A Simplified, Two-Period Version of the Gertler-Kiyotaki Model
by
Lawrence Christiano and Tao Zha
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1. Introduction

The model proposed by Gertler-Kiyotaki is one that many will want to read and study. This note provides the 'CliffsNotes' version of the model. The Gertler-Kiyotaki model has uncertainty, an infinite number of periods, habit persistence in consumption, adjustment costs in investment, an employment decision, some firms that invest and others that do not. We strip away all these features in order to focus on the financial frictions and that lie at the heart of the paper. Our simple two-period framework allows us to clarify the nature of the financial frictions and to derive several of the key policy results in the Gertler-Kiyotaki paper.

Many observers in 2008 were puzzled by the large interest rate spreads in banks, because they seemed too big to be explained by observable default risk. The puzzle was why those spreads weren't arbitraged away by an expansion in bank deposits. Building on Gertler-Karadi, the Gertler-Kiyotaki model describes reasons why spreads might be larger than can be explained by an increase in the incidence of default. Interest rate spreads in the model in fact do reflect concern about default, even though the feared defaults are never in fact observed.

The market's normal way to wipe out bank interest rate spreads is by an expansion in bank deposits and assets. The rise in bank deposits requires an increase in deposit rates and the expansion in bank assets may lead to a general decrease in the rate of return on those assets. In the Gertler-Kiyotaki model, there is a default threshold level of bank deposits such that if deposits expand beyond that level, banks would have an incentive to default. The market reacts to this by not expanding deposits beyond the default threshold and therefore not wiping out bank interest rate spreads. By keeping deposits from expanding beyond the default threshold level, the defaults feared by the market are never observed.

In the model, the default threshold level of deposits is lower, the lower is the level of bank net worth. As a result, one can imagine that in normal times, when bank net worth is relatively high, the default threshold of bank deposits is never binding on the market. That is, in normal times interest rate spreads are fully explicable by costs of intermediation, observed bankruptcies, etc. But it could be that in crisis times, when bank net worth falls to rarely observed levels, the default threshold of deposits falls drastically and becomes binding. In this case, crisis times would be times of high observed bank interest rate spreads that are not explicable by an observed increase in the normal factors driving spreads.

Our two-period environment also allows us to capture other results in the Gertler-Kiyotaki paper. We discuss the implications of several 'unconventional monetary policy interventions' explored in the Gertler-Kiyotaki paper. We explore the consequences of three policy options that have been implemented recently: injecting equity into banks, making loans to banks, and making direct loans to firms.

The first section below lays out the model. The subsequent section uses the model to address policy questions.

2. The Model

In what follows, we first describe the model under the assumption that there are no financial frictions. The following section introduces the financial frictions.

2.1. The No-financial Friction Benchmark

2.1.1. Households

There are many identical households. Each contains a unit measure of members. Of these, a fraction are 'workers' and the complementary fraction are banks. Each member of the household enjoys the same level of consumption because consumption insurance inside the household is perfect. The household's budget constraint in the first period is

$$c + d \leq y. \tag{2.1}$$

Here, all variables in per capita household terms. Thus, c is the level of consumption enjoyed by an individual member of the household. Similarly, d denotes bank deposits, per household member, and y denotes the endowment of output given to workers, expressed in per household member terms. For example, if the fraction of household members that are workers is α, then the amount of income brought in by each worker is y/α. Given the way we set up the model, the parameter, α, does not appear in the analysis.

The household's budget constraint in the second period is:

$$C \leq R^d d + \pi, \tag{2.2}$$

where π denotes bank profits, per household member. The household problem is:

$$\max_{d} u\left(c\right) + \beta u\left(C\right) \tag{2.3}$$

with respect to d, taking R^d and π as given. Optimality of the choice of d implies:

$$\begin{aligned} \frac{u'\left(c\right)}{\beta u'\left(C\right)} &= R^d, \\ c + \frac{C}{R^d} &\leq y + \frac{\pi}{R^d}. \end{aligned}$$

The second equation is obtained by solving out for d from the household's flow budget constraints. It says that the present discounted value of household consumption must equal the present discounted value of its income, i.e., its wealth.

Suppose

$$u\left(c\right) = \frac{c^{1-\gamma}}{1-\gamma},$$

so that the household first order condition implies:

$$C = c\left(\beta R^d\right)^{\frac{1}{\gamma}}. \tag{2.4}$$

So solve for c, substitute into the budge constraint to obtain:

$$c = \frac{y + \frac{\pi}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d} + 1}. \tag{2.5}$$

That is, first period household consumption is a fraction of its wealth, where the fraction is a function of the rate of interest. Note that when $\gamma < 1$ then c definitely drops with a rise in R^d. Note that this is also true in the log-utility case, when $\gamma = 1$, because of the presence of profit income. In our view, the interesting case is where c falls with a rise in R^d, so we impose

$$\gamma \leq 1. \tag{2.6}$$

We obtain an expression for deposits by combining the equation for c with the period 0 budget constraint:

$$d = y - c = y \frac{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d} + 1} - \frac{\frac{\pi}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d} + 1} \tag{2.7}$$

2.1.2. Firms

Production only occurs in the second period. The goods used in the first period appear as endowments. We have already discussed the endowment, y, received in the first period by workers, and in the next subsection we refer to the endowments received by banks. Second period goods are produced by competitive firms. They carry out production using capital, k, that they produce in the first period using goods. Firms have no endowments, and to acquire the ability to purchase the goods to use in producing capital they must issue securities, s. The technology for producing capital from goods is one-for-one. The securities issued by firms entitle the owner to a payment in the second period. In the first period,

$$k = s. \tag{2.8}$$

Firms use k in the second period to produce goods then using the following production function,

$$kR^k. \tag{2.9}$$

Here, R^k is a model parameter. The return on capital, R^k, is simply paid to the holders of the securities, s, in the second period. Firms earn no rents, and so the rate of return on securities is R^k.

2.1.3. Banks

For reasons that are not specified in the model, the only agents that can purchase the securities issued by firms are bankers. In period 0, bankers have an endowment of net worth, N, expressed in per household member terms. Bankers take deposits, d^b. Given that any particular household is small relative to all the households, the deposits by bankers in a given household are with probability one, made by different households. That is why we differentiate at this point between deposits, d, made by a given household and deposits, d^b, taken in by the bankers in that household. Although d and d^b are conceptually distinct objects, they will be the same in equilibrium because of market clearing. In addition, they are measured in the same units, in per household member terms.

A household's bankers combine their own net worth, N, and deposits to purchase a quantity of securities, s,

$$s = N + d^b \tag{2.10}$$

from firms. The quantity, N, is expressed in per household member terms. Like y, N represents goods received as an endowment by bankers.

In period 1, bankers pay off depositors and make profits:

$$\pi = R^k s - R^d d^b. \tag{2.11}$$

The banker's problem is to choose d^b so that

$$\max_{d^b} \pi, \tag{2.12}$$

taking R^d as given.

2.1.4. Resource Constraint and Equilibrium

The resource constraint in periods 0 and 1 are:

$$\begin{aligned} c + k &\leq y + N & (2.13) \\ C &\leq kR^k. & (2.14) \end{aligned}$$

We are now in a position to define our equilibrium condition for the case when there are no financial frictions:

Benchmark Equilibrium: R^d, c, C, d, d^b, π
(i) household problem, (2.3), is solved
(ii) bank problem, (2.12), is solved
(iii) the firm problem, (2.8) and (2.9), is solved
(iv) deposit market clearing: $d = d^b$ and securities market clearing, (2.10)

The resource constraints do not appear here because they are redundant. Substituting out for $d = k - N$ in the household's period 0 budget constraint, (2.1), we have the first period resource constraint, (2.13). Substituting out for bank profits in the household's second period budget constraint, (2.2), we obtain the second period resource constraint, (2.14).

We now turn to the computation of equilibrium. Imposing $d^b = d$ and substituting out for profits, π, from (2.11) into (2.7),

$$\begin{aligned} d &= y\frac{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}+1} - \frac{\frac{NR^k+\left(R^k-R^d\right)d}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}+1} \\ &= y\frac{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}+1} - \frac{\frac{NR^k}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}+1} - \frac{\frac{R^k-R^d}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d}+1}d, \end{aligned}$$

or

$$d = \frac{y\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d} - \frac{NR^k}{R^d}}{\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d} + 1 + \frac{R^k - R^d}{R^d}} = \frac{y\left(\beta R^d\right)^{\frac{1}{\gamma}} - NR^k}{\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^k}. \tag{2.15}$$

We now discuss some of the key properties of our benchmark equilibrium. Let an 'interior equilibrium' be one in which $c, C, d, d^b > 0$. It is easy to verify that an interior equilibrium must satisfy:

$$R^k = R^d. \tag{2.16}$$

The proof is by contradiction. Accordingly, suppose we have an equilibrium with $R^k > R^d$. In this case, the value of d^b that solves the bank problem is $d^b = +\infty$, which exceeds the biggest possible d that is feasible for households ($d \leq y$). Thus, (iii) is not satisfied, contradiction our supposition that we have an equilibrium. Now suppose we have an interior equilibrium with $R^k < R^d$. In this case, the value of d^b that solves the bank problem is $d^b = 0$. This contradicts the assumption of an interior equilibrium. We conclude that if we have an interior equilibrium, then $R^k = R^d$.

We have not imposed on the household the requirement, $0 \leq d \leq y$. Clearly, this condition must be satisfied for the equilibrium to be interesting. According to (2.15), this implies the following restrictions on the basic parameters of the model:

$$0 \leq \frac{y\left(\beta R^k\right)^{\frac{1}{\gamma}} - NR^k}{\left(\beta R^k\right)^{\frac{1}{\gamma}} + R^k} \leq y. \tag{2.17}$$

We assume that the parameters satisfy (2.17). It will also be convenient for us if d is well defined and non-negative for $R^d = (1-\theta) R^k$, and so we add this as an additional restriction on model parameters:

$$y\left(\beta(1-\theta) R^k\right)^{\frac{1}{\gamma}} - NR^k \geq 0. \tag{2.18}$$

There are two implications of the result, (2.16). First, the rate of return perceived by households when they select their deposits corresponds to the true rate of return on the underlying physical activity financed by those deposits. Second, the bankers receive no rent from being bankers. Bankers experience no cost in issuing deposits and they receive no benefits as well. The return to bankers from issuing deposits, d^b, when $R^k = R^d$ is zero. In this case,

$$\pi = NR^k.$$

We can now compute c from (2.5) and C from (2.4). This completes the computation of the equilibrium.

It is easy to see that the no-friction equilibrium is efficient. In particular, consider the problem of a planner who respects only the economy's technology and solves:

$$\max u(c) + \beta u(C)$$

subject to (2.13) and (2.14). The decisions of this planner coincide with corresponding objects in the equilibrium of our economy.

2.2. The Financial Friction

We now change the banker's problem so that the banker earns a rent. In particular, we suppose that the banker now has two options:

- 'no-default': issue deposits, d^b, in period 0, combine these with N to purchase $s = N+d^b$ securities. Receive $sR^k - dR^d$ profits in period 1. This option is the only one envisioned in the no financial friction benchmark.
- 'default': issue deposits, d^b, in period 0, combine these with N to purchase $s = N + d^b$ securities. In period 1 take θsR^k as profits and do not pay $R^d d^b$. Under this option, depositors receive the part of returns not taken by the bank, i.e., $(1-\theta)\, sR^k$.

A bank would only choose to default if this would increase its profits. In particular, the bank chooses the no-default option if, and only if,

$$\left(N + d^b\right) R^k - d^b R^d \geq \theta \left(N + d^b\right) R^k. \tag{2.19}$$

Rearranging this,

$$(1-\theta)\left(N + d^b\right) R^k \geq d^b R^d.$$

The expression on the left is what the depositor receives in the case of default. Thus, the bank chooses the no-default option if, and only if, this implies the depositor receives a rate of return on deposits that exceeds R^d.

We assume that before households place their deposits in a bank, that bank must declare its desired deposit level, d^b. We assume there is an enforcement authority which can compare the number of deposits a bank actually takes with its declared level, d^b. We also suppose that that authority has the ability to apply sanctions that ensure the bank does not have an incentive to exceed its pre-announced level of deposits.

No household would place deposits in a bank that posts a d^b that violates (2.19). This is because such banks would pay a return less than the market rate of return, R^d. From the perspective of a bank, posting a d^b that violates (2.19) is the same as posting $d^b = 0$. These considerations indicate that under the financial friction, the banker's problem is changed from (2.12) to

$$\max_{d^b} \pi, \tag{2.20}$$

subject to (2.19).

The relevant equilibrium condition for this version of the model differs from the benchmark equilibrium only in that we replace the banker problem, (2.12), in the benchmark equilibrium with (2.20):

Financial Friction Equilibrium: R^d, c, C, d, d^b, π
(i) household problem, (2.3), is solved
(ii) bank problem, (2.20), is solved
(iii) the firm problem, (2.8) and (2.9), is solved
(iv) deposit market clearing: $d = d^b$ and securities market clearing, (2.10)

We now discuss some of the properties of a financial friction equilibrium. One distinction from the benchmark equilibrium is that there may now be equilibria in which there is an interest rate spread, i.e., $R^k > R^d$. This is so, despite the complete absence of bankruptcy in the model.

Rewrite (2.19):

$$(1-\theta) N R^k \geq \left[\theta R^k - \left(R^k - R^d\right)\right] d^b. \tag{2.21}$$

We consider two types of equilibria: a no premium equilibrium, in which $R^k = R^d$ and a positive premium equilibrium in which $R^k > R^d$.

Consider the no premium equilibrium, $R^k = R^d$. In this case the banker makes zero profits regardless of the value of d^b, so the equilibrium value of deposits is determined by households, i.e., by the value of d in (2.15). Replacing d^b in (2.21) by the value of d implied by (2.15) and rearranging, we obtain:

$$0 \leq \theta \leq \frac{1}{1+B}, \tag{2.22}$$

where

$$B \equiv \frac{\frac{y}{N}\left(\beta R^k\right)^{\frac{1}{\gamma}} - R^k}{\left(\beta R^k\right)^{\frac{1}{\gamma}} + R^k}.$$

Here, $B > 0$ by (2.17). From (2.22), we see that as long as θ is not too large and/or N is not too small, then (2.21) is satisfied and a no premium equilibrium exists. In this case, the financial friction is non-binding and the financial friction and benchmark equilibria coincide.

Now suppose that (2.21) is violated for $R^k = R^d$ and $d^b = d$. This means that with $R^d = R^k$, the demand for deposits by households exceeds what is consistent with the incentive compatibility constraint. Equilibrium requires that the quantity of deposits demanded by households be reduced. This suggests, given our assumption that d is increasing in R^d (recall (2.6)), that an equilibrium can be found with $R^d < R^k$. In fact, such an equilibrium exists and is unique.

To explain this, we require an expression for the supply of deposits. With $R^b < R^k$, bank profits are strictly increasing in d^b. As a result, profit maximization leads banks to choose the largest value of d^b allowed by the incentive constraint, (2.21):

$$d^b = \frac{(1-\theta) R^k}{R^d - (1-\theta) R^k} N. \tag{2.23}$$

Think of d^b as being a function of R^d, defined over the domain,

$$((1-\theta)R^k, R^k]. \tag{2.24}$$

As R^b converges to the upper point in this range,

$$d^b \to \frac{1-\theta}{\theta} N,$$

which, by hypothesis, is smaller than d at $R^d = R^k$. According to (2.23), the supply of deposits, d^b, is strictly increasing as R^b is reduced and approaches $+\infty$ as $R^d \downarrow (1-\theta) R^k$. At the same time, the demand for deposits, d, is strictly decreasing as R^b falls to $(1-\theta) R^k$.

Deposit demand, d, is a well-defined and positive number in this limit, according to (2.18). Given the continuity and monotonicity of demand and supply, as well as

$$\begin{aligned} d &> d^b \text{ as } R^d \uparrow R^k \\ d &< d^b \text{ as } R^d \downarrow (1-\theta) R^k, \end{aligned}$$

we conclude there is a unique $R^d \in \left((1-\theta) R^k, R^k\right)$ such that $d = d^b$. We find this by solving the following single nonlinear equation in the single unknown, R^d :

$$\frac{y \left(\beta R^d\right)^{\frac{1}{\gamma}} - NR^k}{\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^k} = \frac{(1-\theta) NR^k}{R^d - (1-\theta) R^k}. \tag{2.25}$$

The left side of this expression is the demand for deposits, d in (2.15), and the right side is the supply, d^b in (2.23).

Whether we have a benchmark or a financial friction equilibrium, once we have R^d and d in hand, the remaining equilibrium quantities, c, C, π are easily found.

We summarize these results in the form of a proposition:

Proposition: if (2.22) is satisfied, then the financial friction equilibrium coincides with the benchmark equilibrium. If (2.22) is violated, then the final friction equilibrium is characterized by $R^b < R^k$, and may be solved by finding the unique value of R^d that solves (2.25).

3. Policy Experiments

This section will present the results of tax-financed injections of equity into banks, as well as loans to banks and firms. We will consider the interventions in each case when the financial friction are binding, and not binding.