Notes on Linear Approximations, Equilibrium Multiplicity and E-learnability in the Analysis of the Zero Lower Bound*

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Abstract

We study the properties of the zero lower bound model in Eggertson and Woodford (2003) (EW). EW's analysis is based on the equilibrium conditions after linearization. Working with the actual nonlinear equilibrium conditions and consistent with Braun, Körber and Waki (2012) and Mertens and Ravn (2011), we find the existence of equilibria that are not visible to analyses based on linearization, as well as sunspot equilibria. These findings challenge the findings about the properties of the zlb reported in Christiano, Eichenbaum and Rebelo (2011), Eggertson (2011), EW and others. However, we find that the equilibria that are 'invisible' to analyses using linearization are not E-learnable and so those equilibria may perhaps be treated as mathematical curiosities. In addition, evidence that the quality of linear approximations is poor rests on examples where output deviates by more than 20 percent from its steady state, cases where no one would expect linear approximations to work well. For perturbations of reasonable size, the conclusions arrived at in the zlb analysis that use linear approximations appear to be robust.

1. Introduction

In an influential paper, Eggertsson and Woodford (2003) (EW) studied an equilibrium for a simple New Keynesian model without capital in which the zero lower bound on the nominal rate of interest (zlb) is binding. A well-chosen set of simplifying assumptions on the environment greatly simplified the analysis. EW studied an equilibrium which is characterized by two numbers, inflation and output when the zlb binds, and two equations. In the equilibria they study, the system jumps immediately to steady state as soon as the shock that makes the zlb bind goes away.

Because the equations are linearized, there is a unique solution.

The analysis is so influential in part because its simplicity permits a straightforward analytic derivation of a number of interesting results. These results include (see, for example, EW, Eggertsson (2011) and Christiano, Eichenbaum and Rebelo (2011) (CER)):

- When the zlb binds, the loss in output is potentially substantial.
- The output multiplier on government consumption is larger when the zlb binds than when it does not bind.
- When prices are more flexible, or the expected duration of the zlb is longer, then the magnitude in the drop in output in the zlb is greater and the government consumption multiplier is larger.

In this paper, we restrict ourselves to the type of equilibria considered in EW. That is, we consider equilibria in which the system jumps to a particular steady state as soon as the shock is over. In addition, equilibrium while the zlb binds is characterized by constant numbers. We depart from EW by not linearizing the equilibrium conditions. We do this using the strategy followed in Braun, Körber and Waki (2012), who interpret the price frictions in the EW analysis as stemming from adjustment costs as proposed by Rotemberg (1982). This interpretation is interesting because it implies the same linearized equations that EW study. An alternative approach which also implies the linearized equations studied by EW is based on the price setting frictions proposed by Calvo. The advantage of adopting Rotemberg adjustment costs here is analytic simplicity. The Calvo approach injects an endogenous state variable (past price dispersion), while there is no endogenous state variable in the Rotemberg approach.

We find that the key qualitative results listed above survive our nonlinear analysis. Our argument takes the following form. First, as in Braun, Körber and Waki (2012) we find that there are multiple equilibria, including the sunspot equilibrium documented in Mertens and Ravn (2011). If we look across all the equilibria, one can argue that the conclusions described in the bullets above are not robust.[1] For example, in the sunspot equilibrium the government consumption multiplier is smaller when the zlb binds than when it does not bind. This flatly contradicts a key claim in the literature. Second, we impose the requirement that equilibria be E-learnable. Subject to this refinement the model has a unique equilibrium. Third, the properties of the unique equilibrium have the properties stressed in the existing zlb literature. A caveat to our results is that E-learnability requires taking a stand on a learning mechanism. We are currently exploring alternative learning mechanisms and refinement criteria.

Section 2 below describes our model. Section 3 discusses model steady states and relates our analysis to that of Benhabib, Schmitt-Grohe and Uribe (2001). Section 4 defines the set of equilibria that we study. Section 5 derives the linearized equilibrium conditions of the model. This establishes a baseline for comparison. Section 6 discusses E-learnability and describes the learning rule that we employ. Section 7 presents our results for EW equilibria. Finally, section 8 reviews the sort of calculations that are done when researchers wish to go beyond qualitative results used to develop intuition about the properties of economies in the zlb. In practice, those researchers often apply deterministic simulation methods. We investigate the quality of the of the linear approximations within that context. Conclusions appear in section 9.

[1] It is not even clear how one defines a multiplier when there are multiple equilibria. Before and after the jump in G there are two *sets* of equilibria.

2. Model

A representative household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t\right) - \frac{\chi}{2} h_t^2\right]$$

subject to

$$P_t C_t + B_t \leq \left(1 + R_{t-1}\right) B_{t-1} + W_t h_t + \Pi_t,$$

where Π_t represents lump-sum profits net of lump-sum government taxes. The first order necessary conditions associated with an interior optimum are:

$$\chi h_t C_t = \frac{W_t}{P_t}, \;\; \frac{1}{1+R_t} = \beta E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}.$$

Aggregate output, Y_t, is produced by representative, competitive final good producer using intermediate goods, Y_{jt}, $j \in [0, 1]$. The production function and first order conditions are:

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}, \; \varepsilon \geq 1, \; Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\varepsilon} Y_t.$$

Each intermediate good is produced by a monopolist. The monopolist that produces the j^{th} good has the following objective:

$$E_t \sum_{l=0}^{\infty} \beta^l \upsilon_{t+l} [\left(1+\nu\right) P_{j,t+l} Y_{j,t+l} - \overbrace{s_{t+l} P_{t+l} Y_{j,t+l}}^{\text{labor costs of production}}$$

$$- \overbrace{\frac{\phi}{2} \left(\frac{P_{j,t+l}}{P_{j,t+l-1}} - 1 \right)^2 \left(C_{t+l} + \psi G_{t+l}\right)}^{\text{cost (in terms of final goods) of adjusting prices related to aggregate level of output}} \times P_{t+l}],$$

where υ_t denotes the state and date-contingent value assigned to payments sent to households. When $\psi = 1$, then adjustment costs in changing prices are related to aggregate GDP, $C_t + G_t$, and when $\psi = 0$ they are related to the level of household consumption only. We also allow for intermediate cases, because it is not clear on a priori grounds which specification is more sensible. Finally, ν is a subsidy to firms to address distortions due to monopoly power. We assume

$$1+\nu=\frac{\varepsilon}{\varepsilon-1}.$$

The j^{th} intermediate good producer takes the first order condition of the representative final good producer as its demand curve. The production function and level of firm marginal cost (excluding costs associated with price changes) are given by:

$$Y_{j,t}=\overbrace{h_{j,t}}^{\text{production function}},\ s_t\equiv\overbrace{\frac{W_t}{P_t}}^{\text{real marginal cost}}=\overbrace{\chi h_t C_t}^{\text{household optimization}}$$

The firm is required to satisfy whatever demand occurs at its posted price, so that we can substitute out for Y_{jt} using the firm's demand curve. Doing this and imposing $\upsilon_t=1/(P_tC_t)$:

$$\max_{\{P_{j,l}\}_{l=0}^{\infty}} E_t\sum_{l=0}^{\infty}\beta^l\frac{1}{P_{t+l}C_{t+l}}[(1+\nu)P_{j,t+l}\left(\frac{P_{j,t+l}}{P_{t+l}}\right)^{-\varepsilon}Y_{t+l}$$
$$-P_{t+l}s_{t+l}\left(\frac{P_{j,t+l}}{P_{t+l}}\right)^{-\varepsilon}Y_{t+l}-\frac{\phi}{2}\left(\frac{P_{j,t+l}}{P_{j,t+l-1}}-1\right)^2P_{t+l}\left(C_{t+l}+\psi G_{t+l}\right)].$$

The first order condition is, after rearranging:

$$\begin{aligned}&(1+\nu)\frac{P_{j,t}}{P_t}=\frac{\varepsilon}{\varepsilon-1}s_t+\\&\phi\frac{1}{\varepsilon-1}\left(\frac{P_{j,t}}{P_t}\right)^{\varepsilon}\frac{C_t}{Y_t}[-\left(\frac{P_{j,t}}{P_{j,t-1}}-1\right)\frac{P_{j,t}}{P_{j,t-1}}\frac{(C_t+\psi G_t)}{C_t}\\&+\beta E_t\left(\frac{P_{j,t+1}}{P_{j,t}}-1\right)\frac{P_{j,t+1}}{P_{j,t}}\left(\frac{C_{t+1}+\psi G_{t+1}}{C_{t+1}}\right)]\end{aligned}\tag{2.1}$$

When $\phi=0$, the j^{th} firm simply sets price, $P_{j,t}$, to a markup, $\varepsilon/(\varepsilon-1)$, over marginal cost. If that price is high relative to yesterday's price, then the firm raises

$P_{j,t}$ by less if $\phi > 0$, according to the first term in square brackets. Similarly, the second expression in square brackets implies that if that price is low relative to next period's price, then the firm raises price by more if $\phi > 0$. Impose the equilibrium condition, $P_{j,t} = P_{i,t} = P_t$ for all i, j, and rearrange:

$$\begin{aligned}(\pi_t - 1)\,\pi_t &= \frac{1}{\phi}\left\{\left(1 + \nu - \frac{\varepsilon}{\varepsilon - 1}\right)(1 - \varepsilon) + \varepsilon\,(s_t - 1)\right\}\frac{Y_t}{C_t + \psi G_t} \\ &\quad + \beta E_t\,(\pi_{t+1} - 1)\,\pi_{t+1}\frac{(C_{t+1} + \psi G_{t+1})}{C_{t+1}}\frac{C_t}{C_t + \psi G_t}.\end{aligned}$$

We adopt the assumption that a sufficient subsidy is provided to intermediate goods producers so that, at least in steady state, the monopoly distortion is eliminated:

$$1 + \nu - \frac{\varepsilon}{\varepsilon - 1}.$$

There are three uses of gross final output: household and government consumption, and goods used up in changing prices:

$$C_t + G_t + \frac{\phi}{2}\,(\pi_t - 1)^2\,(C_t + \psi G_t) \leq Y_t.$$

In equilibrium, this is satisfied as an equality because households and government go to the boundary of their budget constraints. Government consumption is an exogenous process discussed below.

The four equilibrium conditions associated with the four unknowns, π_t, C_t, R_t, h_t, are:

$$\frac{1}{R_t} = \frac{1}{1+r_t} E_t \frac{C_t}{\pi_{t+1} C_{t+1}} \tag{2.2}$$

$$(\pi_t - 1)\pi_t = \frac{1}{\phi}\varepsilon (s_t - 1) \frac{Y_t}{C_t + \psi G_t} \tag{2.3}$$

$$+\frac{1}{1+r_t} E_t (\pi_{t+1} - 1)\pi_{t+1} \frac{(C_{t+1} + \psi G_{t+1})}{C_{t+1}} \frac{C_t}{C_t + \psi G_t} \tag{2.4}$$

$$C_t + G_t + \frac{\phi}{2}(\pi_t - 1)^2 (C_t + \psi G_t) = h_t \tag{2.5}$$

$$R_t = \max\left\{1, \frac{1}{\beta} + \alpha(\pi_t - 1)\right\}. \tag{2.6}$$

The last equation is the monetary policy rule.

We suppose that $r_t \in \{r^l, r^h\}$. The economy starts with $r_t = r^l$ in the initial period, and it jumps to r^h ($\equiv 1/\beta - 1$) with constant probability $1 - p$. With probability p, the discount rate remains at its initial, low, value. The higher level, r^h, is an absorbing state for r_t. There are no other stochastic shocks in the system. We consider two types of equilibria. In one, $r^l < 0$ and $r^h > 0$. We call this a 'fundamental equilibrium', because the shock affects preferences. We also consider a sunspot equilibrium, in which $r^l = r^h$, so that the uncertainty does not affect preferences or technology.

3. Model Steady States

Given our assumption about the exogenous shock process, the exogenous randomness settles down eventually in its absorbing state. As a result, the model has a well defined steady state. Benhabib, Schmitt-Grohe and Uribe (2001) drew attention to the fact that a model economy like ours has two steady states. They created a diagram, Figure 1, that makes this particularly clear. Two of the equilibrium conditions of the model include the monetary policy rule and the steady state version of the intertemporal Euler equation:

$$\begin{aligned} R &= \max\left\{1, \frac{1}{\beta} + \alpha(\pi - 1)\right\} \\ R &= \pi/\beta, \end{aligned}$$

respectively. Figure 1 shows that the above two equations have two crossings. The two steady states involve $\pi = 1$ and $\pi = \beta$. Once the inflation rate is selected, then the Phillips curve and aggregate output relations can be used to determine steady state C and h :

$$\begin{aligned} (\pi - 1)\pi &= \frac{1}{\phi}\varepsilon(\chi Ch - 1)\frac{h}{C + \psi G} + \beta(\pi - 1)\pi \\ & C + G + \frac{\phi}{2}(\pi - 1)^2(C + \psi G) = h. \end{aligned}$$

Our baseline parameters are:

$$G = 0.20,\ \beta = 0.99,\ \varepsilon/\phi = 0.03,\ \phi = 100,\ \chi = 1.25,\ \psi = 0.$$

When steady state $\pi = \beta$, then $C = 0.7971$ and $h = 1.001$. When the steady state is $\pi = 1$, then $C = 0.80$ and $h = 1$. The difference is quite small. From here on, we follow EW in assuming that when the shock switches to its high value, the economy jumps to the higher of the two steady states. One rationale for selecting this steady state may be that the monetary policy rule is actually composed of the Taylor rule with an escape clause which specifies that if inflation is not proceeding at its target rate (i.e., inflation is negative versus its target value of zero) then the money growth rate is adjusted. Fleshing out this argument, of course, must be done by introducing money demand and supply. However, with enough separability, this can be done without changing the equilibrium conditions that we work with here (see, for example, Christiano and Rostagno (2001)).

4. An Interior, EW Equilibrium

An interior EW equilibrium is a set of eight numbers:

$$\pi^h, C^h, R^h, h^h, \pi^l, C^l, R^l, h^l,$$

that satisfy the equilibrium conditions for the two values of the exogenous shock, 'h' for when $r_t = r^h$ and 'l' for when $r_t = r^l$. In the low state:

$$\frac{1}{R^l} = \frac{1}{1+r^l}\left[p\frac{C^l}{\pi^l C^l} + (1-p)\frac{C^l}{\pi^h C^h}\right] \tag{4.1}$$

$$\left(\pi^l - 1\right)\pi^l = \frac{1}{\phi}\varepsilon\left(\chi h^l C^l - 1\right)\frac{C^l + G^l + \frac{\phi}{2}\left(\pi^l - 1\right)^2\left(C^l + \psi G^l\right)}{C^l + \psi G^l} \tag{4.2}$$

$$+\frac{1}{1+r^l}\left[p\left(\pi^l - 1\right)\pi^l + (1-p)\left(\pi^h - 1\right)\pi^h\left(1 + \psi\frac{\eta_g}{1-\eta_g}\right)\frac{C^l}{C^l + \psi G^l}\right]$$

$$C^l + G^l + \frac{\phi}{2}\left(\pi^l - 1\right)^2\left(C^l + \psi G^l\right) = h^l \tag{4.3}$$

$$R^l = \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi^l - 1\right)\right\}. \tag{4.4}$$

Following the discussion in section 3, we suppose that the high state is the steady state. We assume that government consumption in that state satisfies:

$$G^h = \eta_g\left(C^h + G^h\right),$$

where η_g is the share of government consumption in GDP. Thus,

$$G^h = \frac{\eta_g}{1-\eta_g}C^h.$$

In addition, it is easy to confirm that the other variables take on the following values in the high state:

$$\pi^h = 1,\ \chi h^h C^h = 1,\ C^h = \left(1 - \eta_g\right)h^h,\ R^h = 1 + r^h = 1/\beta.$$

Also,

$$
\begin{aligned}
h^h &= \left[\frac{1}{\chi\left(1-\eta_g\right)}\right]^{1/2} \\
C^h &= \left(1-\eta_g\right)\left[\frac{1}{\chi\left(1-\eta_g\right)}\right]^{1/2}.
\end{aligned}
$$

These levels of employment and consumption are the 'first-best' allocations, i.e., the allocations a planner would choose, who is constrained only by the preferences and technology and who ignores prices and price adjustment costs.

Conditional on the high-state equilibrium, we can solve for C^l, π^l, h^l using the three low-state equilibrium conditions, (4.1)-(4.4). The algorithm fixes π^l and computes C^l and h^l using (4.1), (4.3) and (4.4). We then evaluate whether (4.2) holds. If we can find a value for π^l such that it holds, we have an equilibrium. If we cannot find such a value of π^l, we say there does not exist an interior EW equilibrium. We use the adjective, 'interior' here, to emphasize that we only consider equilibria in which the first order conditions of the agents hold with equality.

The task of finding an equilibrium or asserting with confidence that one does not exist is greatly simplified by the fact that non-negativity of the C^l and h^l that solve (4.1), (4.3) and (4.4) for given π^l requires that π^l lie in the interior of a particular bounded and convex set, D. We call the set of inflation rates, D, the 'set of candidate equilibrium inflation rates'. We now construct this set.

The monetary policy rule divides the candidate inflation rates into a subinterval in which the zlb is binding and a second one in which it is not:

$$
R^l = \begin{cases} 1 & \pi^l \leq \pi^l_{ub} \\ \frac{1}{\beta} + \alpha\left(\pi^l - 1\right) & \pi^l > \pi^l_{ub} \end{cases}, \quad \pi^l_{ub} = \frac{1+\alpha-\frac{1}{\beta}}{\alpha}. \tag{4.5}
$$

Solving the intertemporal Euler condition for $\pi^l \leq \pi^l_{ub}$ and using (4.5), we obtain:

$$C^l\left(\pi^l\right) = \frac{1 + r^l - p\frac{1}{\pi^l}}{1 - p} C^h, \tag{4.6}$$

after imposing $R^l = \pi^h = 1$. All objects on the right of (4.6) are known, except for π^l. Thus, this equation provides a mapping from π^l to C^l. The fact that we only consider interior equilibria (i.e., those with $h^l, C^l > 0$) implies a lower bound, π^l_{lb}, on π^l :

$$\pi^l_{lb} = \frac{p}{1 + r^l}. \tag{4.7}$$

The function, $C^l(\pi^l)$ in (4.6) is zero at $\pi^l = \pi^l_{lb}$ and is strictly increasing, hence positive, for $\pi^l_{lb} < \pi^l < \pi^l_{ub}$. By (4.3), h^l is positive over this interval too. That $C^l\left(\pi^l\right)$ is increasing for $\pi^l_{lb} < \pi^l < \pi^l_{ub}$ is not surprising since increases in π^l reduce the real rate of interest when the nominal rate is fixed. We conclude:

$$C^l\left(\pi^l\right) > 0 \text{ for } \pi^l_{lb} < \pi^l \leq \pi^l_{ub}. \tag{4.8}$$

Non-negativity of C^l also implies an upper bound on π^l. To see this, solve for C^l using the intertemporal Euler equation for $\pi^l > \pi^l_{ub}$:

$$C^l\left(\pi^l\right) \equiv \frac{C^h}{1-p}\left[\frac{1+r^l}{\frac{1}{\beta}+\alpha\left(\pi^l-1\right)} - p\frac{1}{\pi^l}\right] = \frac{C^h}{1-p}\frac{p\left(\alpha - \frac{1}{\beta}\right) - \left[p\alpha - \left(1+r^l\right)\right]\pi^l}{\left[\frac{1}{\beta}+\alpha\left(\pi^l-1\right)\right]\pi^l}, \tag{4.9}$$

after substituting out for R^l using (4.5). From the fact that the numerator in (4.9) is linear and the denominator is finite for $\pi^l > \pi^l_{ub}$, there is exactly one value of $\pi^l > \pi^l_{ub}$, say $\tilde{\pi}^l$, for which $C^l\left(\tilde{\pi}^l\right) = 0$:

$$\tilde{\pi}^l = \frac{p\left(\alpha - \frac{1}{\beta}\right)}{p\alpha - \left(1 + r^l\right)} = \frac{p\left(\beta\alpha - 1\right)}{p\beta\alpha - \beta\left(1 + r^l\right)}. \tag{4.10}$$

We assume

$$\beta\left(1 + r^l\right) > p, \;\; p\alpha > 1 + r^l, \tag{4.11}$$

so that

$$\tilde{\pi}^l > 1. \tag{4.12}$$

By continuity of $C^l\left(\pi^l\right)$, it must be that

$$C^l\left(\pi^l\right) > 0 \text{ for } \pi^l_{ub} \leq \pi^l < \tilde{\pi}^l. \tag{4.13}$$

It is easy to show that the function, $C^l\left(\pi^l\right)$, converges to zero from below. That $C^l\left(\pi^l\right)$ converges to zero follows from the fact that the denominator is a second order polynomial in π^l while the numerator is linear in π^l. That convergence occurs from below reflects that $C^l\left(\pi^l\right)$ crosses zero exactly once and because of (4.13). Alternatively, the same result can be seen by noting that

$$\lim_{\pi^l\to\infty} \pi^l C^l\left(\pi^l\right) = -\frac{C^h}{1-p}\frac{p\alpha-\left(1+r^l\right)}{\alpha} < 0,$$

by (4.11). Result (4.13), the fact that $C^l\left(\pi^l\right)$ crosses zero exactly once for $\tilde{\pi}^l$ and that $C^l\left(\pi^l\right)$ converges to zero from below implies that[2]

$$C^l\left(\pi^l\right) < 0, \text{ for } \pi^l > \tilde{\pi}^l. \tag{4.14}$$

[2]Differentiating $C^l\left(\pi^l\right)$ with respect to π^l, we obtain results consistent with these observations. In particular, after some algebra,

$$\frac{dC^l\left(\pi^l\right)}{d\pi^l} = \frac{C^h}{1-p}\frac{1}{\left(\left[\frac{1}{\beta}+\alpha\left(\pi^l-1\right)\right]\pi^l\right)^2}\{p\left(\alpha-\frac{1}{\beta}\right)^2-2p\left(\alpha-\frac{1}{\beta}\right)\alpha\pi^l+\alpha\left[p\alpha-\left(1+r^l\right)\right]\left(\pi^l\right)^2\}.$$

so that the slope of the C^l function is negative at $\pi^l = \tilde{\pi}^l$:

$$\frac{dC^l\left(\pi^l\right)}{d\pi^l}|_{\pi^l=\tilde{\pi}^l} = \frac{C^h}{1-p}\frac{1}{\left(\left[\frac{1}{\beta}+\alpha\left(\pi^l-1\right)\right]\pi^l\right)^2}\{-p\left(\alpha-\frac{1}{\beta}\right)^2\frac{\left(1+r^l\right)}{p\alpha-\left(1+r^l\right)}\} < 0.$$

The slope of the C^l function is also negative at $\pi^l = 1$:

$$\frac{dC^l\left(\pi^l\right)}{d\pi^l}|_{\pi^l=1} = \frac{C^h}{1-p}\frac{1}{\left(\left[\frac{1}{\beta}+\alpha\left(\pi^l-1\right)\right]\pi^l\right)^2}\{p\left(\frac{1}{\beta}\right)^2-\alpha\left(1+r^l\right)\} < 0.$$

That the object in braces is negative can be derived from (4.11).

We define the set D as follows:

$$D = \left\{\pi^l : \pi^l_{lb} < \pi^l < \tilde{\pi}^l\right\}, \tag{4.15}$$

where π^l_{lb} is defined in (4.7) and $\tilde{\pi}^l$ is defined in (4.10). We summarize this result in the form of a proposition:

Proposition 4.1. *The non-negativity constraints on C^l and h^l, as well as equations (4.1), (4.3) and (4.4), imply that if an equilibrium exists, $\pi^l \in int\,(D)$.*

Substituting the expressions for C^l and R^l into the Phillips curve, (4.2), we obtain:

$$f\left(\pi^l\right) \equiv \frac{1}{\phi}\varepsilon\left(\chi h^l C^l - 1\right)\frac{C^l + G^l + \frac{\phi}{2}\left(\pi^l - 1\right)^2\left(C^l + \psi G^l\right)}{C^l + \psi G^l} - \left(\pi^l - 1\right)\pi^l\left[1 - \frac{1}{1 + r^l}p\right]. \tag{4.16}$$

Any $\pi^l \in int\,(D)$ with the property, $f\left(\pi^l\right) = 0$, is an interior EW equilibrium. If there is no $\pi^l \in int\,(D)$ such that $f\left(\pi^l\right) = 0$, then an interior EW does not exist.

5. Analysis of Linearized System

Before continuing with the analysis of the nonlinear model, it is useful to briefly summarize the standard log-linear analysis of the system. The equilibrium conditions are given in (2.2)-(2.6), and we impose

$$1 + \nu = \frac{\varepsilon}{\varepsilon - 1}.$$

The steady state is:

$$\pi = 1,\ R = 1/\beta,\ h = \left[\frac{1}{\chi\left(1 - \eta_g\right)}\right]^{1/2},\ C = \left(1 - \eta_g\right)\left[\frac{1}{\chi\left(1 - \eta_g\right)}\right]^{1/2}$$

Linearizing the Phillips curve, we obtain:

$$\hat{\pi}_t = \beta\hat{\pi}_{t+1} + \frac{\varepsilon}{\phi}\frac{1}{\left(1-\eta_g\right)+\psi\eta_g}\hat{s}_t,$$

where

$$\hat{x}_t \equiv dx_t/x \text{ and } dx_t \equiv x_t - x.$$

Turning to marginal cost:

$$ds_t = \chi hC\hat{C}_t + \chi hC\hat{h}_t = \hat{C}_t + \hat{h}_t.$$

Consider the resource constraint:

$$C\hat{C}_t + G\hat{G}_t = h\hat{h}_t,$$

because the adjustment costs are zero when linearized in steady state. Then,

$$\left(1-\eta_g\right)\hat{C}_t + \eta_g\hat{G}_t = \hat{h}_t.$$

Finally, the intertemporal Euler equation is:

$$\hat{C}_t = E_t\left\{\hat{C}_{t+1} - \left[\beta\left(R_t - 1 - r_t\right) - \hat{\pi}_{t+1}\right]\right\},$$

where we have used that R_t and $1+r_t$ both equal $1/\beta$ in steady state.

Let y_t denote GDP, so that $y_t = C_t + G_t$. Then,

$$\hat{y}_t = \left(1-\eta_g\right)\hat{C}_t + \eta_g\hat{G}_t.$$

Note that $\hat{h}_t = \hat{y}_t$. This is because price adjustments disappear in the linearization about steady state. We have

$$\hat{C}_t = \frac{1}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t$$

Substituting this into the intertemporal equation and rearranging:

$$\hat{y}_t - \eta_g \hat{G}_t = \hat{y}_{t+1} - \eta_g \hat{G}_{t+1} - \left(1 - \eta_g\right) \left[\beta \left(R_t - 1 - r_t\right) - \hat{\pi}_{t+1}\right],$$

We have

$$\begin{aligned} \hat{s}_t &= \hat{C}_t + \hat{h}_t = \hat{C}_t + \hat{y}_t = \frac{1}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t + \hat{y}_t \\ &= \frac{2-\eta_g}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t \end{aligned}$$

So, the Phillips curve is:

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{\varepsilon}{\phi} \frac{1}{\left(1-\eta_g\right) + \psi \eta_g} \left[\frac{2-\eta_g}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t\right].$$

In summary, the system is:

$$\hat{y}_t - \eta_g \hat{G}_t = E_t \hat{y}_{t+1} - \eta_g E_t \hat{G}_{t+1} - \left(1 - \eta_g\right) \left[\beta \left(R_t - 1 - r_t\right) - E_t \hat{\pi}_{t+1}\right] \quad (5.1)$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \left[\frac{2-\eta_g}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t\right] \quad (5.2)$$

$$R_t = \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_t - 1\right)\right\}, \quad (5.3)$$

where

$$\kappa \equiv \frac{\varepsilon}{\phi} \frac{1}{1 - \eta_g + \psi \eta_g}.$$

We assume that fiscal policy sets $\hat{G}$ potentially non-zero when $r_t = r^l$ and $\hat{G} = 0$ when $r_t = r^h$. As discussed in section 3 above, we suppose that when r_t reverts to r^h, the system immediately jumps to $\hat{y}_t = \hat{\pi}_t = 0$. We seek the set of deterministic processes, $\hat{y}_t$ and $\hat{\pi}_t$, that satisfy the equilibrium conditions as long as $r_t = r^l$. Under these assumptions the expectations in (5.1), (5.2) reduce as follows:

$$E_t \hat{\pi}_{t+1} = p\hat{\pi}_{t+1}, \;\; E_t \hat{G}_{t+1} = p\hat{G}, \;\; E_t \hat{y}_{t+1} = p\hat{y}_{t+1}$$

Following the approach taken in Carlstrom, Fuerst and Paustian (2012), we collapse the Phillips curve and intertemporal conditions into a single second order difference equation in inflation. Solving the Phillips curve for $\hat{y}_t$:

$$\hat{y}_t = \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[\hat{\pi}_t - \beta p \hat{\pi}_{t+1} + \kappa \frac{\eta_g}{1-\eta_g}\hat{G}_t\right]. \tag{5.4}$$

Leading (5.4) one period and taking expecations,

$$p\hat{y}_{t+1} = \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[p\hat{\pi}_{t+1} - \beta p^2 \hat{\pi}_{t+2} + \kappa \frac{\eta_g}{1-\eta_g}p\hat{G}\right].$$

Substitute the latter two results into the IS equation collect terms and rearrange to obtain

$$\hat{\pi}_t - p\left[\beta + 1 + \left(2-\eta_g\right)\kappa\right]\hat{\pi}_{t+1} + \beta p^2 \hat{\pi}_{t+2} = \kappa\beta\left(2-\eta_g\right)r^l + \kappa\eta_g\left(1-p\right)\hat{G}. \tag{5.5}$$

In (5.5), we have assumed the zero bound binds, so that $R_t = 1$. The full set of solutions to (5.5) is given by

$$\pi_t = \hat{\pi}^l + a_1\lambda_1^t + a_2\lambda_2^t, \tag{5.6}$$

where

$$\hat{\pi}^l = \frac{\kappa\beta\left(2-\eta_g\right)r^l + \kappa\eta_g\left(1-p\right)\hat{G}^l}{\Delta},\ \Delta = \left(1-p\right)\left(1-\beta p\right) - p\left(2-\eta_g\right)\kappa, \tag{5.7}$$

and a_1 and a_2 are arbitrary constants. In (5.6), λ_i, $i = 1, 2$ are the zeros of the following polynomial in λ :

$$1 - \phi p\lambda + \beta p^2\lambda^2 = 0,$$

where

$$\phi \equiv \beta + 1 + \left(2-\eta_g\right)\kappa.$$

Let

$$f(\lambda) \equiv \frac{1}{\lambda} + \beta p^2 \lambda.$$

We seek the λ's that satisfy $f(\lambda) = p\phi$. The function, f, achieves a minimum at

$$\frac{1}{\lambda^2} = \beta p^2 \rightarrow \lambda = \sqrt{\frac{1}{\beta p^2}} > 1.$$

For λ smaller than $\sqrt{1/(\beta p^2)}$, f is decreasing and for larger λ, f is increasing. Note that

$$f(1) = 1 + \beta p^2 = \Delta + p\phi \tag{5.8}$$

Both roots of f exceed unity if, and only if,

$$f(1) > p\phi.$$

Using (5.8), the latter condition is equivalent to

$$\Delta > 0. \tag{5.9}$$

These results, which were obtained in a very similar setting by Carlstrom, Fuerst and Paustian (2012), are summarized as follows:

Proposition 5.1. *Condition (5.9) is necessary and sufficient for (5.6) with* $a_1 = a_2 = 0$ *to be the unique non-explosive solution of (5.5).*

This proposition indicates that, from the perspective of the linear analysis, it is interesting to focus on parameter values for which the condition, (5.9) is satisfied. Interestingly, (5.7) indicates that (5.9) is also necessary for the unique bounded solution to validate our assumption that the zlb is binding. If $\Delta < 0$, then inflation rises in response to the negative value of r^l and the system is pushed away from the zlb. It remains to see what significance (5.9) has, if any, in the nonlinear analysis.

The drop in output is given by:

$$\hat{y} = \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[(1-\beta p)\,\hat{\pi}^l + \kappa\frac{\eta_g}{1-\eta_g}\hat{G}\right].$$

From this expression, we can determine the output multiplier of a change in government spending. Note

$$\frac{d\hat{y}^l}{d\hat{G}} = \frac{d\left(\frac{y^l-y}{y}\right)}{d\left(\frac{G^l-G}{G}\right)} = \frac{G}{y}\frac{dy^l}{dG^l},$$

where G and y denote the steady state levels of government consumption and GDP, respectively. Thus, the multiplier is:

$$\frac{dy^l}{dG^l} = \frac{y}{G}\frac{d\hat{y}^l}{d\hat{G}}.$$

Now,

$$\begin{aligned}\frac{d\hat{y}}{d\hat{G}} &= \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[(1-\beta p)\frac{d\hat{\pi}^l}{d\hat{G}} + \kappa\frac{\eta_g}{1-\eta_g}\right] \\ &= \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[(1-\beta p)\frac{\kappa\eta_g(1-p)}{\Delta} + \kappa\frac{\eta_g}{1-\eta_g}\right]\end{aligned}$$

so that

$$\frac{dy^l}{dG^l} = \frac{1-\eta_g}{2-\eta_g}\left[\frac{(1-\beta p)(1-p)}{\Delta} + \frac{1}{1-\eta_g}\right]$$

The expressions for the drop in output and the government consumption multiplier are very similar to the ones derived for a slightly different model in CER.

6. E-Stability

Rational expectations requires that agents know and act on the exact values of prices and other variables in their environment. In some ways this is an odd sort of an assumption. For example, in our model intermediate good firms choose their price level, $P_{j,t}$, based in part on the value taken on by the aggregate price level, P_t. But, the aggregate price level in turn is a function of their collective price decisions. Obviously, intermediate good firms cannot actually 'know' the current aggregate price level when they choose their own price, in the sense of actually observing it. In practice, one assumes that intermediate good firms form a 'belief' about the aggregate price level at the time they make their decision, and in a rational expectations equilibrium that belief happens to be 'correct'.[3] By correct, we mean they know it without any error at all. Of course, no modeler takes this assumption completely seriously. If we write down a model in which, for example, an agent's belief is off in the 10^9th digit and the equilibrium falls apart, then we surely consider that equilibrium to be uninteresting. Take as an example a pencil and a table top. The pencil has two equilibria. It can lay on its side on the table or it can stand on its head. The second equilibrium, which no doubt exists, has never been observed (at least, by anyone we know!) because the slightest deviation from it causes the pencil's position to diverge from the second equilibrium. For this reason, the second equilibrium is uninteresting and can (perhaps!) be ignored.

To determine what happens when agents do not know the precise values of the prices and other equilibrium variables they must know when they make their decisions, we must make assumptions about what happens when agents' beliefs do not coincide with the beliefs that occur in a rational expectations equilibrium. For this we assume that agents learn from past observations on the variables about which they must form beliefs. Of course, for the analysis to be interesting, the way agents are posited to learn from past observations must be plausible. Here, we adopt the simple assumption that the variables about which they form beliefs will take on the same

[3]Throughout, we use the word 'belief' in contrast to 'forecast'. Belief suggests certainty, while forecast suggests the mean or mode of a conditional distribution.

values that they did in the previous period. The key property of this 'no change', or random walk assumption about forecasting is that agents extrapolate the future from the past. This learning scheme could be extended to what is now the standard approach in the learning literature, where agents use more sophisticated time series methods. We leave that to further work.

We follow the literature in saying that if the economy converges to an equilibrium under a model with learning after a deviation from rational expectation beliefs, then that equilibrium is characterized by E-Stability. Otherwise it is not E-stable and, like the equilibrium with a pencil standing on its head, it is not interesting.

We adopt the following assumptions:

- We suppose agents know the values of the variables outside the zlb and they know the value of government consumption in all periods and states.
- Agents use the 'no change' assumption to forecast inflation, aggregate quantities and their own choices in the future scenario in which the zlb continues to bind. These assumptions are correct in the rational expectations equilibrium.

The assumption that agents form beliefs directly about their own future decisions is questionable, though it is standard in the E-stability literature. Implictily, any rule for forming beliefs about agents' future decisions is equivalent to a rule for forming beliefs about the future values of variables that will determine their decisions. An analytically transparent approach would specify the latter rule and assume that agents' decisions optimize their resulting objective. Later, we plan to implement ways of proceeding that avoid these two shortcomings. We suspect that our basic message E-learnability is robust to alternative approaches.

According to (2.1) and assuming we are in a zlb, the j^{th} firm sets its current price, $P_{j,t}$, according to:

$$(1+\nu)\frac{P_{j,t}}{P_t^e} = \frac{\varepsilon}{\varepsilon-1}s_t^e + \tag{6.1}$$

$$\phi\frac{1}{\varepsilon-1}\left(\frac{P_{j,t}}{P_t^e}\right)^{\varepsilon}\frac{C_t^e}{Y_t^e}[-\left(\frac{P_{j,t}}{P_{j,t-1}}-1\right)\frac{P_{j,t}}{P_{j,t-1}}\frac{(C_t^e+\psi G_t^e)}{C_t^e}$$

$$+\frac{1}{1+r^l}\{p\times\overbrace{\left(\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e-1\right)\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e\left(\frac{C_{t+1}^e+\psi G_{t+1}}{C_{t+1}^e}\right)}^{\text{assuming next period is a zlb period}}$$

$$+(1-p)\times\overbrace{\left(\frac{P_{j,t+1}}{P_{j,t}}-1\right)\frac{P_{j,t+1}}{P_{j,t}}\left(\frac{C_{t+1}^e+\psi G_{t+1}}{C_{t+1}^e}\right)}^{\text{assuming next period is not a zlb}}\}]$$

Here, the superscript, 'e', indicates the firm's belief about the value of the corresponding variable in a zlb. Note that there is a superscript, e, on the current period value of the aggregate price index and of the aggregate consumption and output. As discussed above, when the firm selects a value for its price, $P_{j,t}$, it must do so based on beliefs about the values of P_t, C_t and Y_t and of P_{t+1}, C_{t+1} and Y_{t+1} conditional on being in the zlb in $t+1$. In the case of P_t, C_t and Y_t it is obviously true that they must form beliefs about these variables, rather than 'knowing' them as would be the case if they observed them. This is because P_t, C_t and Y_t are determined in part by the collective price setting actions of the firms. The notation indicates that $P_{j,t}$ is also a function of what price the firm plans to set in period $t+1$. We assume that it expects to set $P_{j,t+1}/P_{j,t}$ in the period $t+1$ state of the world in which the zlb is still binding, equal to $P_{j,t}/P_{j,t-1}$. This is an assumption that is true in the rational expectations equilibrium, though not necessarily true in the expectational equilibrium we now study. Thus, we suppose that

$$\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e = \frac{P_{j,t}}{P_{j,t-1}}. \tag{6.2}$$

The actual realized period t price level, P_t, is just the price set by all the individual firms, i.e., $P_t = P_{j,t}$ for all j. Rearranging (6.1) and implementing this result:

$$\begin{aligned}\left[(1+\nu)(1-\varepsilon)\left(\frac{P_t}{P_t^e}\right)^{-\varepsilon} + s_t\varepsilon\left(\frac{P_t}{P_t^e}\right)^{-\varepsilon-1}\right]\frac{1}{P_t^e}\frac{Y_t^e}{C_t^e} - \phi\left(\pi_t^l - 1\right)\frac{\left(C_t^e + \psi G^l\right)}{P_{t-1}C_t^e} & \\ + \frac{1}{1+r^l}\phi p\left(\pi_t^l - 1\right)\pi_t^l\frac{1}{P_t}\frac{\left(C_{t+1}^e + \psi G^l\right)}{C_{t+1}^e} &= 0,\end{aligned}$$

where (6.2) and the fact, $\pi^h = 1$, has been used. Also,

$$\pi_t^l \equiv \frac{P_t}{P_{t-1}}$$

denotes the inflation rate in the zlb. The above expression can be further simplified, after multiplying both sides by P_t^e and dividing P_t and P_t^e by P_{t-1} :

$$\begin{aligned}\left[(1+\nu)(1-\varepsilon)\left(\frac{\pi_t^l}{\pi_t^e}\right)^{-\varepsilon} + \chi h_t^e C_t^e\varepsilon\left(\frac{\pi_t^l}{\pi_t^e}\right)^{-\varepsilon-1}\right]\frac{Y_t^e}{C_t^e} - \phi\left(\pi_t^l - 1\right)\pi_t^e\frac{\left(C_t^e + \psi G^l\right)}{C_t^e} & \\ + \frac{1}{1+r^l}\phi p\left(\pi_t^l - 1\right)\pi_t^e\frac{\left(C_{t+1}^e + \psi G^l\right)}{C_{t+1}^e} &= 0.\end{aligned}$$

Here,

$$\pi_t^e \equiv \frac{P_t^e}{P_{t-1}}.$$

We assume that intermediate good firm beliefs are set as follows:

$$\pi_t^e = \pi_{t-1}^l,\ C_t^e = C_{t+1}^e = C_{t-1}^l.$$

These 'no change' beliefs are what would be correct if the variables involved were a random walk. Applying these assumptions about beliefs and multiplying by C_t^e/Y_t^e,

$$\begin{aligned}& \phi\left(\pi_t^l - 1\right)\pi_{t-1}^l\left[1 - \frac{1}{1+r^l}p\right] \qquad (6.3)\\ =\ & \left[(1+\nu)(1-\varepsilon)\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon} + \chi h_{t-1}^l C_{t-1}^l\varepsilon\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon-1}\right]\frac{h_{t-1}^l}{C_{t-1}^l + \psi G^l}.\end{aligned}$$

This expression suggests that for given $r^l, \pi_{t-1}^l, h_{t-1}^l, C_{t-1}^l$ there is a unique π_t^l. The left side is increasing in π_t^l while the right side is strictly decreasing in π_t^l.

The other equilibrium conditions in the zlb are:

$$1 = \frac{1}{1+r^l}\left[p\frac{1}{\pi_t^l} + (1-p)\frac{C_t^l}{\pi^h C^h}\right] \tag{6.4}$$

$$C_t^l + G^l + \frac{\phi}{2}\left(\pi_t^l - 1\right)^2\left(C_t^l + \psi G^l\right) = h_t^l \tag{6.5}$$

$$\frac{1}{\beta} + \alpha\left(\pi_t^l - 1\right) \leq 1 \text{ (condition that zlb binds),} \tag{6.6}$$

Here, C_t^l denotes actual period t consumption, and we have used the assumption that households expect consumption in $t+1$ conditional on being in the zlb then to also be C_t^l. In addition, we have used the assumption that they expect inflation in the next period if still in the zlb, π_{t+1}^l, will be the same as it is in the current period.

Equations (6.3)-(6.5) define a mapping from $z_{t-1} = \left[\begin{array}{ccc} C_{t-1}^l & h_{t-1}^l & \pi_{t-1}^l \end{array}\right]'$ to z_t :

$$z_t = f\left(z_{t-1}\right). \tag{6.7}$$

Define

$$F = \left[\frac{df_i\left(z\right)}{dz_j}\right]|_{z=\tilde{z}},\ i,j = 1,2,3,$$

for $\tilde{z}$ associated with the first equilibrium or the second. Using F, we obtain a linear approximation to (6.7):

$$z_t = z^* + F \times \left(z_{t-1} - z^*\right), t = 0,1,2,..., \tag{6.8}$$

where z^* is the point around which the Taylor series expansion is taken. Express F in eigenvector-eigenvalue form,

$$F = P\Lambda P^{-1},\ \Lambda = \left[\begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array}\right],\ P^{-1} = \left[\begin{array}{c} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{array}\right].$$

Then,

$$\tilde{z}_t = \Lambda \tilde{z}_{t-1},\ \tilde{z}_t \equiv P^{-1}\left(z_t - z^*\right),$$

or,

$$\tilde{P}_i\left(z_t - z^*\right) = \lambda_i \tilde{P}_i\left(z_{t-1} - z^*\right),\ i = 1, 2, 3.$$

For our baseline parameterization, we found that there is exactly one eigenvalue, say λ_i, that lies outside the unit circle. Then, if the initial value of $\tilde{P}_i\left(z_t - z^*\right)$ is non-zero, that linear combination of z_t diverges from z^*. Perturbations of initial z_t in which the initial values of the elements of z_t have the property, $\tilde{P}_i\left(z_t - z^*\right) = 0$, do not diverge. However, this is a measure-zero set of perturbations in the initial z_t. In general, a perturbation in the initial z_t will result in $z_t - z^*$ diverging if there is only one eigenvalue of F that is larger than unity in absolute value.

Technically, E-stability is satisfied if only a local perturbation in initial beliefs results in divergence, i.e., if at least one eigenvalue of the linearized learning system exceeds unity in absolute value. However, because we have the actual dynamic equations of the learning system in hand (i.e., not just locally valid linearized equations), it is interesting to do another experiment as well. In particular, we imagine that prior to the shock down in r_t the economy was in its steady state where no one expected the r_t shock to occur. We imagine that when the r_t shock occurs, it does so after intermediate good firms have formed their beliefs about current period aggregate prices and quantities. In this case, the 'no change' forecast, at least of the current variables, is rational. We then investigate where the system evolves to while r_t remains down.

Thus, in our simulation, we suppose that in the first period of the zlb, firm expectations are set as follows:

$$\pi_{-1}^l = \pi^h, C_{-1}^l = C^h,\ Y_{-1}^l = Y^h,\ h_{-1}^l = h^h.$$

In effect, in the first period when the discount rate turns negative, this occurs after the firms have set their price for that period. It is done at a time when they are unaware of the impending switch in the value of the discount rate. To simulate this system, note that equation (6.3) can be used to solve for π_t^l conditional on $C_{t-1}^l, h_{t-1}^l, \pi_{t-1}^l$. Conditional on π_t^l, (6.4) can be solved for C_t^l and then (6.5) can be used to solve for h_t^l. We assume that zlb remains binding, but this must be confirmed at all dates by verifying that (6.6) holds.

7. Numerical Results

7.1. Properties of the EW Equilibrium

We consider the following baseline parameterization of the model:

$$\begin{aligned} \kappa &= 0.03,\ \beta = 0.99,\ \alpha = 1.5,\ p = 0.775, \\ r^l &= -0.02/4,\ \phi = 100,\ \psi = 1,\ \eta_g = 0.2. \end{aligned} \tag{7.1}$$

Figure 2 graphs $f\left(\pi^l\right)$ in (4.16) for two values of $G^l : G^l = G^h$ and $G^l = 1.05 \times G^h$. Recall that values of $\pi^l \in D$ for which $f\left(\pi^l\right) = 0$ correspond to interior EW equilibria, where D is defined in (4.15). Figure 2 displays f for π^l belonging only to a subset of D. We verified that the only values of $\pi^l \in D$ such that $f\left(\pi^l\right) = 0$ are the two depicted in Figure 2. We refer to the first equilibrium on the left as equilibrium #1 and the other one is #2. Note that there is a kink in both f functions. This occurs where the zlb ceases to bind. Various properties of the equilibria are noted at the top of Figure 2. Many of these are also reported in Table 1. Information that pertains to levels of variables refers to the model with the higher value of G^l.

An increase in G^l shifts f up. This has the consequence that the increase in G in the zlb has opposite effects on the inflation rate in the two equilibria. In the first equilibrium, the inflation rate goes down and in the second equilibrium it goes up. The government spending multiplier in the i^{th} equilibrium is computed by taking the ratio of the difference in GDP across the i^{th} equilibria for the two values of G^l, to the corresponding difference of government consumption, for $i = 1, 2$. The multiplier is a little above 2 in the second equilibrium and much smaller in the first, where it is 0.16. The drop in GDP in the zlb in the two equilibria is very different, being 5.4 percent in the second equilibrium and 38 percent in the first. The results based on analyzing the linearized system roughly resemble the properties of the second equilibrium. They imply a multiplier of 2.77. Although this is somewhat larger than the multiplier in the second equilibrium, both multipliers deserve the adjective, 'large', and are in that sense similar. The percent drop in GDP relative to according to the log-linear calculations is 6.0 percent, relatively similar to the drop in output in the second equilibrium

Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
eigenvalues in learning law of motion	$2.28, 1.4 \times 10^{-6}$, 1.9×10^{-3}	$0.71, -2.5 \times 10^{-10}$, -1.1×10^{-4}	
Δ			0.0105
Panel B: Increase in p from 0.775 to 0.793 (longer expected duration of lower bound)			
$\frac{d\text{GDP}}{dG}$	-2.84	5.36	12.41
% drop in GDP	25.49	13.21	38.60
change in inflation rate	-7.42	-3.85	-12.30
eigenvalues in learning law of motion	1.27, -8.7 $\times 10^{-7}$ -9.3×10^{-5}	$0.85, 2.65 \times 10^{-10}$ -1.94×10^{-4}	
Δ			0.00167
Panel C: Increase in κ from 0.03 to 0.0375 (more flexibility in prices)			
$\frac{d\text{GDP}}{dG}$	-2.08	4.54	414.3
% drop in GDP	25.44	12.20	1224.2
change in inflation rate	-8.22	-3.98	-444
eigenvalues in learning law of motion	1.36, -9.3 $\times 10^{-7}$ -1.7×10^{-4}	$0.82, 1.3 \times 10^{-9}$ -2.8×10^{-4}	
Δ			0.000056

We also compute the eigenvalues of the matrix, F, in (6.8). According to the results in Table 1, the first equilibrium is not E-stable while the second one is.

We considered two perturbations on the baseline parameter values. Panel B reports the results of increasing the expected duration of the zero lower bound by raising the value of p from 0.775 to 0.793. The results are also reported in Figure 3. Consistent with the implications of the linear approximation, the multiplier in the second equilibrium is now increased. This matches one of the findings of CER, which discusses the underlying intuition. However, note that the quantitative difference between the results based on linear approximation and the exact nonlinear analysis is substantially larger. The multiplier implied by the linear approximation is now a very large 12.4, more than twice its correct value of 5.36. Part of the reason the multiplier jumped so much in the linear approximation, despite the tiny jump in p, is that Δ in (5.9) falls by one order of magnitude from its value of 0.0105 in the baseline parameterization, to 0.00167. In the first equilibrium, the government spending multiplier falls somewhat with the rise in p, contradicting a basic conclusion in CER that the multiplier and output drop in a binding zlb is more severe, the greater is the expected duration of the zlb. However, Panel B indicates that, as in Panel A, the first equilibrium in Figure 3 is not E-learnable, while the second is E-learnable.

Next, consider Panel C which reports findings based on increasing ε/ϕ, i.e., making prices more flexible (either by reducing adjustment costs or raising the elasticity of demand). The result is that the slope of the Phillips curve rises from 0.03 in the baseline parameterization to 0.0375. The results for this case are also displayed in Figure 4. Based on the implications of the linear approximation, CER argued that increased price flexibility increases the magnitude of the multiplier and of the output collapse in the zlb. There is a simple intuition for this result, which is explained in CER.[4] The second equilibrium confirms the CER result: (i) the output drop increases to roughly 12 percent, which is a little more than double its drop in the baseline parameterization and (ii) the multiplier jumps from 2.18 in the baseline model to 4.54 with more flexible prices. The linear approximation has qualitatively the same implication, though the magnitudes are very different. In particular, according to the linear approximation the output drop jumps from 5.99 percent in the baseline parameterization to 1224 percent with more flexible prices. The multiplier jumps from 2.77 in the baseline parameterization to 414 when prices are more flexible. That the effects in the linear approximation are so very large reflects that in this example Δ is virtually zero, 0.000056. Further drops drive the multiplier and the output drop to plus infinity. Although the linear approximation is clearly far from the mark quantitatively, its implications for the effects of greater price flexibility are qualitatively correct.

Note from the first equilibrium, however, that increased flexibility reduces the multplier and the size of the output collapse in the zero lower bound, directly contradicting the CER finding. At the same time, Panel C indicates that the first equilibrium is not E-stable.

To summarize, the basic qualitative results reported in CER using the log-linear approximation holds up when we consider the nonlinear solution and ignore the first equilibrium on the grounds of not being E-stable. In particular, (i) the government spending multiplier can be considerably bigger than unity when the zlb binds, (ii) as the expected duration of the zlb increases or the degree of flexibility of prices increases, then both the severity of the output collapse in the zlb and the government spending multiplier are larger, (iii) for parameterizations in which the output collapse

[4] For additional intuition, see Werning (2011).

is large, then the government spending multiplier is large too. The implications of the linear approximation become increasingly distorted as parameter values are chosen for which Δ approaches zero and turns negative. Although the properties of the linear approximation change very sharply for Δ's in this region, the properties of the exact nonlinear solution are much less sensitive. However, we found that increases in p and/or in ε which have the effect of reducing Δ eventually result in the non-existence of equilibrium. This is because such parameter changes have the effect of shifting the f function down so that it does not intersect zero at any of the values of π^l considered. We discuss an example of this below.

7.2. Sunspot Equilibria

We consider sunspot equilibria simply by setting $r^l = r^h = 1/\beta - 1$ when evaluating the f function. As before, the economy starts out in the 'low' state and escapes with constant probability, $1 - p$. In terms of Figure 2, we implement one change in the f function, by setting $r^l = 1/\beta - 1$. The resulting f functions (one each for the steady state and high values of G) are exhibited in Figure 5.[5] In terms of general shape and number of crossings of the zero line, the f functions resemble the ones depicted in Figures 2-4. In the present case, however, the second equilibrium is simply the steady state equilibrium itself when G is constant and it is the steady state perturbed by G in the case where G is temporarily high, in which the sunspot is ignored. In the equilibrium #1 the sunspot has a non-trivial impact on prices and allocations, so we call it the 'sunspot equilibrium'.

As stressed in Mertens and Ravn (2011), the sunspot equilibrium can be characterized as a situation in which the shock driving the economy into a binding zlb is a

[5] As in Figures 2-4, the higher curve corresponds to the case in which $G^l = 1.05 \times G^h$ while the sunspot shock is 'low' and the lower curve corresponds to the case in which $G^l = G^h$.

loss in confidence. Agents anticipate deflation, creating the perception that the real interest rate is high. Households respond with a reduction in expenditures and thus drive the economy into a recession. The reduced level of economic activity results in a drop in marginal cost as the wage rate falls with the lower demand for labor. Reduced marginal costs create downward pressure on the price level. This pressure is manifested in the form of a sustained fall in the price level over time because of the presence of price-setting frictions. In this way, the initial fear of deflation is self-fulfilling. Mertens and Ravn (2011) propose this non-fundamental 'loss of confidence' shock as an alternative to the type of fundamental shock that is in practice assumed to push the economy into the zlb. For example, CER consider a discount rate shock, as in this paper, as well as a shock to financial intermediation.

CER stress their finding the government spending multiplier is high when the zlb is binding, but this conclusion is reversed if the cause of the binding zlb is a confidence shock. To see this, consider Table 2. According to the result in the table, the government spending multiplier is 0.41 in the sunspot equilibrium ('equilibrium #1') and it is 0.81 in equilibrium #2. To understand the implications of this result, note from Figure 5 that equilibrium #2 is to the right of the kink point for both values of G. This shows that the zlb is not binding in that equilibrium, whether G remains at its steady state value or is increased. Because the zlb is not binding in equilibrium #2, the multiplier there corresponds to the government spending multplier in 'normal times'. That multiplier is small compared to its value in the equilibria of type #2 considered in Table 1 because the Taylor rule increases the rate of interest in response to the increase in inflation that follows the rise in G (note from Table 2 that a rise in G raises inflation in equilibrium #2). This mechanism is not present when the zlb is sufficiently binding, as it is in Table 1.[6] The multiplier in the

[6] We say that the zlb is very binding when the 'shadow interest rate', the second argument in

sunspot equilibrium is essentially half of what it is in equilibrium #2. This suggests that the CER conclusion is not robust to which shock drives the economy into a binding zlb. We now examine other features of equilibrium #1 and #2, including E-learnability.

According to the table, the percent drop in GDP when G is expansionary, relative to what GDP is in the high state, is a dramatic 43 percent (the GDP drop is a little bigger in equilibrium #1 when G is held to its steady state value). This reflects the enormous size of the drop in anticipated inflation (15 percent, at a *quarterly* rate!). In the case of equilibrium #2, the drop in GDP is negative relative to steady state. This reflects that in the absence of an expansion in G, the second equilbrium simply *is* the steady state equilibrium. So, the percent change in GDP in equilibrium #2 reported in Table 2 reflects the positive impact of a rise in G on GDP. Similarly, the rise in inflation reflects that an increase in G pushes the inflation rate up. The Taylor rule responds by raising the nominal rate of interest and this is what keeps the multiplier small in equilibrium #2.

We considered the learnability of the equilibria in Figure 5. According to Table 2, one of the three eigenvalues of the matrix F in (6.8) associated with the sunspot equilibrium exceeds unity. As a result, the equilibrium has the property that if agents initially conjecture that aggregate quantities and prices deviate by a tiny amount from their equilibrium values, the system will diverge from the sunspot equilibrium. Moreover, the explosive eigenvalue is not just larger than unity, but it is enormous. The system would propel itself away from the sunspot equilibrium at an extremely

the max operator in the policy rule, (2.5), is very negative. As a result, though the shadow interest rate rises in response to the rise in G, it does not rise by enough to put the shadow interest rate into the positive region. The outcome is that the actual interest rate remains constant after a rise in G.

rapid rate. Equilibrium #2, on the other hand, is characterized by E-stability.

Table 2: Properties of Sunspot Equilibrium		
Baseline parameters, except $r^l = 1/\beta - 1$		
	equilibrium #1	equilibrium #2
$\frac{d\text{GDP}}{dG}$	0.41	0.81
% drop in GDP	43.1	-0.81
change in inflation rate	-14.76	0.07
eigenvalues in learning law of motion	$3.82, 2.6 \times 10^{-6}$, 6.1×10^{-3}	$0.83, 2.7 \times 10^{-11}$, -8.8×10^{-5}

7.3. Absence of an Interior, EW Equilibrium

Expression (4.16) indicates that as p increases, for $\pi^l < 1$, the f function shifts down. Given the shape of that function in Figures 2, 3, 4, it is not surprising that for p large enough, the function ceases to cross the zero line for $\pi^l \in D$, where D is defined in (4.15). Indeed, when we adjust the baseline parameter values by setting $p = 0.8$, this is what we find. There does not exist a $\pi^l \in D$ for which $f\left(\pi^l\right) = 0$ (see Figure 6). In this case the model may have some other type of equilibrium, but it does not have an interior, EW equilibrium. We found the same to be true when we change parameters in a way that causes κ to increase. To see this, note from (4.16) and using (4.11), that real marginal cost, $\chi h^l C^l$, is less than unity when $\pi^l < 1$. As a result, increases in ε drive the function, f, down for $\pi^l < 1$. This is suggestive of our numerical finding that equilibrium ceases to exist as κ increases.[7]

There is a loose connection between this absence of equilibrium and the extreme behavior of the linearized economy as Δ converges to zero and then goes negative.

[7]These results are related to non-existence results obtained by Rhys Mendes (2011).

In our numerical experiments we found that equilibrium ceases to exist for values of κ or p that put Δ very slightly into the negative region. For example, in the computation in Figure 6, $\Delta = -0.0016$.

8. Deterministic Simulations

In practice, the framework described in the preceding sections is primarily used to build intuition about the qualitative economic implications of the zlb. When it is of interest to obtain quantitative results, researchers have typically turned to deterministic simulations in which the shock that drives the economy into the zlb is 'on' for a known period of time.[8] For example, CER use the EW framework to build intuition about the dynamics of an economy in a binding zlb, but turn to deterministic simulations of an estimated medium-sized DSGE model to investigate the quantitative properties of an economy in a binding zlb. This section investigates the quality of the linear approximation for these deterministic simulations. We do so using the model of section 2, where the exact (and, unique) deterministic equilibrium is easy to compute.[9]

We suppose that the economy begins in period $t = 1$ with a negative discount rate, $r_t = r^l$. It is known that the discount rate will remain negative until $t = T - 1$ and that when $t = T$ it switches permanently back to its steady state value of $1/\beta - 1$. For the reasons explained in section 3, we assume that the system jumps to its steady state in period T. To compute the equilibrium, we simulate the equilibrium conditions backwards, starting from steady state in period T. Since the model

[8] The wisdom of this strategy is supported by the recent analysis of Carlstrom, Fuerst and Paustian (2012). They stress that the quantitative results based on the stochastic process for the discount rate proposed by EW can be misleading.

[9] Uniqueness reflects in part our assumption about steady state in section 3.

has no initial conditions, this is a particularly straightforward strategy for finding the equilibrium.[10] To see how we do this, rewrite the deterministic version of the equilibrium conditions of the model, (2.2)-(2.6):

$$
\begin{aligned}
C_t &= \frac{1+r_t}{\max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_t - 1\right)\right\}} \pi_{t+1} C_{t+1} && (8.1) \\
h_t &= C_t + G_t + \frac{\phi}{2}\left(\pi_t - 1\right)^2 \left(C_t + \psi G_t\right) && (8.2) \\
\left(\pi_t - 1\right)\pi_t &= \frac{1}{\phi}\varepsilon\left(\chi C_t h_t - 1\right)\frac{Y_t}{C_t + \psi G_t} \\
&\quad + \frac{1}{1+r_t}\left(\pi_{t+1} - 1\right)\pi_{t+1}\frac{\left(C_{t+1} + \psi G_{t+1}\right)}{C_{t+1}}\frac{C_t}{C_t + \psi G_t}, && (8.3)
\end{aligned}
$$

for $t = 0, 1, ..., T-1$. Here, G_t, r_t are exogenous for $t = 1, ..., T$. This system of equations induces a mapping from π_{t+1}, C_{t+1} to π_t and C_t. We initiate the mapping with π_T and C_T set to their steady state values. For given π_{t+1} and C_{t+1} (8.1)-(8.3) can be reduced to one equation in the one unknown, π_t. For given π_t, we use (8.1) and (8.2) to compute C_t and h_t, respectively. The inflation rate, π_t, is adjusted until (8.3) is satisfied using a numerical zero-finding algorithm. An interior equilibrium requires $\pi_t > 0$. If gross inflation were not positive, sign restrictions on the price level and consumption (see (8.1)) would be violated. That each backward simulation step requires finding the zero of a nonlinear function draws attention to two possibilities: (i) there may be no $\pi_t > 0$ that solves the nonlinear equation for some t, in which case there is no interior (i.e., where the efficiency conditions hold with equality) equilibrium and (ii) there may be multiple values of $\pi_t > 0$ that solve the nonlinear equation. To ensure that we do not miss (ii), we initiate the zero-finding for each t

[10] When there is a given initial state, then this backward solution strategy requires backward 'shooting', as in Christiano, Braggion and Roldos (2009). To see how this backward approach works in a stochastic setting when the equilibrium conditions have been linearized, see http://faculty.wcas.northwestern.edu/~lchrist/course/Korea_2012/fixing_interest_rate.pdf

by first placing a fine grid on a range of values of π_t that extends from nearly zero to 1.02. As before, we solve the model twice. The first time, we solve it for the case where G_t is unchanged relative to its value in steady state. In the other case, G_t is increased by 5 percent in each of periods $t = 1, ..., T-1$. This second computation allows us to deduce the government spending multiplier.

We now turn to the solution of the log-linearized system, (5.1)-(5.3). It is convenient to express the Phillips curve in terms of $\hat{y}_t$, as in (5.4) (except, here $p = 1$):

$$\hat{y}_t = \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[\hat{\pi}_t - \beta\hat{\pi}_{t+1} + \kappa\frac{\eta_g}{1-\eta_g}\hat{G}_t\right]. \tag{8.4}$$

Conditional on $\hat{\pi}_{t+1}$, this represents a linear restriction across $\hat{y}_t$ and $\hat{\pi}_t$. The IS curve and policy rule (i.e., (5.1) and (5.3)) are:

$$\begin{aligned} \hat{y}_t - \eta_g\hat{G}_t &= \hat{y}_{t+1} - \eta_g\hat{G}_{t+1} - \left(1-\eta_g\right)\left[\beta\left(R_t - 1 - r_t\right) - \hat{\pi}_{t+1}\right] && (8.5) \\ R_t &= \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_t - 1\right)\right\}, && (8.6) \end{aligned}$$

respectively. We solve (8.4)-(8.6) by applying the same backward simulation strategy just described for (8.1)-(8.3). The calculations are initiated by setting $\hat{\pi}_T = \hat{y}_T = 0$. At the t^{th} step, we take $\hat{\pi}_{t+1}$ and $\hat{y}_{t+1}$ as given and compute $\hat{\pi}_t$ and $\hat{y}_t$ that solve (8.4)-(8.6). We convert the problem of solving (8.4)-(8.6) into that of solving one (nonlinear, because of (8.6)) equation in one unknown, $\hat{\pi}_t$. Given $\hat{\pi}_t$ we use (8.4) to solve for $\hat{y}_t$ and (8.6) to solve for R_t. Finally, we adjust $\hat{\pi}_t$ until (8.5) is satisfied, if this is possible.

We compute the government consumption multiplier at each date by $\left(\tilde{y}_t - y_t\right) / \left(\tilde{G}_t - G_t\right)$, where y_t denotes GDP ($\equiv C_t + G_t$) and a tilde over a variable indicates

$$G_t = \begin{cases} 1.05 \times G^h & t = 1, ..., T-1 \\ G^h & t = T \end{cases},$$

where G^h denotes the value of government consumption in steady state. Absence of a tilde indicates that $G_t = G^h$ for all t.

We found that for our baseline parameter values, there is a unique equilibrium for the non-linear equilibrium conditions. As a result, we have a unique representation of the level of GDP at each date. In the linear approximation, there are two locally valid representations of the level of GDP, corresponding to two interpretations of $\hat{y}_t$ that are equivalent local to steady state:

$$\hat{y}_t = \begin{cases} \frac{y_t - y}{y} \\ \log(y_t/y) \end{cases}.$$

Here, y denotes the steady state value of y_t.[11] This gives rise to two ways of computing the level of GDP :

$$y_t = \begin{cases} (\hat{y}_t + 1)\, y \\ y \exp(\hat{y}_t) \end{cases}.$$

Similarly, there are two representations of the government consumption multiplier:

$$\frac{dy_t}{dG_t} = \begin{cases} \frac{\widetilde{\hat{y}}_t - \hat{y}_t}{\eta_g \left[\widetilde{\hat{G}}_t - \hat{G}_t\right]} \\ \frac{\exp\left(\widetilde{\hat{y}}_t\right) - \exp(\hat{y}_t)}{\eta_g \left[\exp\left(\widetilde{\hat{G}}_t\right) - \exp\left(\hat{G}_t\right)\right]} \end{cases}.$$

As before, a tilde signifies the equilibrium in which government consumption is 5 percent above steady state for $t = 1, ..., T-1$ and absence of a tilde indicates government spending is always at its steady state. Close to steady state, these two ways of computing levels and the multiplier are equivalent. However, far from steady state they are very different. For example, in the case of the exponential transformation,

[11] To see that these are equivalent, for y_t close to y, note that under the first interpretation:

$$1 + \hat{y}_t = \frac{y_t}{y},$$

and that for $\hat{y}_t$ small, $\log(1 + \hat{y}_t) \simeq \hat{y}_t$.

y_t is guaranteed to be non-negative, while in the case of the other transformation non-negativity could be violated.

A numerical simulation is reported in Figure 7. We suppose that $T = 22$ and the model parameters values are set at their baseline values, (7.1) (of course, $p = 1$ here). In the figure, the solid line represents the exact equilibrium, computed using the nonlinear equations. The starred line represents the equilibrium approximated using the linear approximation, with exponential transformation used to compute the level of GDP (see 'linear, exponential'). The line with circles indicates the effects of using the linear transformation on $\hat{y}_t$ to compute the level of GDP (see 'linear, non-exponential').

Consider the exact equilibrium first. Note that in the first period, output is roughly 0.3, which is a substantial 70 percent below steady state. In that first period there is a massive deflation, amounting to -60 percent per quarter. The government spending multiplier is 4 in the first period, and declines monotonically thereafter. Although the shock driving the economy into the zlb does not lift until period 22, the zlb ceases to bind in period 18. The figure can be used to infer the equilibrium for other values of T. For example, to see the equilibrium for $T = 15$, simply treat $t = 8$ as the first period. Similarly, for larger values of T, the initial period extrapolates the results in the figure to the left. From this it is not surprising that as T is increased a little beyond $T = 22$, there ceases to exist a interior equilibrium.[12] For larger values of T, gross inflation 'wants' to go negative in the initial period. This is not an interior equilibrium because it formally implies a negative price level and negative consumption (see (8.1)). The fact that the output drop in the initial periods is greater for larger values of T is another manifestation of the finding in previous

[12] Recall, by interior equilibrium we mean one in which prices and quantities are non-zero and efficiency conditions hold with equality.

sections that the zlb is more severe for larger values of p. For the intuition behind this result, see CER.

Consider now the performance of the linear-approximate solution. As one would expect, when the system gets very far away from steady state, the accuracy of the approximation deteriorates. From this perspective, it is very surprising that the deterioration is negligible for inflation and the interest rate. In terms of output and the multiplier, the deterioration in performance is very severe for the non-exponential transformation. For example, output is negative in periods 1 to 4. The multiplier is over twice as large as its correct value in the first few periods. Interestingly, approximation based on the exponential transformation of $\hat{y}_t$ is quite good, even very far away from steady state.

Figure 7 suggests that if we confine ourselves to the portion of the figure where output is within 20 percent of its steady state value (i.e., periods after $t = 10$), then the approximation works very well, regardless of whether the linear or exponential transformation of $\hat{y}_t$ is used. For example, in the exact equilibrium, output is 0.81 in period $t = 10$, or 19 percent below steady state. According to the linear approximation, with non-exponential transformation on $\hat{y}_t$ output is 0.77 in $t = 0$ and with the exponential transformation on $\hat{y}_t$ output is 0.80 in period $t = 10$. These errors in approximation are small, particularly when we bear in mind that output is so very far from steady state at $t = 10$. If we consider dates when output is closer to steady state approximation error essentially vanishes. For example, in period $t = 13$ output is 0.89, after rounding, in the exact equilibrium as well as in the two versions of the linear approximation. As long as we use the exponential transformation on $\hat{y}_t$, the approximation works well even when output is 50 or 60 percent away from its steady state value. On the whole, Figure 7 provides substantial evidence in favor of the accuracy of working with linear approximations.

9. Conclusion

We listed three conclusions reached by the literature on the zlb, obtained using linearized equilibrium conditions. We find that these conclusions are robust to working with non-linear equilibrium conditions and allowing for sunspot equilibria. Our finding rests crucially on the use of E-learnability as an equilibrium selection device. The plausibility of the E-learning criterion depends on the plausibility of the model of learning used. We have explored one model of learning. A caveat to our analysis is that there may be another model of learning that changes our results. We are currently exploring other such approaches to learning.

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Figure 1: BSGU Demonstration of Two Steady States

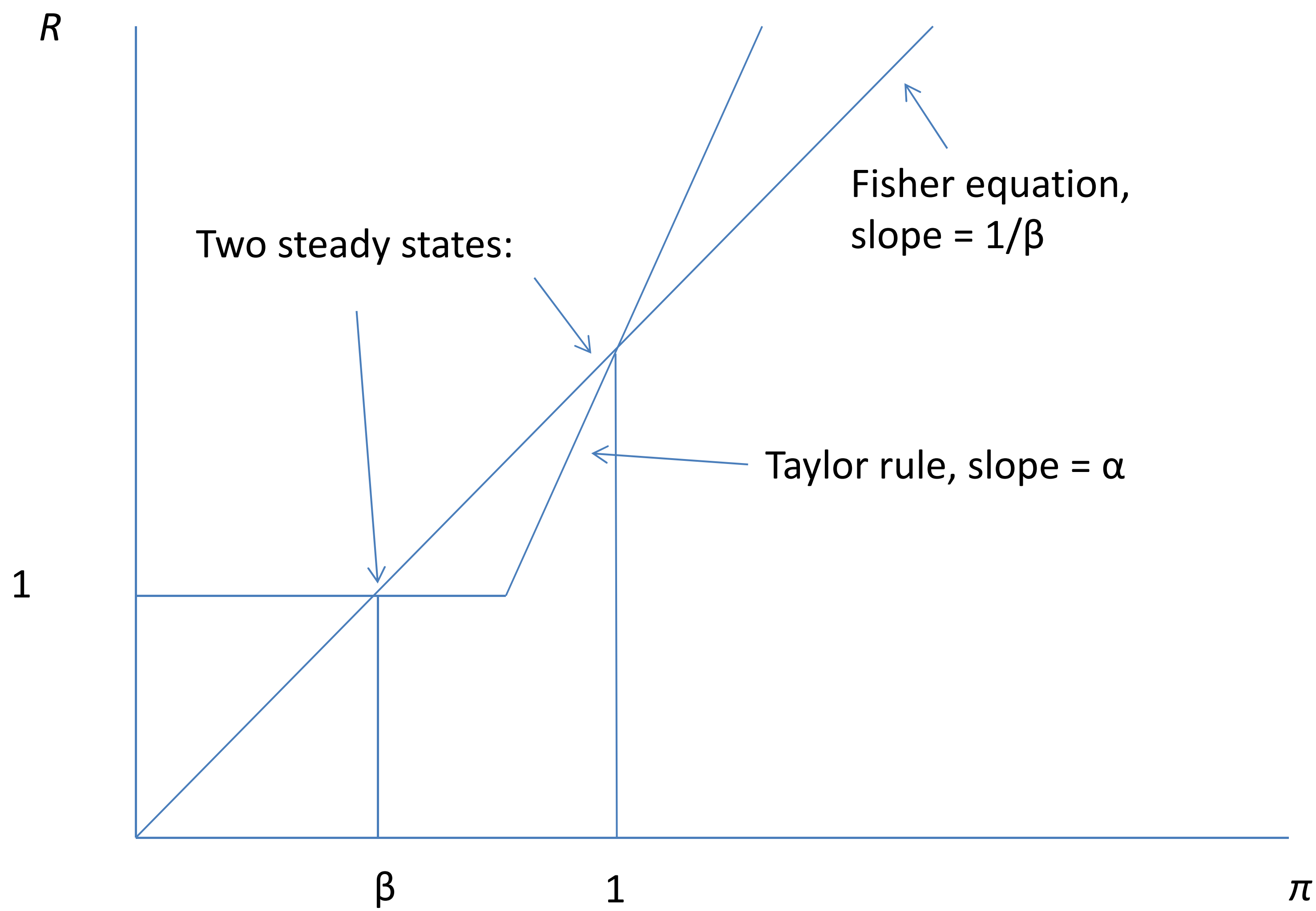


Figure 2: EW Equilibria

Interval of candidate EW equilibrium inflation rates: [0.78,2.27]. There are no other zeros.

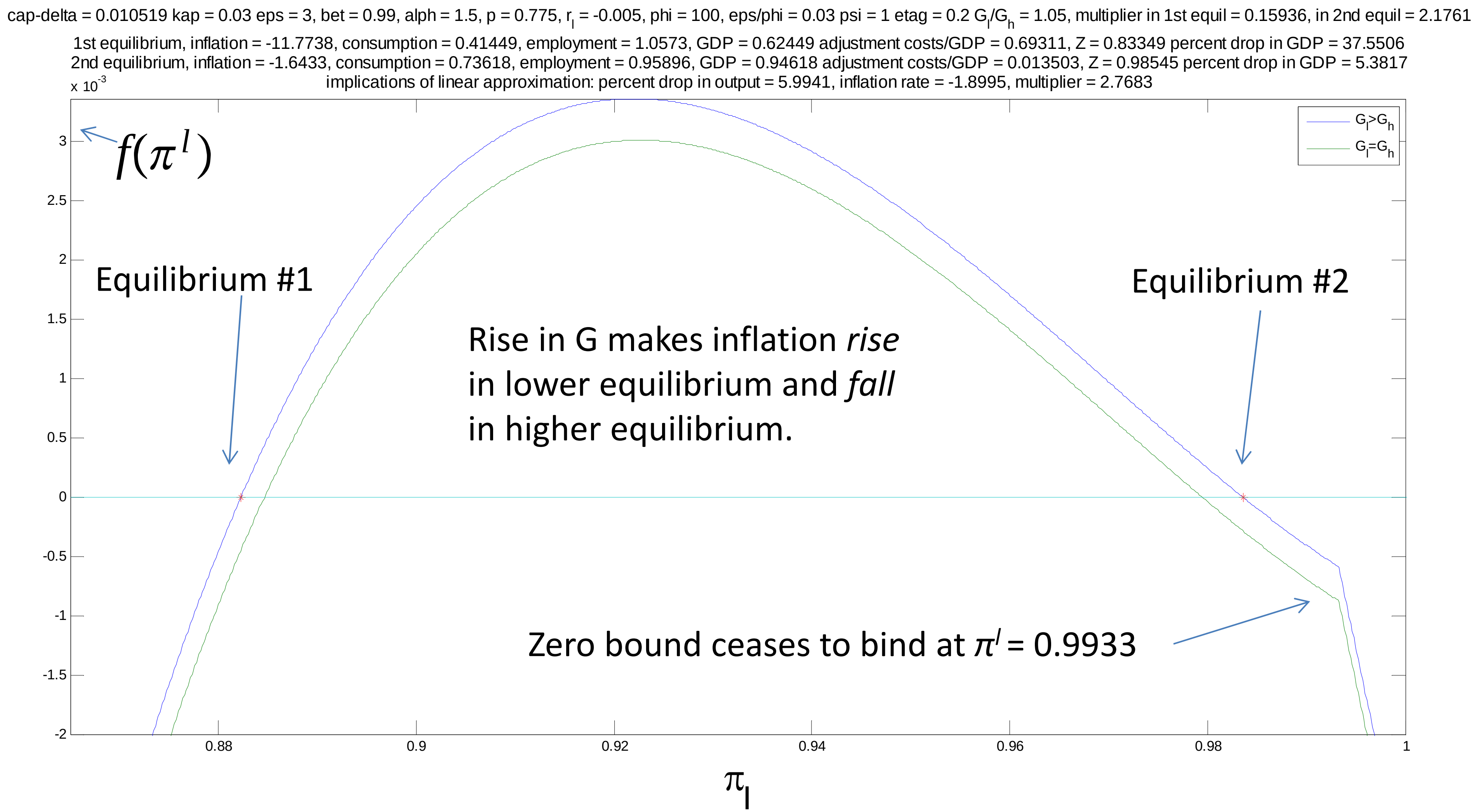
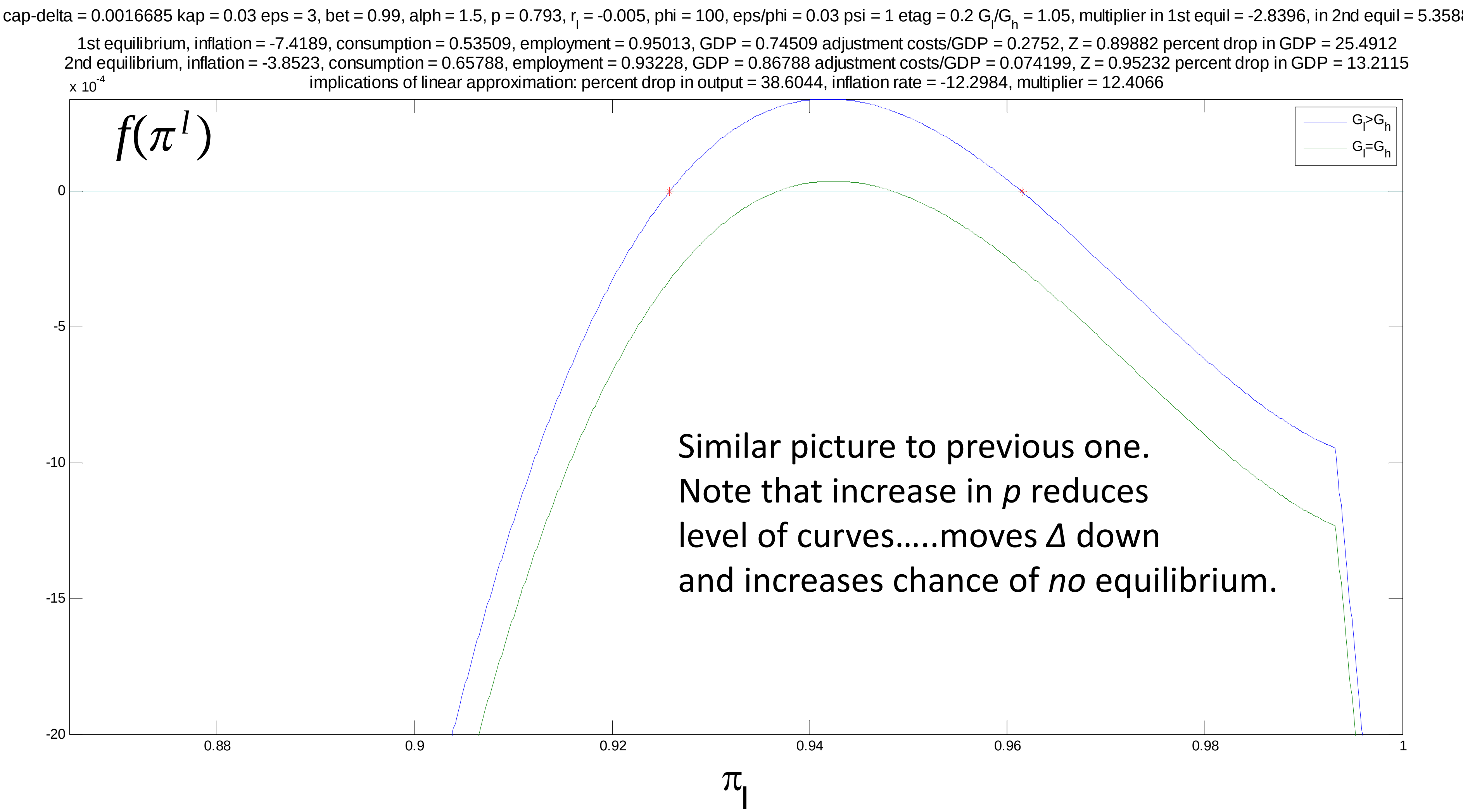


Figure 3: Longer Expected Duration



Similar picture to previous one.
Note that increase in *p* reduces
level of curves…..moves *Δ* down
and increases chance of *no* equilibrium.

Figure 4: More Flexible Prices

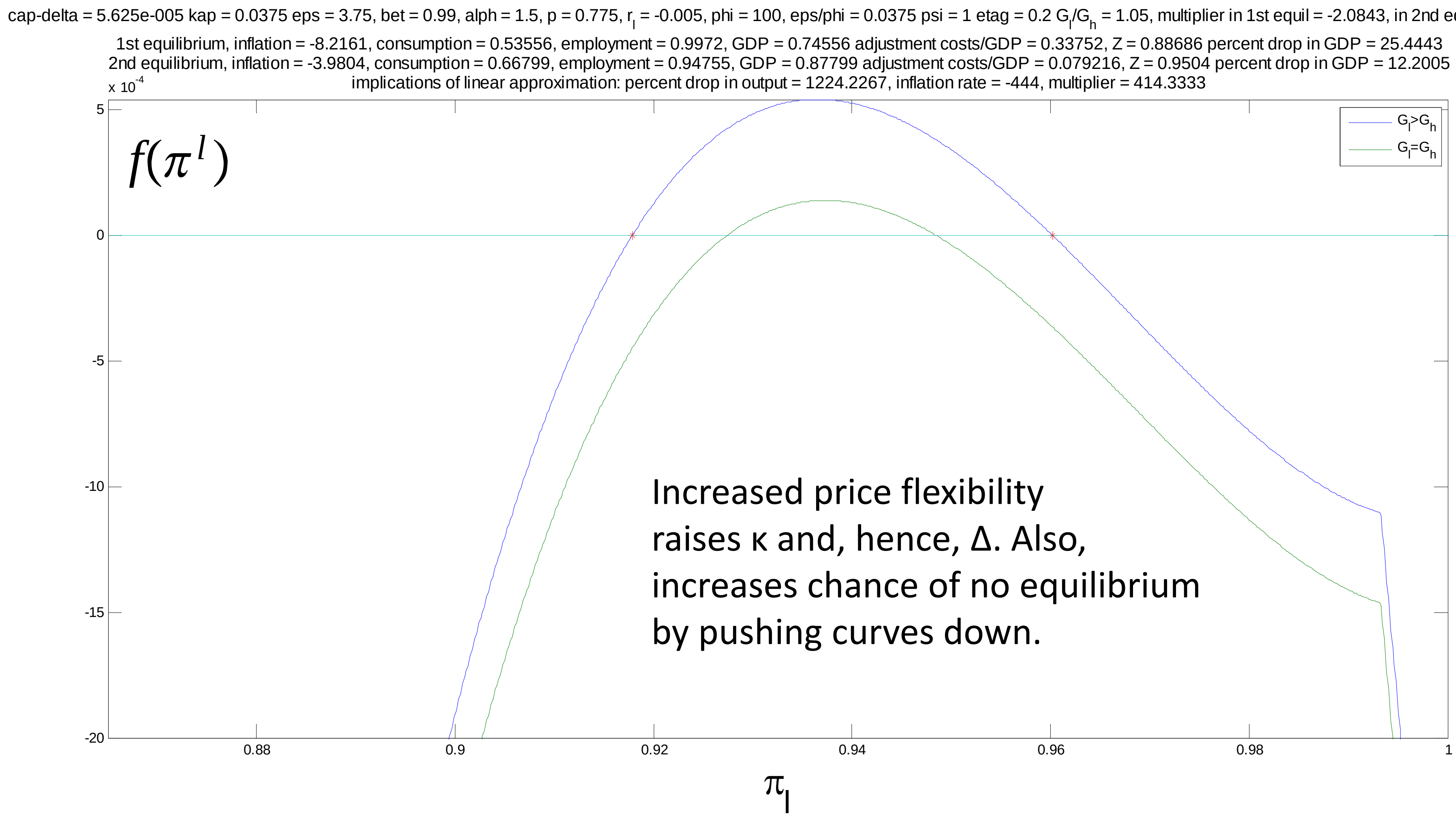


Figure 5: Sunspot Equilibrium

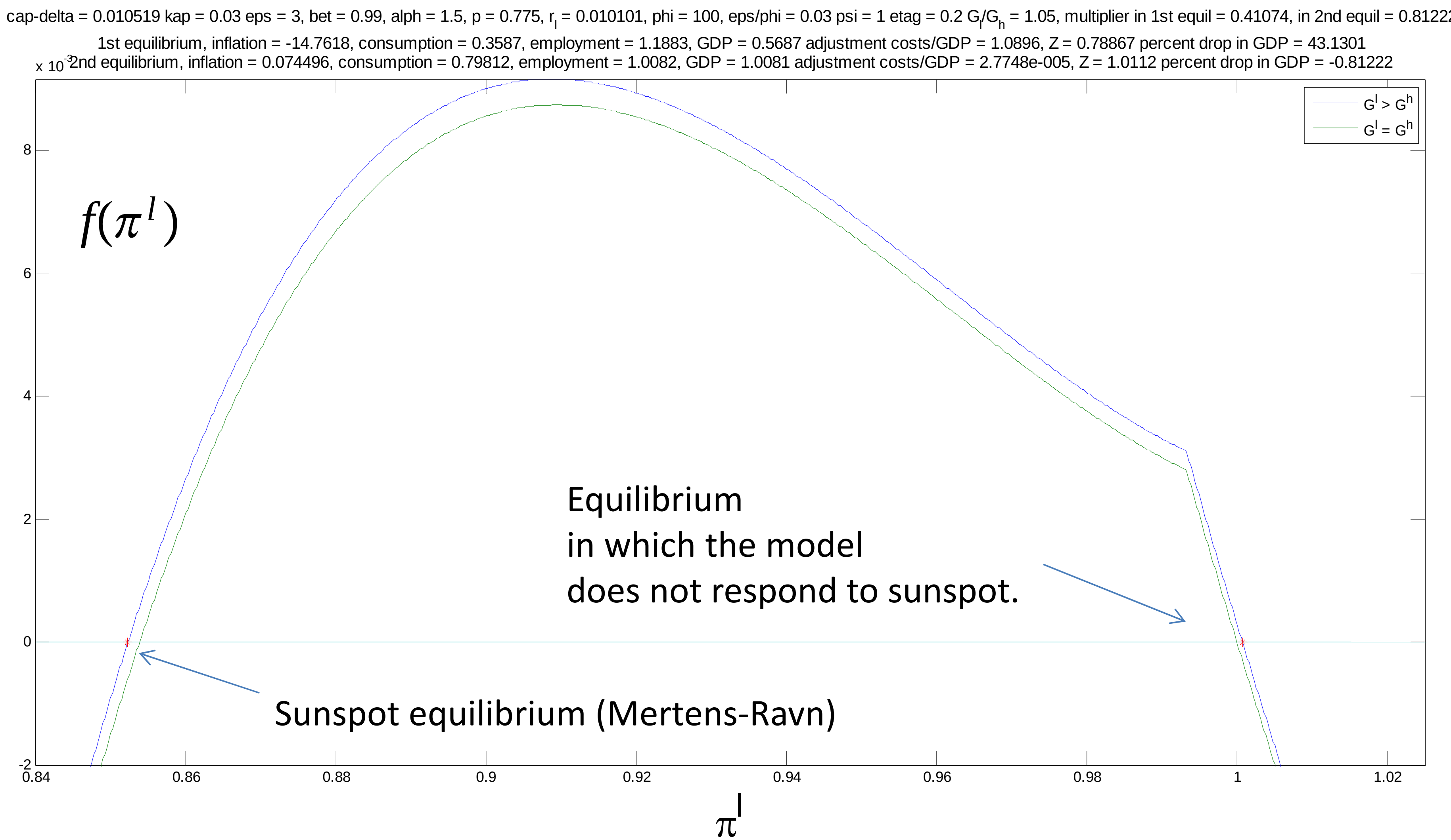


Figure 6: No Equilibrium

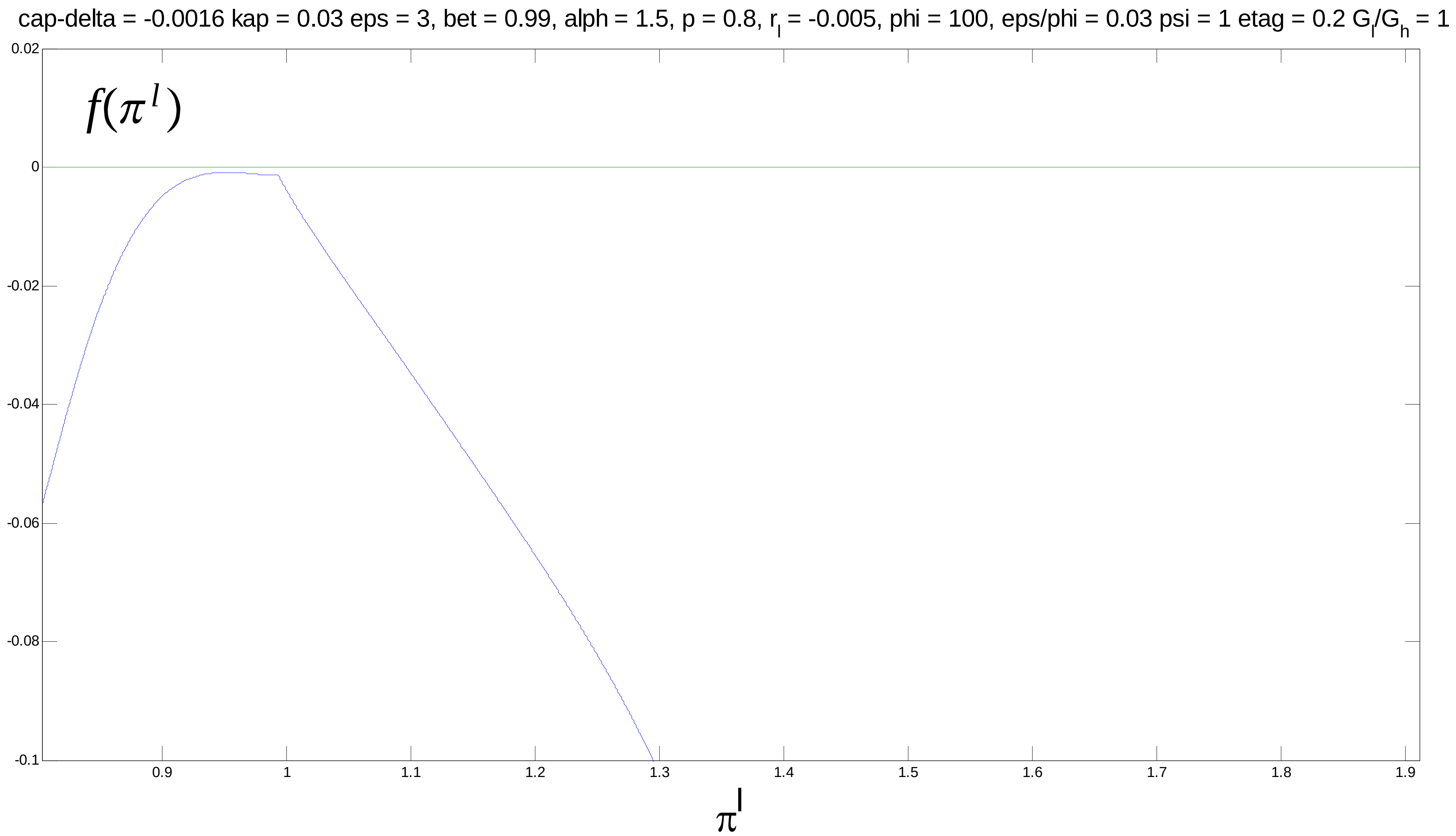


Figure 7: Deterministic Simulation