\begin{pmatrix}L_1 z_t \\ L_2 z_t\end{pmatrix}, \\ D_t &= -Q\left[d_t+\alpha_2 z_{t-1}\right],\end{aligned}$$

where $H_0$ and $H_1$ are upper triangular and the diagonal elements of $G_0$ are non-zero while the $l$ diagonal elements of $H_0^{22}$ are all zero. It is necessary to verify numerically that all the elements of $H_0^{22}$ are zero. We assume that the diagonal elements of $H_1^{22}$ are all non-zero. Also,

$$d_t=d+\beta_0 s_{t+1}+\beta_1 s_t$$

Then, the system is written

$$\begin{aligned}
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\gamma_2^1 \\ \gamma_2^2\end{bmatrix}+\begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22}\end{bmatrix}\begin{bmatrix}\gamma_1^1 \\ \gamma_1^2\end{bmatrix} &= \begin{bmatrix}D_1^1 \\ D_1^2\end{bmatrix} \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\gamma_3^1 \\ \gamma_3^2\end{bmatrix}+\begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22}\end{bmatrix}\begin{bmatrix}\gamma_2^1 \\ \gamma_2^2\end{bmatrix} &= \begin{bmatrix}D_2^1 \\ D_2^2\end{bmatrix} \\
&\quad \cdots \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\gamma_{T-1}^1 \\ \gamma_{T-1}^2\end{bmatrix}+\begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22}\end{bmatrix}\begin{bmatrix}\gamma_{T-2}^1 \\ \gamma_{T-2}^2\end{bmatrix} &= \begin{bmatrix}D_{T-2}^1 \\ D_{T-2}^2\end{bmatrix} \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\gamma_T^1 \\ \gamma_T^2\end{bmatrix}+\begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22}\end{bmatrix}\begin{bmatrix}\gamma_{T-1}^1 \\ \gamma_{T-1}^2\end{bmatrix} &= \begin{bmatrix}D_{T-1}^1 \\ D_{T-1}^2\end{bmatrix},
\end{aligned}$$

where we have imposed $H_0^{22}=0$. We have found that, numerically, this is a property of our model. To simulate this system forward note first that $D_1$ is determined because $\gamma_0=0$. Fix a value for $\gamma_1^1$ and compute:

$$\gamma_1^2=\left(H_1^{22}\right)^{-1}D_1^2.$$

For $t=1:$

$$\begin{aligned}D_2 &= -Q\left(d_2+Q\alpha_2 Z\gamma_1\right) \\ \gamma_2^2 &= \left(H_1^{22}\right)^{-1}D_2^2 \\ \gamma_2^1 &= -G_0^{-1}\left[H_0^{12}\gamma_2^2+G_1\gamma_1^1+H_1^{12}\gamma_1^2-D_1^1\right]\end{aligned}$$

For $t=1,...,T-1:$

$$\begin{aligned}D_{t+1} &= -Q\left(d_{t+1}+\alpha_2 Z\gamma_t\right) \\ \gamma_{t+1}^2 &= \left(H_1^{22}\right)^{-1}D_{t+1}^2 \\ \gamma_{t+1}^1 &= -G_0^{-1}\left[H_0^{12}\gamma_{t+1}^2+G_1\gamma_t^1+H_1^{12}\gamma_t^2-D_t^1\right]\end{aligned}$$

We now have $\gamma_T^1$, $\gamma_T^2$, $D_T$.Recall that in period $T+1$,

$$z_{T+1} = Az_T,$$

so that, after multiplying by $Z'$ :

$$\gamma_{T+1} = \tilde{A}\gamma_T,\ \tilde{A} = Z'AZ,$$

or,

$$\begin{pmatrix} \gamma_{T+1}^1 \\ \gamma_{T+1}^2 \end{pmatrix} = \begin{bmatrix} \tilde{A}_1 \\ \cdots \\ \tilde{A}_2 \end{bmatrix} \gamma_T.$$

We must still satisfy the $t = T$ equilibrium conditions:

$$\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{T+1}^1 \\ \gamma_{T+1}^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix} \begin{bmatrix} \gamma_T^1 \\ \gamma_T^2 \end{bmatrix} = \begin{bmatrix} D_T^1 \\ D_T^2 \end{bmatrix}.$$

Note, however, that the bottom set of equations are satisfied because of the way $\gamma_T^2$ was chosen and because $\gamma_{T+1}^2$ does not enter these equations. The first set of equations need not be satisfied, however, and so we use the requirement that these be satisfied to pin down $\gamma_1^1$. In particular, we adjust $\gamma_1^1$ until the following expression is satisfied:

$$G_0\gamma_{T+1}^1 + H_0^{12}\gamma_{T+1}^2 + G_1\gamma_T^1 + H_1^{12}\gamma_T^2 = D_T^1.$$

Note that this is a number of equations equal to the dimension of $\gamma_1^1$.

### 2.0.3. Extending the Algorithm

We now address the possibility that $t_1 \geq 1$ and $t_2 \leq T$. That is, the lower bound starts to bind in some period after the discount rate goes negative and before it turns positive again. Thus, the lower bound is not binding for $t = 1, ..., t_1 - 1$, it is binding for $t = t_1, ..., t_2$ and it is not binding for $t > t_2$. Because we assume $t_2 \leq T$, we can apply a straightforward adaptation of the algorithm in the previous section. First, the $d_t$ sequence needs to be adjusted so that the constant vector, $d$ in (**??**), is only turned on for $t = t_1, ..., t_2$. Second, we require the QZ decomposition of the system both for the time when the lower bound is binding and the time when it is binding:

$$\begin{aligned} Q\alpha_0 Z &= H_0,\ Q\alpha_1 Z = H_1 \\ \tilde{Q}\alpha_0\tilde{Z} &= \tilde{H}_0,\ \tilde{Q}\tilde{\alpha}_1\tilde{Z} = \tilde{H}_1 \\ \gamma_t &= Z' z_t,\ \tilde{\gamma}_t = \tilde{Z}' z_t. \end{aligned}$$

**2.0.3.1.The Initial Non-Binding Regime** For $t = 1, ..., t_1 - 1$ :

$$\alpha_0 z_{t+1} + \alpha_1 z_t = -\left[\alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right].$$

To solve these equations, proceed as before. After applying the QZ decomposition:

$$\begin{aligned}
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_2^1 \\ \gamma_2^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix}\begin{bmatrix} \gamma_1^1 \\ \gamma_1^2 \end{bmatrix} &= \begin{bmatrix} D_1^1 \\ D_1^2 \end{bmatrix} \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_3^1 \\ \gamma_3^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix}\begin{bmatrix} \gamma_2^1 \\ \gamma_2^2 \end{bmatrix} &= \begin{bmatrix} D_2^1 \\ D_2^2 \end{bmatrix} \\
&\cdots \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_{t_1-1}^1 \\ \gamma_{t_1-1}^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix}\begin{bmatrix} \gamma_{t_1-2}^1 \\ \gamma_{t_1-2}^2 \end{bmatrix} &= \begin{bmatrix} D_{t_1-2}^1 \\ D_{t_1-2}^2 \end{bmatrix} \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma_{t_1}^1 \\ \gamma_{t_1}^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix}\begin{bmatrix} \gamma_{t_1-1}^1 \\ \gamma_{t_1-1}^2 \end{bmatrix} &= \begin{bmatrix} D_{t_1-1}^1 \\ D_{t_1-1}^2 \end{bmatrix}.
\end{aligned}$$

The basic idea of the simulation is that we start a given period, $t$, with $(\gamma_t^1, \gamma_t^2)$ and use the period $t$ equation to solve forward to obtain $\left(\gamma_{t+1}^1, \gamma_{t+1}^2\right)$. This would be completely straightforward and standard if the lead matrix in the QZ decomposition of the dynamic equation were invertible. It is not. So, to do the simulation, we compute $\gamma_{t+1}^2$ using the $t+1$ equation. Then, there is only $\gamma_{t+1}^1$ to compute using the period $t$ equation. This computation is possible because the relevant block in the lead matrix is invertible.

To begin the simulation, note first that $D_1$ is determined because $z_0 = 0$. Fix a value for $\gamma_1^1$ and compute:

$$\gamma_1^2 = \left(H_1^{22}\right)^{-1} D_1^2.$$

For $t = 1$ :

$$\begin{aligned}
D_2 &= -Q\left(d_2 + \alpha_2 Z \gamma_1\right) \text{ (constant term in period 2 equation)} \\
\gamma_2^2 &= \left(H_1^{22}\right)^{-1} D_2^2 \text{ (solving period 2 equation for } \gamma_2^2\text{)} \\
\gamma_2^1 &= -G_0^{-1}\left[H_0^{12}\gamma_2^2 + G_1\gamma_1^1 + H_1^{12}\gamma_1^2 - D_1^1\right] \text{ (using period 1 equation to find } \gamma_2^1\text{)}
\end{aligned}$$

We proceed in this way in each period, $t = 1, ..., t_1 - 2$ :

$$\begin{aligned}
D_{t+1} &= -Q\left(d_{t+1} + \alpha_2 Z \gamma_t\right) \\
\gamma_{t+1}^2 &= \left(H_1^{22}\right)^{-1} D_{t+1}^2 \\
\gamma_{t+1}^1 &= -G_0^{-1}\left[H_0^{12}\gamma_{t+1}^2 + G_1\gamma_t^1 + H_1^{12}\gamma_t^2 - D_t^1\right]
\end{aligned}$$

Period $t = t_1 - 1$ requires special treatment because the subsequent period's equation belongs to a different regime. We first solve for $\tilde{\gamma}_{t_1}^2$ using the $t_1$ equation. The 'constant term' in the $t_1$ equation is:

$$\tilde{D}_{t_1} = -\tilde{Q}\left(d_{t_1} + \alpha_2 Z \begin{pmatrix} \gamma_{t_1-1}^1 \\ \gamma_{t_1-1}^2 \end{pmatrix}\right).$$

Note that the $\tilde{Q}$ here belongs to the binding regime, while $Z$ belongs to the non-binding regime. We require the $Z$ from the non-binding regime because what is needed to construct the constant term is $z_{t_1-1}$, and what we actually have in hand at this point is $\gamma_{t_1-1} = Z' z_{t_1-1}$. With $\tilde{D}_{t_1}$ in hand, we compute $\tilde{\gamma}_{t_1}^2$ :

$$\tilde{\gamma}_{t_1}^2 = \left(\tilde{H}_1^{22}\right)^{-1} \tilde{D}_{t_1}^2.$$

We now return to the $t_1 - 1$ equation to seek $\tilde{\gamma}^1_{t_1}$. The equation is:

$$\alpha_0 z_{t_1} + \alpha_1 z_{t_1-1} = -\left[\alpha_2 z_{t_1-2} + \beta_0 s_{t_1} + \beta_1 s_{t_1-1}\right],$$

or,

$$\alpha_0 \tilde{Z} \tilde{\gamma}_{t_1} + \alpha_1 z_{t_1-1} = -\left[\alpha_2 z_{t_1-2} + \beta_0 s_{t_1} + \beta_1 s_{t_1-1}\right].$$

Now, apply the QZ decomposition relevant for the non-binding regime:

$$Q\alpha_0 Z Z' \tilde{Z} \tilde{\gamma}_{t_1} + Q\alpha_1 Z Z' z_{t_1-1} = D_{t_{1-1}},\ D_{t_{1-1}} = -\left[\alpha_2 z_{t_1-2} + \beta_0 s_{t_1} + \beta_1 s_{t_1-1}\right]$$

or,

$$H_0 Z' \tilde{Z} \tilde{\gamma}_{t_1} + H_1 \gamma_{t_1-1} = D_{t_{1-1}}.$$

Let

$$Z'\tilde{Z} \equiv M = \begin{bmatrix} \underbrace{M_{11}}_{(m-l)\times(m-l)} & \underbrace{M_{12}}_{(m-l)\times l} \\ \underbrace{M_{21}}_{l\times(m-l)} & \underbrace{M_{22}}_{l\times l} \end{bmatrix},$$

where $m$ denotes the length of $z_t$, and $m-l$ is the rank of $\alpha_0$. Then the previous system can be written:

$$\begin{bmatrix} \underbrace{G_0}_{(m-l)\times(m-l)} & \underbrace{H_0^{12}}_{(m-l)\times l} \\ \underbrace{0}_{l\times(m-l)} & \underbrace{0}_{l\times l} \end{bmatrix} \begin{bmatrix} \underbrace{M_{11}}_{(m-l)\times(m-l)} & \underbrace{M_{12}}_{(m-l)\times l} \\ \underbrace{M_{21}}_{l\times(m-l)} & \underbrace{M_{22}}_{l\times l} \end{bmatrix} \begin{bmatrix} \underbrace{\tilde{\gamma}^1_{t_1}}_{(m-l)\times 1} \\ \underbrace{\tilde{\gamma}^2_{t_1}}_{l\times 1} \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix} \begin{bmatrix} \gamma^1_{t_1-1} \\ \gamma^2_{t_1-1} \end{bmatrix} = \begin{bmatrix} D^1_{t_1-1} \\ D^2_{t_1-1} \end{bmatrix},$$

or,

$$\begin{bmatrix} G_0 M_{11} + H_0^{12} M_{21} & G_0 M_{12} + H_0^{12} M_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}^1_{t_1} \\ \tilde{\gamma}^2_{t_1} \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix} \begin{bmatrix} \gamma^1_{t_1-1} \\ \gamma^2_{t_1-1} \end{bmatrix} = \begin{bmatrix} D^1_{t_1-1} \\ D^2_{t_1-1} \end{bmatrix}.$$

We are now in a position to solve for $\tilde{\gamma}^1_{t_1}$. Writing out the first of the above equations:

$$\left[G_0 M_{11} + H_0^{12} M_{21}\right] \tilde{\gamma}^1_{t_1} + \left[G_0 M_{12} + H_0^{12} M_{22}\right] \tilde{\gamma}^2_{t_1} + G_1 \gamma^1_{t_1-1} + H_1^{12} \gamma^2_{t_1-1} = D^1_{t_1-1},$$

so that

$$\tilde{\gamma}^1_{t_1} = -\left[G_0 M_{11} + H_0^{12} M_{21}\right]^{-1} \left[\left(G_0 M_{12} + H_0^{12} M_{22}\right) \tilde{\gamma}^2_{t_1} + G_1 \gamma^1_{t_1-1} + H_1^{12} \gamma^2_{t_1-1} - D^1_{t_1-1}\right].$$

**2.0.3.2.The Binding Regime** We now turn to the period when the lower bound constraint on the interest rate is binding, $t = t_1, ..., t_2$. The equation to be solved is:

$$\alpha_0 z_{t+1} + \tilde{\alpha}_1 z_t = -\left[\alpha_2 z_{t-1} + d + \beta_0 s_{t+1} + \beta_1 s_t\right].$$

Multiply by $\tilde{Q}$

$$\tilde{H}_0 \tilde{\gamma}_{t+1} + H_1 \tilde{\gamma}_t = \tilde{D}_t,$$

or,

$$\begin{array}{rcl}
\begin{bmatrix} \tilde{G}_0 & \tilde{H}_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_1+1}^1 \\ \tilde{\gamma}_{t_1+1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 & \tilde{H}_1^{12} \\ 0 & \tilde{H}_1^{22} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_1}^1 \\ \tilde{\gamma}_{t_1}^2 \end{bmatrix} & = & \begin{bmatrix} \tilde{D}_{t_1}^1 \\ \tilde{D}_{t_1}^2 \end{bmatrix} \\
\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_1+2}^1 \\ \tilde{\gamma}_{t_1+2}^2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 & \tilde{H}_1^{12} \\ 0 & \tilde{H}_1^{22} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_1+1}^1 \\ \tilde{\gamma}_{t_1+1}^2 \end{bmatrix} & = & \begin{bmatrix} \tilde{D}_{t_1+1}^1 \\ \tilde{D}_{t_1+1}^2 \end{bmatrix} \\
& \cdots & \\
\begin{bmatrix} \tilde{G}_0 & \tilde{H}_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_2-2}^1 \\ \tilde{\gamma}_{t_2-2}^2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 & \tilde{H}_1^{12} \\ 0 & \tilde{H}_1^{22} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_2-3}^1 \\ \tilde{\gamma}_{t_2-3}^2 \end{bmatrix} & = & \begin{bmatrix} \tilde{D}_{t_2-3}^1 \\ \tilde{D}_{t_2-3}^2 \end{bmatrix} \\
\begin{bmatrix} \tilde{G}_0 & \tilde{H}_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_2-1}^1 \\ \tilde{\gamma}_{t_2-1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 & \tilde{H}_1^{12} \\ 0 & \tilde{H}_1^{22} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_2-2}^1 \\ \tilde{\gamma}_{t_2-2}^2 \end{bmatrix} & = & \begin{bmatrix} \tilde{D}_{t_2-2}^1 \\ \tilde{D}_{t_2-2}^2 \end{bmatrix},
\end{array}$$

We have $\tilde{\gamma}_{t_1}$ in hand. Consider $t = t_1$ first. As before, we must compute $\tilde{\gamma}_{t_1+1}^2$ using the period $t_1 + 1$ equation. Thus,

$$\begin{aligned}
\tilde{D}_{t_1+1} &= -\tilde{Q}\left(d_{t_1+1} + \alpha_2 \tilde{Z} \tilde{\gamma}_{t_1}\right) \\
\gamma_{t_1+1}^2 &= \left(\tilde{H}_1^{22}\right)^{-1} \tilde{D}_{t_1+1}^2.
\end{aligned}$$

Then, using the period $t_1$ equation:

$$\gamma_{t_1+1}^1 = -\tilde{G}_0^{-1}\left[\tilde{H}_0^{12}\tilde{\gamma}_{t_1+1}^2 + \tilde{G}_1\tilde{\gamma}_{t_1}^1 + \tilde{H}_1^{12}\tilde{\gamma}_{t_1}^2 - \tilde{D}_{t_1}^1\right]$$

We proceed in this way in each period, $t = t_1, ..., t_2 - 1$ :

$$\begin{aligned}
\tilde{D}_{t+1} &= -\tilde{Q}\left(d_{t+1} + \alpha_2 \tilde{Z} \tilde{\gamma}_t\right) \\
\tilde{\gamma}_{t+1}^2 &= \left(\tilde{H}_1^{22}\right)^{-1} \tilde{D}_{t+1}^2 \\
\tilde{\gamma}_{t+1}^1 &= -\tilde{G}_0^{-1}\left[\tilde{H}_0^{12}\tilde{\gamma}_{t+1}^2 + \tilde{G}_1\tilde{\gamma}_t^1 + \tilde{H}_1^{12}\tilde{\gamma}_t^2 - \tilde{D}_t^1\right].
\end{aligned}$$

The $t = t_2$ equation requires special adjustments analogous to the ones used at the end of the non-binding regime, because solving the $t = t_2$ equation requires working with the $t_2 + 1$ equation first. The ‘constant term’ in the $t_2 + 1$ equation is:

$$D_{t_2+1} = -Q\left(d_{t_2+1} + \alpha_2 \tilde{Z}\begin{pmatrix} \tilde{\gamma}_{t_2}^1 \\ \tilde{\gamma}_{t_2}^2 \end{pmatrix}\right).$$

As before, $Q$ is part of the QZ decomposition relevant to the non-binding regime, but $\tilde{Z}$ belongs to the binding regime because we have $\tilde{\gamma}_{t_2}$ in hand, and this must be converted to $z_{t_2}$.

With $D_{t_2+1}$ in hand, we compute $\gamma_{t_2+1}^2$ as follows:

$$\gamma_{t_2+1}^2 = \left(H_1^{22}\right)^{-1} D_{t_2+1}^2.$$

We now return to the $t_2$ equation:

$$\alpha_0 z_{t_2+1} + \tilde{\alpha}_1 \tilde{Z}\tilde{\gamma}_{t_2} = -\left[\alpha_2 \tilde{Z}\tilde{\gamma}_{t_2} + d + \beta_0 s_{t_2+1} + \beta_1 s_{t_2}\right],$$

or, after multiplying by $\tilde{Q}$ :

$$\tilde{H}_0 \tilde{Z}' Z \gamma_{t_2+1} + \tilde{H}_1 \tilde{\gamma}_{t_2} = \tilde{D}_{t_2},$$

where $\tilde{D}_{t_2}$ is available from the previous computations. Writing this out more carefully,

$$\begin{bmatrix} \tilde{G}_0 & \tilde{H}_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} \gamma_{t_2+1}^1 \\ \gamma_{t_2+1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 & \tilde{H}_1^{12} \\ 0 & \tilde{H}_1^{22} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t_2}^1 \\ \tilde{\gamma}_{t_2}^2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{t_2}^1 \\ \tilde{D}_{t_2}^2 \end{bmatrix},$$

where

$$\tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} = \tilde{Z}' Z \ \ (= M').$$

Writing out the first of the above equations:

$$\left[\tilde{G}_0 \tilde{M}_{11} + \tilde{H}_0^{12} \tilde{M}_{21}\right] \gamma_{t_2+1}^1 + \left[\tilde{G}_0 \tilde{M}_{12} + \tilde{H}_0^{12} \tilde{M}_{22}\right] \gamma_{t_2+1}^2 + \tilde{G}_1 \tilde{\gamma}_{t_2}^1 + \tilde{H}_1^{12} \tilde{\gamma}_{t_2}^2 = \tilde{D}_{t_2}^1,$$

or,

$$\gamma_{t_2+1}^1 = -\left[\tilde{G}_0 \tilde{M}_{11} + \tilde{H}_0^{12} \tilde{M}_{21}\right]^{-1} \left[\left(\tilde{G}_0 \tilde{M}_{12} + \tilde{H}_0^{12} \tilde{M}_{22}\right) \gamma_{t_2+1}^2 + \tilde{G}_1 \tilde{\gamma}_{t_2}^1 + \tilde{H}_1^{12} \tilde{\gamma}_{t_2}^2 - \tilde{D}_{t_2}^1\right].$$

**2.0.3.3.The Final Non-Binding Regime** Given $\left(\gamma_{t_2+1}^1, \gamma_{t_2+1}^2\right)$, we now solve the equations in the non-binding regime, $t = t_2 + 1, ..., T$. Consider period $t_2 + 1$ first:

$$\alpha_0 z_{t_2+2} + \alpha_1 z_{t_2+1} = -\left[\alpha_2 z_{t_2} + \beta_0 s_{t_2+2} + \beta_1 s_{t_2+1}\right].$$

Multiplying by $Q$ and applying the QZ decomposition:

$$Q \alpha_0 Z Z' z_{t_2+2} + Q \alpha_1 Z Z' z_{t_2+1} = -Q\left[\alpha_2 z_{t_2} + \beta_0 s_{t_2+2} + \beta_1 s_{t_2+1}\right],$$

or,

$$H_0 \gamma_{t_2+2} + H_1 \gamma_{t_2+1} = D_{t_2+1},$$

or,

$$\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{t_2+2}^1 \\ \gamma_{t_2+2}^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix} \begin{bmatrix} \gamma_{t_2+1}^1 \\ \gamma_{t_2+1}^2 \end{bmatrix} = \begin{bmatrix} D_{t_2+1}^1 \\ D_{t_2+1}^2 \end{bmatrix}.$$

To solve for $\left(\gamma_{t_2+2}^1, \gamma_{t_2+2}^2\right)$, we first obtain $\gamma_{t_2+2}^2$ using the $t_2 + 2$ equation:

$$\begin{aligned} D_{t_2+2} &= -Q\left(d_{t_2+2} + \alpha_2 Z \gamma_{t_2+1}\right) \ \text{(constant term in period } t_2+2 \text{ equation)} \\ \gamma_{t_2+2}^2 &= \left(H_1^{22}\right)^{-1} D_{t_2+2}^2 \ \text{(solving period } t_2+2 \text{ equation for } \gamma_{t_2+2}^2) \end{aligned}$$

Then,

$$\gamma_{t_2+2}^1 = -G_0^{-1}\left[H_0^{12} \gamma_{t_2+2}^2 + G_1 \gamma_{t_2+1}^1 + H_1^{12} \gamma_{t_2+1}^2 - D_{t_2+1}^1\right].$$

For $t = t_2 + 1, ..., T - 1$ :

$$\begin{aligned} D_{t+1} &= -Q\left(d_{t+1} + \alpha_2 Z \gamma_t\right) \\ \gamma_{t+1}^2 &= \left(H_1^{22}\right)^{-1} D_{t+1}^2 \\ \gamma_{t+1}^1 &= -G_0^{-1}\left[H_0^{12} \gamma_{t+1}^2 + G_1 \gamma_t^1 + H_1^{12} \gamma_t^2 - D_t^1\right] \end{aligned}$$

We now have $\gamma_T^1$, $\gamma_T^2$, $D_T$. Recall that $s_t = 0$ for $t \geq T+1$.

$$z_{T+1} = Az_T,$$

so that, after multiplying by $Z'$ :

$$\gamma_{T+1} = \tilde{A}\gamma_T, \ \tilde{A} = Z'AZ,$$

or,

$$\begin{pmatrix} \gamma_{T+1}^1 \\ \gamma_{T+1}^2 \end{pmatrix} = \begin{bmatrix} \tilde{A}_1 \\ \cdots \\ \tilde{A}_2 \end{bmatrix} \gamma_T.$$

We must still satisfy the $t = T$ equilibrium conditions:

$$\begin{bmatrix} G_0 & H_0^{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{T+1}^1 \\ \gamma_{T+1}^2 \end{bmatrix} + \begin{bmatrix} G_1 & H_1^{12} \\ 0 & H_1^{22} \end{bmatrix} \begin{bmatrix} \gamma_T^1 \\ \gamma_T^2 \end{bmatrix} = \begin{bmatrix} D_T^1 \\ D_T^2 \end{bmatrix}.$$

Note, however, that the bottom set of equations are satisfied because of the way $\gamma_T^2$ was chosen and because $\gamma_{T+1}^2$ does not enter these equations. The first set of equations need not be satisfied, however, and so we use the requirement that these be satisfied to pin down $\gamma_{T+1}^1$. In particular, we adjust $\gamma_1^1$ until the following expression is satisfied:

$$G_0\gamma_{T+1}^1 + H_0^{12}\gamma_{T+1}^2 + G_1\gamma_T^1 + H_1^{12}\gamma_T^2 = D_T^1.$$

Note that this is a number of equations equal to the dimension of $\gamma_1^1$.