How Severe is the Time Inconsistency Problem in Monetary Policy?

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Abstract

We analyze two monetary economies - a cash-credit good model and a limited participation model. In our models, monetary policy is made by a benevolent policymaker who cannot commit to future policies. We define and analyze Markov equilibrium in these economies. We show that there is no time inconsistency problem for a wide range of parameter values.

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1. Introduction

The history of inflation in the United States and other countries has occasionally been quite bad. Are the bad experiences the consequence of policy errors? Or, does the problem lie with the nature of monetary institutions? The second possibility has been explored in a long literature which starts at least with Kydland and Prescott (1977) and Barro and Gordon (1982). This paper seeks to make a contribution to that literature.

The Kydland-Prescott and Barro-Gordon literature focuses on the extent to which monetary institutions allow policymakers to commit to future policies. A key result is that if policymakers cannot commit to future policies, inflation rates are higher than if they can commit. That is, there is a time inconsistency problem which introduces a systematic inflation bias. This paper investigates the magnitude of the inflation bias in two standard general equilibrium models. One is the cash-credit good model of Lucas and Stokey (1983). The other is the limited participation model of money described in Christiano, Eichenbaum and Evans (1997). We find that, for a large range of parameter values, there is no inflation bias.

In the Kydland-Prescott and Barro-Gordon literature, equilibrium inflation in the absence of commitment is the outcome of an interplay between the benefits and costs of inflation. For the most part, this literature consists of reduced form models. Our general equilibrium models[1] incorporate the kinds of benefits and costs that seem to motivate the reduced form specifications. To understand these benefits and costs, it is necessary first to explain why money is not neutral in our models. In each case, at the time the monetary authority sets its money growth rate some nominal variable in the economy has already been set. In the cash-credit good model, this variable is the price of a subset of intermediate goods. As in Blanchard and Kiyotaki (1987), some firms must post prices in advance and are required to meet all demand at their posted price. In the limited participation model, a portfolio choice variable is set in advance. In each case, higher than expected money growth tends - other things the same - to raise output. The rise in output raises welfare because the presence of monopoly power in our model economies implies output and employment are below their efficient levels. These features give incentives to the monetary authority to make money

[1]See Chari, Christiano and Eichenbaum (1998), Ireland (1997) and Neiss (1999) for related general equilibrium models.

growth rates higher than expected.

Turning to the costs of inflation, we first discuss the cash-credit good model. We assume that cash good consumption must be financed using money carried over from the previous period.[2] If the money growth rate is high, the price of the cash good is high and the quantity of cash goods consumed is low. This mechanism tends to reduce welfare as the money growth rate rises. The monetary authority balances the output-increasing benefits of high money growth against the costs of the resulting fall in cash good consumption. Somewhat surprisingly, we find that there is a large subset of parameter values where the costs of inflation dominate the benefits at all levels of inflation and money growth above the ex ante optimal rate. As a result, for these parameter values, the unique equilibrium yields the same outcome as under commitment.

In our limited participation model, at all interest rates higher than zero, increases in money growth tend to stimulate employment by reducing the interest rate. As a result, there is no equilibrium with a positive interest rate. When the interest rate is already zero, further reductions are not possible. In this case, additional money generated by the monetary authority simply accumulates as idle balances at the financial intermediary. The unique Markov equilibrium in this model has a zero interest rate. Again, there is no time inconsistency problem and no inflation bias.

Should we conclude from our examples that lack of commitment in monetary policy cannot account for the bad inflation outcomes that have occurred? We think such a conclusion is premature. Research on the consequences of lack of commitment in dynamic general equilibrium models is still in its infancy. Elsewhere, in Albanesi, Chari and Christiano (2000), we have displayed a class of empirically plausible models in which lack of commitment may in fact lead to high and volatile inflation. The key difference between the model in that paper and the models studied here lies in the modeling of money demand. Taken together, these findings suggest that a resolution of the importance of time inconsistency in monetary policy depends on the details of money demand. As our understanding about the implications for time inconsistency in dynamic models grows, we may discover other features of the economic

[2] We assume a timing structure as in Svensson (1985) rather than in Lucas and Stokey (1983). See also Nicolini (1998).

environment that are crucial for determining the severity of the time inconsistency problem. It is too soon to tell whether the ultimate conclusion will be consistent with the implications of the models studied in this paper.

The paper is organized as follows. In Section 2, we analyze a cash credit goods model with arbitrary monetary policies. This section sets up the basic framework for analyzing purposeful monetary policy. Interestingly, we also obtain some new results on multiplicity of equilibria under mild deflations. In section 3 we analyze the model in section 2 when monetary policy is chosen by a benevolent policy maker without commitment. In Section 4, we analyze a limited participation model. Section 5 concludes.

2. Equilibrium in A Cash-Credit Good Model With Arbitrary Monetary Policy

In this section we develop a version of the Lucas and Stokey's cash-credit good model. There are three key modifications: we introduce monopolistic competition, as in Blanchard and Kiyotaki (1987); we modify the timing in the cash in advance constraint as indicated in the introduction; and we consider non-stationary equilibria. The agents in the model are a representative household and representative intermediate and final good producers. A policy for the monetary authority is a sequence of growth rates for the money supply. We consider arbitrary monetary policies and define and characterize the equilibrium. We show that in the best equilibrium with commitment, monetary policy follows the Friedman rule in the sense that the nominal interest rate is zero. Following Cole and Kocherlakota (1998), we show that there is a non-trivial class of monetary policies which support the best equilibrium. Interestingly, we show that only one of the policies in this class is robust, while the others are fragile. Specifically, we show that only the policy in which money growth deflates at the pure rate of time preference supports the best equilibrium as the unique outcome. We show that the other policies are fragile in the sense that there are many equilibria associated with them.

2.1. Households

The household's utility function is:

$$\sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, n_t), \;\; u(c_1, c_2, n) = \log c_{1t} + \log c_{2t} + \log(1-n) \tag{2.1}$$

where c_{1t}, c_{2t} and n_t denote consumption of cash goods, consumption of credit goods, and employment, respectively.

The sequence of events in the period is as follows. At the beginning of the period, the household trades in a securities market in which it allocates nominal assets between money and bonds. After trading in the securities market, the household supplies labor and consumes cash and credit goods.

For securities market trading, the constraint is

$$A_t \geq M_t + B_t, \tag{2.2}$$

where A_t denotes beginning-of-period t nominal assets, M_t denotes the household's holdings of cash, B_t denotes the household's holdings of interest bearing bonds, and A_0 is given. Cash goods must be paid for with currency from securities market trading. The cash in advance constraint is given by:

$$P_{1t} c_{1t} \leq M_t. \tag{2.3}$$

The household's sources of cash during securities market trading are cash left over from the previous period's goods market, $M_{t-1} - P_{1t-1} c_{1t-1}$, earnings on bonds accumulated in the previous period, $R_{t-1} B_{t-1}$, transfers received from the monetary authority, T_{t-1}, labor income in the previous period, $W_{t-1} n_{t-1}$, and profits in the previous period, D_{t-1}. Let P_{1t-1}, P_{2t-1} and R_{t-1} denote the period $t-1$ prices of cash and credit goods, and the gross interest rate, respectively. Finally, the household pays debts, $P_{2t-1} c_{2t-1}$, owed from its period $t-1$ purchases of credit goods during securities market trading. These considerations are summarized in the following securities market constraint:

$$A_t = W_{t-1}n_{t-1} - P_{2t-1}c_{2t-1} + (M_{t-1} - P_{1t-1}c_{1t-1}) + R_{t-1}B_{t-1} + T_{t-1} + D_{t-1}. \tag{2.4}$$

We place the following restriction on the household's ability to borrow:

$$A_{t+1} \geq -\frac{1}{q_{t+1}} \sum_{j=1}^{\infty} q_{t+j+1} \left[W_{t+j} + T_{t+j} + D_{t+j} \right], \text{ for } t = 0, 1, 2, ..., \tag{2.5}$$

where $q_t = \prod_{j=0}^{t-1} 1/R_j$, $q_0 \equiv 1$. Condition (2.5) says that the household can never borrow more than the maximum present value future income.

The household's problem is to maximize (2.1) subject to (2.5)-(2.2) and the nonnegativity constraints, n_t, c_{1t}, c_{2t}, $1 - n_t \geq 0$. If $R_t < 1$ for any t, this problem does not have a solution. We assume throughout that $R_t \geq 1$.

2.2. Firms

We adopt a variant of the production framework in Blanchard and Kiyotaki (1987). In developing firm problems, we delete the time subscript. In each period, there are two types of perfectly competitive, final goods firms: those that produce cash goods and those that produce credit goods. Their production functions are:

$$y_1 = \left[\int_0^1 y_1(\omega)^{\lambda} d\omega \right]^{\frac{1}{\lambda}}, \; y_2 = \left[\int_0^1 y_2(\omega)^{\lambda} d\omega \right]^{\frac{1}{\lambda}}, \tag{2.6}$$

where y_1 denotes output of the cash good, y_2 denotes output of the credit good, and $y_i(\omega)$ is the quantity of intermediate good of type ω used to produce good i, and $0 < \lambda < 1$. These firms solve:

$$\max_{y_i, \{y_i(\omega)\}} P_i y_i - \int_0^1 P_i(\omega) y_i(\omega) d\omega, \; i = 1, 2.$$

Solving this problem leads to the following demand curves for each intermediate good:

$$y_i(\omega) = y_i \left[\frac{P_i}{P_i(\omega)} \right]^{\frac{1}{1-\lambda}}, \ i = 1, \ 2. \tag{2.7}$$

Intermediate good firms are monopolists in the product market and competitors in the market for labor. They set prices for their goods and are then required to supply whatever final good producers demand at those prices. The intermediate good firms solve

$$\max_{y_i(\omega)} P_i(\omega) y_i(\omega) - W n_i(\omega), \ i = 1, 2,$$

where W is the wage rate, subject to a production technology, $y_i(\omega) = n_i(\omega)$ and the demand curve in (2.7). Profit maximization leads the intermediate good firms to set prices according to a markup over marginal costs:

$$P_1(\omega) = \frac{W}{\lambda}, \ P_2(\omega) = \frac{W}{\lambda}. \tag{2.8}$$

2.3. Monetary Authority

At date t, the monetary authority transfers T_t units of cash to the representative household. It finances the transfers by printing money. Let g_t denote the growth rate of the money supply. Then, $T_t = (g_t - 1) M_t$, where M_0 is given and $M_{t+1} = g_t M_t$. A monetary policy is an infinite sequence, g_t, $t = 0, 1, 2, \ldots$.

2.4. Equilibrium

We begin by defining an equilibrium, given an arbitrary specification of monetary policy. We then discuss the best equilibrium achievable by some monetary policy. This equilibrium is one in which the nominal interest rate is zero. Thus, the Friedman rule is optimal in this model. We go on to discuss the set of policies which support the best equilibrium.

Definition 2.1. *A Private Sector Equilibrium is a set of sequences,* $\{P_{1t},\ P_{2t},\ W_t,\ R_t,\ c_{1t},$

c_{2t}, n_t, B_t, M_t, $g_t\}$ *with the properties:*

1. *Given the prices and the government policies the quantities solve the household problem.*

2. *The firm optimality conditions in (2.8) hold*

3. *The various market clearing conditions hold:*

$$c_{1t} + c_{2t} = n_t, \;\; B_t = 0, \;\; M_{t+1} = M_t g_t. \tag{2.9}$$

Next, we define the best equilibrium.

Definition 2.2. *A Ramsey Equilibrium is a Private Sector Equilibrium with the highest level of utility.*

We now develop a set of equations that, together with (2.7)-(2.9), allow us to characterize a private sector equilibrium. From (2.8) it follows that $P_{1t} = P_{2t}$. Let $P_t = P_{1t} = P_{2t}$. Combining the household and the firm first order conditions we get:

$$\frac{c_{2t}}{1 - c_{1t} - c_{2t}} = \lambda, \;\text{ all } t. \tag{2.10}$$

$$P_{t+1}c_{1t+1} = \beta R_t P_t c_{1t}, \;\text{ all } t. \tag{2.11}$$

$$R_t = \frac{c_{2t}}{c_{1t}} \geq 1, \;\text{ all } t. \tag{2.12}$$

$$P_t c_{1t} - M_t \leq 0, \;\; (R_t - 1)(P_t c_{1t} - M_t) = 0. \tag{2.13}$$

In equilibrium, with $B_t = 0$, the household's transversality condition is:

$$\lim_{t\to\infty} \beta^t \frac{M_t}{P_t c_{1t}} = 0. \tag{2.14}$$

The nonnegativity constraint on leisure implies:

$$c_{1t} + c_{2t} \leq 1. \tag{2.15}$$

We summarize these results in the form of a proposition.

Proposition 2.3. *(Characterization Result) A sequence, $\{P_{1t},\ P_{2t},\ W_t,\ R_t,\ c_{1t},\ c_{2t},\ n_t,\ B_t,\ M_t,\ g_t\}$, is an equilibrium if and only if (2.8)-(2.15) and $P_{1t} = P_{2t} = P_t$ are satisfied. Furthermore, for any $R_t \geq 1$, there exists a private sector equilibrium with employment and consumption allocations uniquely determined by:*

$$n_{1t} = c_{1t} = \frac{\lambda}{\lambda + (1+\lambda)R_t},\ n_{2t} = c_{2t} = R_t c_{1t},\ \text{all } t. \tag{2.16}$$

Proof: Equations (2.10) - (2.15) are the resource constraints and the necessary and sufficient conditions for household and firm optimization. Necessity and sufficiency in the case of the firms is obvious and in the case of the households, the results are derived formally in the Appendix.

We now turn to the second part of the proposition. We need to verify that prices and a monetary policy can be found such that, together with the given sequence of interest rates and (2.16), they constitute a private sector equilibrium. First, by construction of (2.16), it can be verified that (2.9), (2.10) and (2.12) are satisfied. It can also be verified that (2.15) is satisfied. Second, let $P_0 = M_0/c_{1,0}$, and use this and (2.11) to compute P_t, for $t = 1, 2, 3, ...$. This construction assures that (2.11) for all t and (2.13) for $t = 0$ are satisfied. Next, we compute $M_t = P_t c_{1t}$ for $t = 1, 2, ...$, so that (2.13) is satisfied for all t. Finally, (2.14) is satisfied because $0 < \beta < 1$ and $M_t/(P_t c_{1t}) = 1.$■

We use this proposition to characterize the Ramsey equilibrium:

Proposition 2.4. *(Ramsey Equilibrium Yields Friedman Rule) Any Ramsey equilibrium has the property $R_t = 1$ for all t and employment and consumption allocations given in (2.16).*

Proof: The Ramsey equilibrium solves:

$$\max_{\{R_t \geq 1\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[2\log(c_{1t}) + \log R_t + \log\left(1 - (1+R_t)c_{1t}\right)\right].$$

where c_{1t} is given by (2.16). This problem equivalent to the static problem $\max_{R\geq 1} f(R)$, where $f(R) = 2\log(c_1) + \log R + \log\left(1 - (1+R)c_1\right),\ c_1 = \lambda/(\lambda + (1+\lambda)R)$. This function is concave in R and is maximized at the corner solution $R = 1.$■

We now turn to the set of policies that are associated with a Ramsey equilibrium. The next proposition shows that there is a continuum of such policies. It is the analog of Proposition 2 in Cole and Kocherlakota (1998).

Proposition 2.5. *(Policies Associated with Ramsey Equilibrium) There exists a private sector equilibrium with $R_t = 1$ for all t if and only if,*

$$\frac{M_t}{\beta^t} \geq \kappa, \kappa > 0 \text{ for all } t \tag{2.17}$$

$$\lim_{T \to \infty} M_T \to 0 \tag{2.18}$$

Proof: Consider necessity of (2.17) and (2.18). Suppose we have an equilibrium satisfying $R_t = 1$ and (2.8)-(2.15). From (2.12) and (2.10) letting $c_t \equiv c_{1t} = c_{2t}$, we obtain

$$c_t = c = \frac{\lambda}{1 + 2\lambda} \text{ for all } t. \tag{2.19}$$

From (2.11) we obtain

$$P_t c_t = \beta^t P_0 c > 0. \tag{2.20}$$

Substituting (2.20) into (2.14), we get

$$\lim_{t \to \infty} \beta^t \frac{M_t}{P_t c_t} = \lim_{t \to \infty} \beta^t \frac{M_t}{\beta^t P_0 c} = 0,$$

so that (2.18) is satisfied. From the cash in advance constraint in (2.13),

$$\beta^t P_0 c \leq M_t, \text{ for each } t,$$

which implies (2.17).

Consider sufficiency. Suppose (2.17) and (2.18) are satisfied, and that $R_t = 1$. We must verify that the other nonzero prices and quantities can be found that satisfy (2.8) - (2.15). Let $c_{1t} = c_{2t} = c$ in (2.19) for all t. Let $P_t = \beta^t P_0$, where $P_0 > 0$ will be specified below. These two specifications guarantee (2.10), (2.12), (2.11). Condition (2.18), together with the given specification of prices and consumption guarantee (2.14). Finally, it is easily verified that by setting $0 < P_0 \leq \kappa / c$, the cash in advance constraint in (2.13) holds for each t.■

The previous proposition shows that there are many policies that implement the Ramsey outcome. However, many of these policies are fragile in the sense that they can yield worse outcomes than the Ramsey outcome. The next proposition characterizes the set of equilibria associated with mild monetary deflations in which the (stationary) growth rate of the money supply satisfies $\beta < g < 1$.

Proposition 2.6. *(Fragility of Mild Monetary Deflations) If $\beta < g < 1$, the following are equilibrium outcomes:*

(i) $R_t = 1$, $c_{1t} = c_{2t} = \lambda/[1+2\lambda]$ for all t, $P_{t+1}/P_t = \beta$, $M_t/P_t \to \infty$.

(ii) $R_t = g/\beta$, $c_{1t} = \lambda/[\lambda + (1+\lambda)g/\beta]$, $c_{2t} = (g/\beta)c_{1t}$, $P_{t+1}/P_t = g$, for all t, M_t/P_t independent of t.

(iii) $R_t = g/\beta$ for $t \leq t^*$, $R_t = 1$ for $t > t^*$ for $t^* = 0, 1, 2, ...$, $c_{1t} = \lambda/[\lambda + (1+\lambda)R_t]$, $c_{2t} = R_t c_{1t}$,

$$\frac{P_{t+1}}{P_t} = \begin{cases} g, \ t = 0, 1, ..., t^* - 1, \ (\text{for } t^* > 0), \\ \frac{(1+2\lambda)g}{\lambda + \frac{g}{\beta}(1+\lambda)}, \ t = t^*, \\ \beta, \ t = t^* + 1, t^* + 1, ... \end{cases}$$

Proof: That these are all equilibria may be confirmed by verifying that (2.8) - (2.15) are satisfied.■

This proposition does not characterize the entire set of equilibria that can occur with $\beta < g < 1$. It gives a flavor of the possibilities, however. For example, (iii) indicates that there is a countable set of equilibria (one for each possible t^*) in which the consumption and employment allocations are not constant and the interest rate switches down to unity after some date. Although there do exist equilibria in which consumption and employment are not constant, they appear to be limited. For example, it can be shown that there is no equilibrium in which the interest rate switches up from unity at some date, that is, there does not exist an equilibrium in which $R_{t^*} = 1$ and $R_{t^*+1} > 1$ for some t^*. To see this, suppose the contrary. Then, from (2.11), $\beta P_{t^*} c_{1t^*} = P_{t^*+1} c_{1t^*+1} = M_{t^*+1}$, since cash in advance constraint must be binding in period $t^* + 1$. But, $M_{t^*} \geq P_{t^*} c_{1t^*}$ implies $\beta \geq g$, contradiction. Also, we can also show that there do not exist equilibria in which the interest rate changes and it is always greater than unity, i.e., in which $R_t \neq R_{t+1} \neq 1$. So, although the set of equilibria with non-constant interest rates (and, hence, nonconstant consumption) is limited, Proposition 2.6 indicates that they do exist.

The preceding proposition indicates that mild monetary deflations are fragile. It turns out, however, that a deflationary policy of the kind advocated by Milton Friedman is robust in the sense that it always yields the Ramsey outcome.

Proposition 2.7. *(Robustness of Friedman Deflation) Suppose $g_t = \beta$. Then, all equilibria are Ramsey equilibria.*

Proof: To show that if $g_t = \beta$, $R_t = 1$, suppose the contrary. That is, $R_t > 1$ for some t. Therefore, $P_t c_{1t} = M_t$. Also, $P_{t+1} c_{1t+1} \leq M_{t+1}$. By (2.11) we find $1/(P_t c_{1t}) = \beta R_t / (P_{t+1} c_{1t+1})$, so that $(1/M_t) \geq \beta R_t (1/M_{t+1})$, or, $g_t \geq \beta R_t$, which is a contradiction.■

It is worth pointing out that since the interest rate is constant so are real allocations. There are, however, a continuum of equilibria in which the price level is different. In all of these equilibria, $P_{t+1}/P_t = \beta$. These equilibria are indexed by the initial price level, P_0, which satisfies $P_0 \leq M_0[1 + 2\lambda]/\lambda$ and $P_{t+1}/P_t = \beta$.

3. Markov Equilibrium in a Cash-Credit Good Model

In this section we analyze a version of the model presented above in which a benevolent government chooses monetary policy optimally. We consider a more general utility function of the CES form:

$$u(c_1, c_2, n) = \frac{1}{1-\sigma} \left[(\alpha c_1^\rho + (1-\alpha) c_2^\rho)^{\frac{1}{\rho}} (1-n)^\gamma \right]^{1-\sigma}.$$

Note that this utility function is a generalization of the one used in the previous section. Here, we focus on the Markov equilibrium of this model. The timing is as follows. A fraction, μ_1, of intermediate good producers in the cash good sector and a fraction, μ_2, of intermediate good producers in the credit good sector set prices at the beginning of the period. These firms are referred to as sticky price firms. We show below that all sticky price firms set the same price. Denote this price by P^e. This price, all other prices, and all nominal assets in this section are scaled by the aggregate, beginning of period money stock. Then, the monetary authority chooses the growth rate of the money supply. Finally, all other decisions are made.

The state of the economy at the time the monetary authority makes its decision is P^e.[3] The monetary authority makes its money growth decision conditional on P^e. We denote the

[3] Notice that we do not include the aggregate stock of money in the state. In our economy, all equilibria are neutral in the usual sense that if the initial money stock is doubled, there is an equilibrium in which real

gross money growth rate by G and the policy rule by $X(P^e)$. The state of the economy after the monetary authority makes its decision is $S = (P^e, G)$. With these definitions of the economy's state variables, we proceed now to discuss the decisions of firms, households and the monetary authority.

Recall that profit maximization leads intermediate good firms to set prices as a markup over the wage rate (see equation (2.8).) The price set by the $1 - \mu_1$ intermediate good producers in the cash good sector and $1 - \mu_2$ intermediate good firms in the credit good sector which set their prices after the monetary authority makes its decision ('flexible price firms') is denoted by $\hat{P}(S)$. For the μ_1 and μ_2 sticky price cash and credit good firms, respectively, and the $1-\mu_1$ and $1-\mu_2$ flexible price cash and credit good firms, respectively, the markup rule implies:

$$\begin{aligned} P^e &= \frac{W(P^e, X(P^e))}{\lambda}, \\ \hat{P}(S) &= \frac{W(S)}{\lambda},\ 0 < \lambda < 1, \end{aligned} \tag{3.1}$$

where $W(S)$ denotes the nominal wage rate. In this model of monopolistic competition, output and employment are demand-determined. That is, output and employment are given by (2.8). Let $P_i(S)$ denote the price of the cash and credit good for $i = 1, 2$, respectively. Let $y_{ij}(S)$, $i, j = 1, 2$, denote the output of the intermediate goods firms, where the first subscript denotes whether the good is a cash good ($i = 1$) or a credit good ($i = 2$), and the second subscript indicates whether it is produced by a sticky price ($j = 1$) or a flexible price ($j = 2$) producer.

In terms of the household's problem, it is convenient to write the constraints in recursive form. The analog of (2.2) is

$$M + B \leq A, \tag{3.2}$$

allocations and the interest rate are unaffected and all nominal variables are doubled. This consideration leads us to focus on equilibria which are invariant with respect to the initial money stock. We are certainly mindful of the possibility that there can be equilibria which depend on the money stock. For example, if there are multiple equilibria in our sense, it is possible to construct 'trigger strategy-type' equilibria which are functions of the initial money stock. In our analysis we exclude such equilibria and we normalize the aggregate stock of money at the beginning of each period to unity.

where, recall, A denotes beginning of period nominal assets, M denotes the household's holdings of cash, and B denotes the household's holdings of interest-bearing bonds. Here, nominal assets, money and bonds are all scaled by the aggregate stock of money. We impose a no-Ponzi constraint of the form $B \leq \bar{B}$, where $\bar{B}$ is a large, finite, upper bound. The household's cash in advance constraint is

$$M - P_1(S)c_1 \geq 0, \tag{3.3}$$

where c_1 denotes the quantity of the cash good. Nominal assets evolve over time as follows:

$$\begin{aligned} 0 \leq\ & W(S)n + (1 - R(S))M - P_1(S)c_1 - P_2(S)c_2 \\ & + R(S)A + (G - 1) + D(S) - GA', \end{aligned} \tag{3.4}$$

where c_2 denotes the quantities of credit goods purchased. In (3.4), $R(S)$ denotes the gross nominal rate of return on bonds, and $D(S)$ denotes profits after lump sum taxes. Finally, B has been substituted out in the asset equation using (3.2). Notice that A' is multiplied by G. This modification is necessary because of the way we have scaled the stock of nominal assets.

Consider the household's asset, goods and labor market decisions. Given that the household expects the monetary authority to choose policy according to X in the future the household solves the following problem:

$$v(A, S) = \max_{n, M, A', c_i; i=1,2} u(c_1, c_2, n) + \beta v(A', P^e, X(P^e)) \tag{3.5}$$

subject to (3.2), (3.3), (3.4), and non-negativity on allocations. In (3.5), v is the household's value function. The solution to (3.5) yields decision rules of the form $n(A, S)$, $M(A, S)$, $A'(A, S)$ and $c_i(A, S)$, $i = 1, 2$. We refer to these decision rules, together with the production decisions of firms, $y_{ij}(S)$, $i, j = 1, 2$, as *private sector allocation rules.* We refer to the collection of prices, P^e, $\hat{P}(S)$, $W(S)$, $R(S)$, $(P_i(S), i = 1, 2)$, as *pricing rules.*

3.1. Monetary Authority

The monetary authority chooses the current money growth rate, G, to solve the problem:

$$\max_{G} v(1, S), \tag{3.6}$$

where, recall, $S = (P^e, G)$. Let $X(P^e)$ denote the solution to this problem. We refer to this solution as the *monetary policy rule.*

3.2. Markov Equilibrium

We now define a Markov equilibrium. This equilibrium requires that households and firms optimize and markets clear.

Definition 3.1. *A Markov equilibrium is a set of private sector allocation rules, pricing rules, a monetary policy rule and a value function for households such that*
(i) the value function, v, and the private sector rules solve (3.5)
(ii) intermediate good firms optimize, i.e., (3.1) is satisfied, final good prices satisfy

$$P_i(S) = \left[\mu_i \left(P^e\right)^{\frac{\lambda}{\lambda-1}} + (1-\mu_i)\widehat{P}(S)^{\frac{\lambda}{\lambda-1}}\right]^{\frac{\lambda-1}{\lambda}}, \text{ for } i = 1, 2,$$

and the output of intermediate good firms, $y_{ij}(S)$, is given by the analog of (2.7),
(iii) asset markets clear, i.e., $A'(1, S) = 1$ and $M(1, S) = 1$,
(iv) the labor market clears, i.e.,

$$n(1, S) = \mu_1 y_{11}(S) + (1-\mu_1)y_{12}(S) + \mu_2 y_{21}(S) + (1-\mu_2)y_{22}(S),$$

(v) the monetary authority optimizes, i.e., $X(P^e)$ solves (3.6).

Notice that our notion of Markov equilibrium has built into it the idea of sequential optimality captured in game-theoretic models by subgame perfection. In particular, we require that for any deviation by the monetary authority from $X(P^e)$, the resulting allocations be the ones that would actually occur. That is, the ones that would be in the best interests of households and firms and would clear markets.

We now define a Markov equilibrium outcome:

Definition 3.2. *A Markov equilibrium outcome is a set of numbers, n, c_1, c_2, y_{ij}, $i, j = 1, 2$, P^e, W, R, P_1, P_2, g, satisfying $n = n(1, P^e, g)$, $c_1 = c_1(1, P^e, g)$, ..., and $g = X(P^e)$.*

3.3. Analysis of Markov Equilibrium

In this section, we characterize the Markov equilibrium. In particular, we provide sufficient conditions for the Ramsey outcomes to be Markov equilibrium outcomes. We also provide sufficient conditions for the Markov equilibrium to be unique. Combining these conditions, we obtain sufficient conditions for the unique Markov equilibrium to yield the Ramsey outcomes.

In developing these results, we find it convenient to recast the monetary authority's problem as choosing $\hat{P}$, rather than G. First, we analyze the private sector allocation rules and pricing functions. Then, we analyze the monetary authority's problem.

We use the necessary and sufficient conditions of private sector maximization and market clearing to generate the private sector allocation rules and pricing functions. The conditions are given by:

$$-\frac{u_3}{u_2} = \lambda \frac{\widehat{P}}{P_2}, \tag{3.7}$$

$$\left(\frac{1}{P_1} - c_1\right)(R - 1) = 0, \tag{3.8}$$

$$R = \frac{u_1}{u_2}\frac{P_2}{P_1}, \tag{3.9}$$

$$n_i = c_i \left[\mu_i \left(\frac{P_i}{P^e}\right)^{\frac{1}{1-\lambda}} + (1 - \mu_i)\left(\frac{P_i}{\widehat{P}}\right)^{\frac{1}{1-\lambda}}\right], \text{ for } i = 1, 2, \tag{3.10}$$

$$n = n_1 + n_2, \tag{3.11}$$

$$P_i = \left[\mu_i \left(P^e\right)^{\frac{\lambda}{\lambda-1}} + (1 - \mu_i)\widehat{P}^{\frac{\lambda}{\lambda-1}}\right]^{\frac{\lambda-1}{\lambda}}, \text{ for } i = 1, 2. \tag{3.12}$$

$$\frac{Gu_1}{P_1} = \beta R v_1(1, P^e, X(P^e)). \tag{3.13}$$

Notice that the growth rate of the money supply, G, appears only in (3.13). Equations (3.7)-(3.12) constitute 8 equations in the 8 unknowns, c_1, c_2, n_1, n_2, n, P_1, P_2, R. Given values for P^e and $\hat{P}$, these equations can be solved to yield functions of the form:

$$c_1(P^e,\hat{P}),\ c_2(P^e,\hat{P}),...,\ R(P^e,\hat{P}). \tag{3.14}$$

Replacing $\hat{P}$ in (3.14) by a pricing function, $\hat{P}(P^e,G)$, we obtain the allocation rules and pricing functions in a Markov equilibrium.

The pricing function, $\hat{P}(P^e,G)$, is obtained from equation (3.13). This equation can be thought of as yielding a function, $G(P^e,\hat{P})$. The pricing function, $\hat{P}(P^e,G)$, is obtained by inverting $G(P^e,\hat{P})$. It is possible that the inverse of $G(P^e,\hat{P})$ is a correspondence. In this case, $\hat{P}(P^e,G)$ is a selection from the correspondence. Any such selection implies a range of equilibrium prices, $\hat{P}$. Denote this range by D.

Given the function, $\hat{P}(P^e,G)$, the monetary authority's problem can be thought of in either of two equivalent ways: either it chooses G or it chooses $\hat{P}$. The government's decision problem is simplified in our setting because its choice of $\hat{P}$ has no impact on future allocations. As a result, the government faces a static problem.

The allocation functions in (3.14) can be substituted into the utility function to obtain:

$$U\left(P^e,\hat{P}\right)=u\left[c_1\left(P^e,\hat{P}\right),c_2\left(P^e,\hat{P}\right),n\left(P^e,\hat{P}\right)\right]. \tag{3.15}$$

Then, define

$$P\left(P^e\right)=\arg\max_{\hat{P}\in D} U\left(P^e,\hat{P}\right).$$

The function, $P(P^e)$, is the monetary authority's best response, given P^e. Equilibrium requires that $P(P^e)=P^e$. This procedure determines the expected price P^e, the actual price $\hat{P}$ and the eight allocations and other prices described above. Given these values, we can determine the equilibrium growth rate of the money supply by evaluating $G(P^e,P^e)$.

In what follows, we assume that the first order conditions to the monetary authority's problem characterize a maximum. In quantitative exercises we have done using these models, we have found that the first order conditions in the neighborhood of a Ramsey outcome do in fact characterize the global maximum of the monetary authority's problem.

Next we show that for a class of economies the Ramsey outcomes are Markov equilibrium outcomes. Recall that a Ramsey equilibrium is a private sector equilibrium with $R = 1$. In the appendix, we prove the following result:

Proposition 3.3. *(Markov is Ramsey) Suppose*

$$(1-\rho)(1-\mu_1) \geq \mu_2 \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}}.$$

then, there exists a Markov equilibrium with $R = 1$.

The intuition for this proposition is as follows. A benefit of expansionary monetary policy is that it leads to an increase in demand for goods whose prices are fixed. This increase in demand tends to raise employment. Other things the same, welfare rises because employment is inefficiently low. A principal cost of expansionary monetary policy is that it tends to reduce employment in the cash good sector. The reason for this reduction in employment is that nominal consumption of the cash good is predetermined, while its price rises due to the increase in flexible intermediate good prices. It is possible that the reduction in employment in the cash good sector is so large that overall employment and welfare fall. Indeed, it can be shown that if the sufficient condition of the proposition is met, employment falls with an increase in the money growth rate in the neighborhood of the Ramsey equilibrium. The monetary authority has an incentive to contract the money supply. This incentive disappears only if the nominal interest rate is zero.

In what follows, we assume that $\hat{P}(P^e, G)$ is a continuous function of G. This restriction is not innocuous. We have constructed examples where, for a given value of G, there is more than one value of private sector allocations and prices which satisfy the conditions for private sector optimization and market clearing.[4] Thus, it is possible to construct private sector allocation rules and pricing functions which are discontinuous functions of G. The

[4] Specifically, we found numerical examples in which the function, $G(P^e, \hat{P})$, displayed an inverted 'U' shape when graphed for fixed P^e with G on the vertical axis and $\hat{P}$ on the horizontal. In these examples, each fixed $\hat{P}$ implied a unique G. However, there are intervals of values of G where a fixed G maps into two distinct $\hat{P}$'s.

assumption of continuity plays an important role in the proof of uniqueness below. In the next proposition we provide sufficient conditions for uniqueness of the Markov equilibrium:

Proposition 3.4. *(Uniqueness of Markov Equilibrium) Suppose:*
(i) $\rho = 0$, $\sigma = 1$
(ii) $1 - \mu_1 > \mu_2 \frac{1-\alpha}{\alpha}$
(iii) $(\lambda + \frac{\gamma\alpha}{1-\alpha})(1 - \mu_1) \geq \frac{(1-\lambda)\gamma\mu_2}{\alpha}$
then in the class of Markov equilibria in which $\hat{P}(P^e, G)$ *is a continuous function of* G*:*
(a) there exists an equilibrium with $R = 1$*and*
(b) there is no equilibrium with outcome $R > 1$.

We conjecture that if we allow a discontinuous pricing function, $\hat{P}(P^e, G)$, then there exist Markov equilibria with $R > 1$, even under the conditions of this proposition.

4. Markov Equilibrium in A Limited Participation Model

In this section we analyze the set of Markov equilibria in a limited participation model. We briefly describe the model. The sequence of events is as follows. Households start each period with nominal assets and they must choose how much to deposit in a financial intermediary. The monetary authority then chooses its transfer to the financial intermediary. The financial intermediary makes loans to firms, who must borrow the wage bill before they produce. Households make their consumption and labor supply decision and firms make production decisions. Money is not neutral because households cannot change their deposit decision after the monetary authority chooses its transfer. Let Q denote the aggregate deposits made by households and let G denote the growth rate of the money supply chosen by the monetary authority. In this section, as in the previous section, all prices and quantities of nominal assets are scaled by the aggregate stock of money. Let $S = (Q, G)$ denote the state of the economy after these decisions are made.

The household's utility function is:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \; u(c, n) = \log(c) + \gamma \log(1 - n),$$

where, c_t and n_t denote date t consumption labor, respectively. We write the household's problem recursively. We start with the problem solved by the household after the monetary authority has made its transfer. Let A denote the household's beginning-of-period nominal assets. Let q denote its deposits. Both variables have been scaled by the aggregate, beginning-of-period stock of money. The consumption, employment and asset accumulation decisions solve

$$w(A, q, S) = \max_{c,n,M'} u(c, n) + \beta v(A'),$$

subject to

$$P(S)c \leq W(S)n + A - q,$$

and

$$GA' = R(S)(q + (G - 1)) + D(S) + W(S)n + A - q - P(S)c.$$

Here, v is the value function at the beginning of the next period, before the household makes next period's deposit decision. Also, $R(S)$ is the gross interest rate, $P(S)$ is the price of the consumption good, $W(S)$ is the wage rate, $D(S)$ is profits from firms. The choice of q solves the following dynamic programming program

$$v(A) = \max_{q} w(A, q, S^e),$$

where S^e is the state if the monetary authority does not deviate, i.e., $S^e = (Q, X(Q))$, where $X(Q)$ is the monetary authority's policy function.

The production sector is exactly as in the cash-credit good model, with one exception. To pay for the labor that they hire during the period, intermediate good producing firms must borrow in advance from the financial intermediary at gross interest rate $R(S)$. Thus, the marginal dollar cost of hiring a worker is $R(S)W(S)$, so that, by the type of reasoning in the cash credit good model, we find $R(S)W(S)/P(S) = \lambda$.

The financial intermediary behaves competitively. It receives Q from households, and $G - 1$ on households' behalf from the central bank. When $R(S) > 1$, it lends all these funds in the loan market. When $R(S) = 1$, it supplies whatever is demanded, up to the funds it has available. We shall say that when $R(S) = 1$ and demand is less than available funds, then there is a 'liquidity trap'. At the end of the period the financial intermediary returns its earnings, $R(S)(Q+G-1)$, to the households. Finally, if $R(S)<1$, the financial intermediary lends no funds, and returns $Q+G-1$ to households. Loan demand by firms is given by $W(S)n(S)$. Therefore loan market clearing requires:

$$W(S)n(S) \leq Q+G-1,$$

with equality if $R(S)>1$.

The monetary authority's policy function, $X(Q)$, solves

$$X(Q) \in \arg\max_{G} w(1,Q,Q,G).$$

A recursive private sector equilibrium and a Markov equilibrium are defined analogously to those in the previous section.

It is useful to begin with an analysis of outcomes under commitment. It is easy to show, as in section 2 above, that the Ramsey equilibrium has $R=1$ and can be supported by a policy that sets the growth rate of the money supply equal to β. Let c^*, n^*, W^*, R^*, P^*, Q^* denote this Ramsey equilibrium. These variables solve the following system of equations:

$$\begin{aligned}
\frac{\gamma c^*}{1-n^*} &= \frac{W^*}{P^*},\ \frac{W^*}{P^*}=\frac{\lambda}{R^*},\ R^*=1, \\
W^*n^* &= Q^*+\beta-1,\ P^*c^*=W^*n^*+1-Q^* \qquad (4.1)\\
c^* &= n^*.
\end{aligned}$$

It is straightforward to verify that the usual non-negativity constraints are satisfied. Notice that the first equation is the household's first order condition for labor, the second results from firm optimization, the third corresponds to the intertemporal Euler equation, the fourth corresponds to money market clearing, the fifth is the household's cash in advance constraint and the last equation corresponds to goods market clearing.

Next, we analyze the Markov equilibria of our model. The necessary and sufficient conditions for allocations and pricing functions to constitute a recursive private sector equilibrium are:

$$
\begin{aligned}
\frac{\gamma n(S)}{1-n(S)} &= \frac{W(S)}{P(S)}, &(4.2)\\
\frac{W(S)}{P(S)} &= \frac{\lambda}{R(S)}, &(4.3)\\
W(S)n(S) &\leq \begin{cases} Q+G-1, & \text{if } R(S)\geq 1 \\ 0 & \text{if } R(S)<1 \end{cases}, &(4.4)\\
P(S)n(S)-W(S)n(S) &\leq 1-Q, &(4.5)
\end{aligned}
$$

where (4.4) holds with equality if $R(S) > 1$. As noted above, if $R(S) < 1$, the supply of funds in the loan market is zero. Also, (4.5) holds with equality if $R(Q, X(Q)) > 1$ and $S = (Q, X(Q))$. That is, if along the Markov equilibrium path the net interest rate is strictly positive, the household's cash in advance constraint is satisfied as a strict equality. In a deviation from the Markov equilibrium path, the cash in advance constraint must hold as a weak inequality, regardless of the realized interest rate.

We now establish the following Proposition:

Proposition 4.1. *(All Markov Equilibria are Ramsey) In any Markov equilibrium,* $R(Q, X(Q)) = 1$*, and the allocations and prices on the equilibrium path are the Ramsey outcomes given in (4.1).*

Proof. We prove this proposition in two parts. First, we construct a Markov equilibrium in which $R(Q, X(Q)) = 1$. Then, we show that there is no equilibrium with $R(Q, X(Q)) > 1$. Our constructed Markov equilibrium is as follows. Let $Q = Q^*$, where Q^* solves (4.1). On the equilibrium path, the monetary authority's decision rule is $X(Q^*) = \beta$. The allocation and pricing functions, $c(S)$, $n(S)$, $W(S)$, $P(S)$, $R(S)$, in a recursive private sector equilibrium are defined as follows. For all S, $c(S) = n(S)$. For $G \leq \beta$, $R(S) = 1$, $n(S) = n^*$, $W(S)$ is obtained from (4.4) with equality, $P(S) = W(S)/\lambda$. It is then easy to show that (4.5) holds with inequality. For $G > \beta$ the functions are defined as follows $n(S) = n^*$, $W(S) = w^*$, $R(S) = R^* = 1$, $P(S) = P^*$, where the variables with the asterisk are those associated with the Ramsey equilibrium, (4.1). Notice that these allocation and pricing rules satisfy (4.2), (4.3) and (4.5) with equality and (4.4) with inequality.

Next we show by contradiction that there does not exist a Markov equilibrium with $R(Q, X(Q)) > 1$. Suppose, to the contrary that there did exist such an equilibrium. Notice that it is always possible to construct a private sector equilibrium for arbitrary $G \geq \beta$ by simply setting (4.2)-(4.5) to equality. Therefore, the domain of deviation that needs to be considered includes all $G > X(Q)$. Consider such a deviation. We will show that, in the private sector equilibrium associated with this deviation, $R(Q, G) < R(Q, X(Q))$. This argument is also by contradiction. Thus, suppose $R(Q, G) \geq R(Q, X(Q))$. Since $R(Q, X(Q)) > 1$, (4.4) must hold as an equality at the deviation. Substituting for $P(S)$ from (4.3) and $W(S)n(S)$ from (4.4), the left side of (4.5) becomes

$$(R(S)/\lambda - 1)\,(Q + G - 1)\,,$$

which is larger than $(R(S)/\lambda - 1)\,(Q + X(Q) - 1)$. On the equilibrium path, (4.5) must hold as an equality. Therefore, at the deviation (4.5) must be violated. We have established that, in any deviation of the form $G > X(Q)$, $R(Q, G) < R(Q, X(Q))$. But, from (4.2) and (4.3), this raises employment towards the efficient level, contradicting monetary authority optimization. We have established the desired contradiction. ■

Notice that in the Markov equilibrium we have constructed there is a 'liquidity trap'. If the monetary authority deviates and chooses a growth rate for the money supply greater than β, the resulting transfers of money of money are simply hoarded by the financial intermediary and not lent out to firms. All allocations and prices are unaffected by such a deviation.

5. Conclusion

In this paper we worked with an environment that, with one exception, is similar in spirit to the one analyzed in the Kydland-Prescott and Barro-Gordon literature. The exception is that we are explicit about the mechanisms that cause unanticipated monetary injections to generate benefits and distortions. We found that, for two standard models, there is no inflation bias at all.

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A. Necessary and Sufficient Conditions for Household Optimization in the Cash-Credit Good Model

This appendix develops necessary and sufficient conditions for optimality of the household problem in the cash-credit good model of section 2. It is included here for completeness. Many of the results here can be found in the literature. See, for example, Woodford (1994).

In what follows, we assume:

$$P_{1t},\ P_{2t},\ W_t > 0,\ R_t \geq 1,\ lim_{t\to\infty} \sum_{j=0}^{t} q_{j+1}\left[W_j + T_j + D_j\right] \text{ finite.} \tag{A.1}$$

If these conditions did not hold, there could be no equilibrium. We begin by proving a proposition which allows us to rewrite the household's budget set in a more convenient form. We show that

Proposition A.1. *Suppose (2.2), (2.4) and (A.1) are satisfied. The constraint given in (2.5) is equivalent to*

$$\lim_{T\to\infty} q_T A_T \geq 0. \tag{A.2}$$

Proof It is useful to introduce some new notation. Let I_t and S_t be defined by

$$I_t \equiv W_t + T_t + D_t. \tag{A.3}$$

and

$$S_t \equiv (R_t - 1)M_t + P_{1t}c_{1t} + P_{2t}c_{2t} + W_t(1 - n_t),$$

respectively. It is straightforward to show that household nominal assets satisfy:

$$A_{t+1} = I_t + R_t A_t - S_t. \tag{A.4}$$

We establish that (A.2) implies (2.5). Recursively solving for assets using (A.4) and

(2.2) from t to T yields:

$$q_T A_T \leq \sum_{j=0}^{T-t-1} q_{t+j+1} I_{t+j} + q_t A_t - \sum_{j=0}^{T-t-1} q_{t+j+1} S_{t+j}. \tag{A.5}$$

Taking into account $q_{t+j+1}S_{t+j} \geq 0$ and rewriting this expression, we obtain

$$q_t A_t \geq q_T A_T - \sum_{j=0}^{T-t-1} q_{t+j+1} I_{t+j}.$$

Fixing t, taking the limit, $T \rightarrow \infty$, and using (A.2) yields (2.5).

We now show that (2.5) implies (A.2). Note first that the limit in (A.1) being finite implies

$$\lim_{t\rightarrow\infty} \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j} = 0.$$

Using this result and (2.5), (A.2) follows trivially.■

Following is the main result of this appendix:

Proposition A.2. *A sequence, $\{c_{1t},\ c_{2t},\ n_t,\ M_t, B_t\}$, solves the household problem if and only if the following conditions are satisfied. The Euler equations are:*

$$\frac{u_{1t}}{P_{1t}} = R_t \frac{u_{2t}}{P_{2t}}, \tag{A.6}$$

$$\frac{-u_{3t}}{u_{2t}} = \frac{W_t}{P_{2t}}, \tag{A.7}$$

$$\frac{u_{1t}}{P_{1t}} = \beta R_t \frac{u_{1t+1}}{P_{1t+1}}, \tag{A.8}$$

$$(R_t - 1)(P_{1t}c_{1t} - M_t) = 0. \tag{A.9}$$

The transversality condition is (A.2) with equality:

$$\lim_{t\rightarrow\infty} q_t A_t = 0. \tag{A.10}$$

Proof We begin by showing that if a sequence $\{c_{1t},\ c_{2t},\ n_t,\ M_t, B_t\}$ satisfies (A.6)-(A.10), then that sequence solves the household's problem. That is, we show that:

$$D = \lim_{T\to\infty}\left[\sum_{t=0}^{T}\beta^t u(c_{1t}, c_{2t}, n_t) - \sum_{t=0}^{T}\beta^t u(c'_{1t}, c'_{2t}, n'_t)\right] \geq 0.$$

where $\{c'_{1t}, c'_{2t}, n'_t, M'_t, B'_t\}_{t=0}^{\infty}$ is any other feasible plan. Note first that the Euler equations imply:

$$\begin{aligned}
\beta^t u_{1,t} &= q_t P_{1t}\frac{u_{1,0}}{P_{1,0}},\\
\beta^t u_{2,t} &= q_{t+1} P_{2t}\frac{u_{1,0}}{P_{1,0}},\\
\beta^t u_{3,t} &= -q_{t+1} W_t\frac{u_{1,0}}{P_{1,0}},
\end{aligned}$$

where $u_{i,t}$ is the derivative of u with respect to its i^{th} argument. By concavity and the fact that the candidate optimal plan satisfies (A.6) and (A.7) we can write:

$$\begin{aligned}
D &\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T}\left[q_t P_{1t}(c_{1t}-c'_{1t}) + q_{t+1}P_{2t}(c_{2t}-c'_{2t}) - q_{t+1}W_t(n_t - n'_t)\right]\\
&= \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T} q_t\left[\frac{S_t}{R_t} + \frac{1-R_t}{R_t}(M_t - P_{1t}c_{1t}) - \frac{S'_t}{R_t} - \frac{1-R_t}{R_t}(M'_t - P_{1t}c'_{1t})\right]\\
&\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T}\left[q_{t+1}S_t - q_{t+1}S'_t\right], \qquad \text{(A.11)}
\end{aligned}$$

where the equality is obtained by using the definition of S_t and the second inequality is obtained by using $R_t \geq 1$, and $(1-R_t)(M'_t - P_{1t}c'_{1t}) \leq 0$ (see (2.3)). Iterating on (A.4) for the two plans we can rewrite (A.11) as

$$\begin{aligned}
D &\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\left[\sum_{t=0}^{T} q_{t+1}S_t + q_{T+1}A'_{T+1} - \sum_{t=0}^{T} q_{t+1}I_t - A_0\right]\\
&\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\left[\sum_{t=0}^{T} q_{t+1}S_t - \sum_{t=0}^{T} q_{t+1}I_t - A_0\right]\\
&= \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}} q_{T+1}A_{T+1} \geq 0,
\end{aligned}$$

by (A.10).

Now we establish that if $\{c_{1t},\ c_{2t},\ n_t,\ M_t, B_t\}$ is optimal, then (A.6)-(A.10) is true. That (A.6)-(A.9) are necessary is obvious. It remains to show that (A.10) is necessary. Suppose (A.10) is not true. We show this contradicts the hypothesis of optimality.

We need only consider the case where $\lim_{T\to\infty} q_T A_T$ is strictly positive. The strictly negative case is ruled out by the preceding proposition. So, suppose

$$\lim_{T\to\infty} q_T A_T = \Delta > 0.$$

We construct a deviation from the optimal sequence which is consistent with the budget constraint and results in an increase in utility. Fix some particular date, τ. We replace $c_{1\tau}$ by $c_{1\tau} + \varepsilon/P_{1\tau}$, where $0 < \varepsilon \le \Delta/q_\tau$. Consumption at all other dates and $c_{2\tau}$ are left unchanged, as well as employment at all dates. We finance this increase in consumption by replacing M_τ^d with $M_\tau^d + \varepsilon$ and B_τ with $B_\tau - \varepsilon$. Money holdings at all other dates are left unchanged. Debt and wealth after t, B_t, A_t, $t > \tau$ are different in the perturbed allocations. We denote the variables in the perturbed plan with a prime. From (A.4)

$$\begin{aligned}
A'_{\tau+1} - A_{\tau+1} &= -R_\tau \varepsilon = -\frac{q_\tau}{q_{\tau+1}}\varepsilon \\
A'_{\tau+j} - A_{\tau+j} &= -R_{\tau+j-1}\cdots R_\tau \varepsilon = -\frac{q_\tau}{q_{\tau+j}}\varepsilon.
\end{aligned}$$

Multiplying this last expression by $q_{\tau+j}$ and setting $T = \tau + j$:

$$q_T\left(A'_T - A_T\right) = -q_\tau \varepsilon.$$

Taking the limit, as $T \to \infty$

$$\lim_{T\to\infty} q_T A'_T = \Delta - q_\tau \varepsilon \ge 0.$$

We conclude that the perturbed plan satisfies (A.2). But, utility is clearly higher in the perturbed plan. We have a contradiction.■

B. Properties of Markov Equilibrium

In this appendix we prove Proposition 3.3. We establish the result by constructing a Markov equilibrium which supports the Ramsey outcomes. Specifically, we construct a P^e, a set of private sector allocations rules, a set of pricing functions and a monetary policy rule, all of which satisfy the conditions for a Markov equilibrium. In section 3.3 it is shown that private sector allocation rules and pricing functions can equivalently be expressed as functions of the growth rate of the money supply, G, or of $\hat{P}$, the price of the flexibly priced intermediate goods. Since these representations are equivalent and it is convenient to work with $\hat{P}$, we do so here.

The construction of the Markov equilibrium is as follows. Let c_1^*, c_2^*, W^*, R^*, P^*, P_1^*, P_2^* solve (3.7)-(3.12) with $R = 1$ and with the cash in advance constraint holding with equality. That is, they are given by

$$c_1^* = \left[1 + \left(1 + \frac{\gamma}{\lambda}\right)\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + \frac{\gamma}{\lambda}\right]^{-1}, \tag{B.1}$$

$$c_2^* = c_1^* \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}}, \tag{B.2}$$

$R^* = 1$, $P_1^* = P_2^* = P^* = 1/c_1^*$, $W^* = \lambda P^*$. Let $P^e = P^*$. For $\hat{P} > P^e$, let the allocation rules and pricing functions solve (3.7)-(3.12) with (3.8) replaced by $c_1 = 1/P_1$. For $\hat{P} < P^e$ let allocation rules and pricing functions solve (3.7)-(3.12) with $R = 1$. By construction, P^e and these allocation and pricing functions satisfy private sector optimality and market clearing. We need only check optimality of the monetary authority.

Denote the derivative of U in (3.15) with respect to $\hat{P}$ by L, where:

$$L = u_1 c_1' + u_2 c_2' + u_n n', \tag{B.3}$$

where u_1, u_2, u_n denote derivatives of the utility function with respect to the cash good, the credit good and employment, respectively. In addition, c_1', c_2', and n' denote derivatives of the allocation rules defined in (3.14) with respect to $\hat{P}$. These derivatives and all others in this appendix are evaluated at $\widehat{P} = P^e$. Let L^+ be the right derivative and L^- be the left derivative associated with L. We show that when our sufficient conditions are met, $L^+ \leq 0$ and $L^- \geq 0$.

Note that:

$$P_i' = (1 - \mu_i),\ \ i = 1, 2. \tag{B.4}$$

Using (B.4) and grouping terms in (B.3), we obtain

$$\begin{aligned} L &= u_2 \left[\frac{u_1}{u_2} c_1' + c_2' + \frac{u_3}{u_2} \left(c_1' + c_2' \right) \right] \\ &= (1 - \lambda)\, u_2 c_1 \left[\frac{c_1'}{c_1} + \frac{c_2}{c_1} \frac{c_2'}{c_2} \right], \end{aligned} \tag{B.5}$$

since $u_1/u_2 = R = 1$ and $-u_3/u_2 = \lambda$ when $\hat{P} = P^e$.

B.1. Right Derivative

We now establish that when our sufficient conditions are met, $L^+ \leq 0$. In order to evaluate the derivatives in (B.5) we require expressions for c_1'/c_1 and c_2'/c_2. The first of these is obtained by differentiating the binding cash in advance constraint:

$$\frac{c_1'}{c_1} = -\frac{1 - \mu_1}{P^e}. \tag{B.6}$$

To obtain c_2'/c_2, note that the static labor Euler equation is given by:

$$\frac{\gamma c_2}{1 - n} \left(\frac{c}{c_2} \right)^{\rho} = \lambda \frac{\widehat{P}}{P_2},$$

or, substituting for c and rearranging,

$$\frac{\gamma}{1 - \alpha} \left[\alpha \left(\frac{c_1}{c_2} \right)^{\rho} + 1 - \alpha \right] = \lambda \frac{\widehat{P}}{P_2} \frac{1 - n_1 - n_2}{c_2}. \tag{B.7}$$

Differentiating both sides of this expression with respect to $\hat{P}$ and taking into account $d(\hat{P}/P_2)/d\hat{P} = \mu_2/P_2$ when $\hat{P} = P^e$, we obtain, after some manipulations

$$\left[\lambda \frac{1-c_1}{c_1} - \gamma\rho\right] \frac{c_2'}{c_2} = \lambda \frac{\mu_2}{P_2} \frac{1-c_1-c_2}{c_1} - [\lambda + \gamma\rho] \frac{c_1'}{c_1}. \tag{B.8}$$

Substitute for c_1'/c_1 and c_2'/c_2 from (B.6) and (B.8), respectively, into (B.5), we obtain

$$\begin{aligned} L^+ &= \frac{(1-\lambda)u_2 c_1}{\lambda \frac{1-c_1}{c_1} - \gamma\rho} \{ \left[\frac{c_2}{c_1} (\lambda + \gamma\rho) - \left(\lambda \frac{1-c_1}{c_1} - \gamma\rho \right) \right] \frac{1-\mu_1}{P^e} \\ & + \frac{c_2}{c_1} \left(\lambda \frac{\mu_2}{P_2} \frac{1-c_1-c_2}{c_1} \right) \}. \end{aligned} \tag{B.9}$$

The denominator of (B.9) is positive. To see this, use (B.1) to show

$$\lambda \frac{1-c_1}{c_1} - \gamma\rho = \lambda \left(1 + \frac{\gamma}{\lambda}\right) \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + \gamma (1-\rho) > 0, \tag{B.10}$$

since $\rho \leq 1$. We can rewrite (B.9) as

$$L^+ = \frac{u_2 c_2 (1-\lambda)}{P_2 \left[\lambda \frac{1-c_1}{c_1} - \gamma\rho \frac{g}{\beta}\right]} \left\{ - \left[\lambda \frac{1-c_1-c_2}{c_2} - \gamma\rho \frac{c_1+c_2}{c_2} \right] (1-\mu_1) + \lambda \mu_2 \frac{1-c_1-c_2}{c_1} \right\}, \tag{B.11}$$

Substituting for c_1 from (B.1) and c_2/c_1 from (B.2), we obtain

$$\frac{1-c_1-c_2}{c_1} = \frac{\gamma}{\lambda} \left[\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + 1 \right]. \tag{B.12}$$

Also,

$$\frac{1-c_1-c_2}{c_2} = \frac{\gamma}{\lambda} \left[1 + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}} \right],$$

and

$$\frac{c_1+c_2}{c_2} = \left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}} + 1.$$

Substituting these results into (B.11) we obtain

$$
\begin{aligned}
L^{+} &= \frac{u_2 c_2 (1-\lambda)}{P_2\left[\lambda \frac{1-c_1}{c_1} - \gamma\rho \frac{g}{\beta}\right]} \{ - \left[\lambda \left(\frac{\gamma}{\lambda}\left[1 + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}}\right]\right) - \gamma\rho \left(\left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}} + 1\right)\right](1-\mu_1) \\
&\quad + \lambda\mu_2 \left(\frac{\gamma}{\lambda}\left[\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + 1\right]\right) \}.
\end{aligned}
$$

Simplifying

$$
L^{+} = \frac{\gamma u_2 c_2 (1-\lambda)\left[1 + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}}\right]}{P_2\left[\lambda \frac{1-c_1}{c_1} - \gamma\rho \frac{g}{\beta}\right]} \left\{ -(1-\rho)(1-\mu_1) + \mu_2 \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} \right\}.
$$

Since the term in front of the braces is positive, it follows that $L^{+} \leq 0$ if and only if

$$
(1-\rho)(1-\mu_1) \geq \mu_2 \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}}. \tag{B.13}
$$

B.2. Left Derivative

Next, we establish that under the sufficient conditions of the proposition, $L^{-} \geq 0$. The expression for c_2'/c_2 is still given by (B.8). To obtain c_1'/c_1, we differentiate (3.9) with $R = 1$ and use (B.4) to obtain:

$$
\frac{\alpha}{1-\alpha}(1-\rho)\left(\frac{c_2}{c_1}\right)^{1-\rho}\left(\frac{c_2'}{c_2} - \frac{c_1'}{c_1}\right) = \frac{\mu_2 - \mu_1}{P} \tag{B.14}
$$

or, since $(\alpha/(1-\alpha))(c_2/c_1)^{1-\rho} = 1$:

$$
\frac{c_2'}{c_2} - \frac{c_1'}{c_1} = \frac{\mu_2 - \mu_1}{(1-\rho)P}
$$

Substituting for c_1'/c_1 from here into (B.8) and collecting terms, we obtain, after simplifying

$$\lambda \frac{1}{c_1} \frac{c_2'}{c_2} = \lambda \frac{\mu_2}{P_2} \frac{1 - c_1 - c_2}{c_1} + (\lambda + \gamma\rho) \frac{\mu_2 - \mu_1}{(1-\rho) P}.$$

Then, using (B.12), we obtain

$$\lambda \frac{1}{c_1} \frac{c_2'}{c_2} = \frac{\mu_2}{P_2} \gamma \left[\left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1-\rho}} + 1 \right] + (\lambda + \gamma\rho) \frac{\mu_2 - \mu_1}{(1-\rho) P}.$$

Now substitute out for c_1'/c_1 and c_2'/c_2 into (B.5) to obtain, after simplifying

$$L^{-} = (1-\lambda) \frac{u_2 c_1 \gamma}{P^e (\lambda + \gamma)} \left\{ \mu_2 \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1-\rho}} + \mu_1 \right\} > 0.$$

C. Uniqueness Result

We prove Proposition 3.4 by contradiction. Suppose that there exists a Markov equilibrium outcome with $R > 1$. The contradiction is achieved in two steps. First, we establish that a deviation down in $\hat{P}$ can be accomplished by some feasible deviation in G. We then establish that such a deviation is desireable. That a Markov equilibrium exists follows from Proposition 3.3.

C.1. Feasibility of a Downward Deviation in $\hat{P}$

Let P^e denote the expected price level in the Markov equilibrium, and let G^e denote the money growth rate in the corresponding equilibrium outcome, i.e., $G^e = X(P^e)$. We establish that for any $\hat{P}$ in a neighborhood, U, of P^e, there exists a G belonging to a neighborhood, V, of G^e, such that $\hat{P} = \hat{P}(P^e, G)$. Here, $\hat{P}(P^e, G)$ is the price allocation rule in the Markov equilibrium.

Substituting from (3.9) into (3.13) and using the assumptions, $\sigma = 1$ and $\rho = 0$, we obtain:

$$G(P^e,\hat{P}) = P_2(P^e,\hat{P})c_2(P^e,\hat{P})\frac{\beta}{1-\alpha}v_1(1,P^e,X(P^e)). \tag{C.1}$$

From the analogs of (B.6) and (B.8) obtained for the case $g/\beta \geq 1$ and using $\rho = 0$, it can be determined that $c_2(P^e,\hat{P})$ is a strictly increasing function of $\hat{P}$ for $\hat{P}$ in a sufficiently small neighborhood, U, of P^e. It is evident from (3.12) that P_2 is globally increasing in $\hat{P}$. This establishes that $G(P^e,\hat{P})$ is strictly increasing for $\hat{P} \in U$. By the inversion theorem $G(P^e,\hat{P})$ has a unique, continuous inverse function mapping from $V = G(P^e,U)$ to U. By continuity of $\hat{P}(P^e,G)$, this inverse is $\hat{P}(P^e,G)$ itself. This establishes the desired result.

C.2. Desireability of a Downward Deviation in $\hat{P}$

To show that a deviation, $\hat{P} < P^e$, is desireable we first establish properties of the private sector allocation rules and pricing functions in Markov equilibria in which the interest rate is strictly greater than one. Let

$$x^a(P^e,\hat{P}) \equiv \left[c_1^a(P^e,\hat{P}),\ c_2^a(P^e,\hat{P}),...,\ R^a(P^e,\hat{P})\right],$$

denote the solutions to (3.7)-(3.12) with (3.8) replaced by the cash in advance constraint holding with equality. Let

$$x^b(P^e,\hat{P}) \equiv \left[c_1^b(P^e,\hat{P}),\ c_2^b(P^e,\hat{P}),...,\ R^b(P^e,\hat{P})\right],$$

denote the solutions to (3.7)-(3.12) with (3.8) replaced by $R = 1$. For any $\hat{P}$, P^e, private sector allocations and prices must be given either by $x^a(P^e,\hat{P})$ or $x^b(P^e,\hat{P})$. We now show that for all $\hat{P}$ in a neighborhood of P^e the private sector allocations and prices must be given by $x^a(P^e,\hat{P})$.

Consider $\hat{P} = P^e$. Solving (3.7)-(3.12) with (3.8) replaced by the cash in advance constraint holding with equality and with $\hat{P} = P^e$, we obtain:

$$c_1^a(P^e, P^e) = \left[1 + \left(1 + \frac{\gamma}{\lambda}\right)\left(\frac{1-\alpha}{\alpha}R\right)^{\frac{1}{1-\rho}} + \frac{\gamma}{\lambda}\right]^{-1}.$$

Solving the analogous equations for $c_1^b(P^e, P^e)$, we obtain:

$$c_1^b(P^e, P^e) = \left[1 + \left(1 + \frac{\gamma}{\lambda}\right)\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + \frac{\gamma}{\lambda}\right]^{-1}.$$

Evidently, $P^e c_1^b(P^e, P^e) > P^e c_1^a(P^e, P^e)$. By continuity, for all $\hat{P}$ in some neighborhood of P^e, $\hat{P}c_1^b(\hat{P}, P^e) > \hat{P}c_1^a(\hat{P}, P^e)$. Since $\hat{P}c_1^a(\hat{P}, P^e) = 1$, it follows that $\hat{P}c_1^b(\hat{P}, P^e) > 1$ for all $\hat{P}$ in a neighborhood of P^e. Since the cash in advance constraint is violated, $x^b(\hat{P}, P^e)$ cannot be part of a Markov equilibrium. We have established that for $\hat{P}$ in a neighborhood of P^e, private sector allocation rules and prices must be given by $x^a(\hat{P}, P^e)$.

With these pricing and allocation functions, the derivative of the utility function with respect to $\hat{P}$, evaluated at $P^e = \hat{P}$ can be shown to be:

$$L = \frac{u_2 c_2 / P_2}{\lambda \frac{1-c_1}{c_1} - \gamma\rho\frac{g}{\beta}}\{-a(\frac{g}{\beta}) + b(\frac{g}{\beta})\}$$

where g is the growth rate of money at the supposed outcome and

$$\begin{aligned} a(\frac{g}{\beta}) &= \left(\frac{g}{\beta} - \lambda\right)\left(\lambda + \gamma + \gamma\left(\frac{g}{\beta}\right)^{\frac{-\rho}{1-\rho}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{-1}{1-\rho}}(1-\rho)\right)(1-\mu_1) \\ b(\frac{g}{\beta}) &= (1-\lambda)\gamma\mu_2\left[\left(\frac{g}{\beta}\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}} + \frac{g}{\beta}\right] + (1-\mu_1)(1-\lambda)\left(\lambda + \gamma\rho\frac{g}{\beta}\right). \end{aligned}$$

Condition (ii) guarantees that $a(1) \geq b(1)$.[5] In addition under (i) a and b are linear with

[5] It is easily verified that this is equivalent to the condition, (B.13).

slopes, a' and b' respectively, given by:

$$\begin{aligned} a' &= (\lambda + \frac{\gamma\alpha}{1-\alpha})(1-\mu_1) \\ b' &= \frac{(1-\lambda)\gamma\mu_2}{\alpha}. \end{aligned}$$

Given (iii), it is trivial to verify that $L < 0$ for all $g/\beta > 1$. Thus, the supposition that there is an outcome with $R > 1$ leads to the implication that the monetary authority can raise utility by reducing $\hat{P}$. This contradicts monetary authority maximization. We conclude that there are no Markov equilibrium outcomes with $R > 1$.