For Online Publication: Theoretical Results Appendix to "Financialization in Commodity Markets",

V.V. Chari and Lawrence Christiano

A Appendix: Proofs of Results for Model

A1. Share of Commodities in Production

In the text, we adopt the following production function for the final good, Q, as a function of the commodity, q :

$$Q = q^{\delta},\ \frac{1}{2} < \delta < 1.$$

Our assumption on the size of δ may give the impression that the share of the commodity in the production of final goods must be very high for our analysis to be relevant. Here, we point out that our δ is consistent with the notion that the share of commodities in production is quite low. To do this, we extend the model to include another input, say x. Thus, suppose that the production function for Q is given by

$$Q = q^{\tilde{\delta}} x^{\omega},\ \omega, \tilde{\delta} \geq 0,\ 0 < \omega + \tilde{\delta} < 1. \tag{A.1}$$

The market price of q and x are P and w, respectively. Note that the technology exhibits decreasing returns to scale, which affects our implicit assumption that in addition to q and x, the producer also possesses a fixed amount of another factor (e.g., managerial talent). Profits represent the return on that factor.

Profits are given by

$$P^Q g(q, x) - Pq - wx.$$

Profit maximization leads to the following first order conditions:

$$\tilde{\delta} P^Q Q = Pq \tag{A.2}$$

$$\omega P^Q Q = wx, \tag{A.3}$$

so that, after taking ratios,

$$\frac{\omega}{\tilde{\delta}} Pq = wx.$$

Using this and (A.2), we can express profits as follows

$$P^Q Q - Pq - wx = \left(\frac{1}{\delta} - 1\right) Pq,$$

where

$$\delta \equiv \frac{\tilde{\delta}}{1-\omega}.$$

Thus, $\delta \in (1/2, 1)$ is consistent with a small share of the commodity, $q,$ in final good production.

A2. Deriving Equilibrium Pricing Functions

Substituting for H^w,H^b and H^o into (15) from (7), (12) and (14), respectively, we obtain

$$\frac{E\left(P-F\right)}{\alpha var\left(P-F\right)} - \lambda q \frac{1}{\delta} + \left(1-2\lambda\right) s \frac{\sigma_\eta^2}{\sigma_\varepsilon^2} = 0. \tag{A.4}$$

Reproducing (11) we have,

$$P = D_0 - D_q q + \theta + \varepsilon. \tag{A.5}$$

We guess that

$$F = F_0 + F_\theta \theta + F_s s, \tag{A.6}$$

where F_0, F_θ, F_s are to be determined.

Substituting for P and F from (A.5) and (A.6) into (A.4), using $var\left(P-F\right) = \sigma_\varepsilon^2,$ we obtain

$$D_0 - F_0 + \left(1-F_\theta\right)\theta - dq + \left[\left(1-2\lambda\right)\alpha\sigma_\eta^2 - F_s\right] s = 0, \tag{A.7}$$

where

$$d \equiv D_q + \lambda\alpha\sigma_\varepsilon^2/\delta. \tag{A.8}$$

We guess that $q = q_0 + q_\theta\theta + q_s s.$ We use the first order condition for q, (6), to express q_0, q_θ and q_s in terms of the equilibrium future's pricing function given in (A.6):

$$F_0 + F_\theta\theta + F_s s = \bar{c} + cq_0 + cq_\theta\theta + cq_s s$$

From this we see that

$$q_0 = \frac{F_0 - \bar{c}}{c},\ q_\theta = \frac{F_\theta}{c},\ q_s = \frac{F_s}{c}. \tag{A.9}$$

Substituting, from (A.9) into (A.7) and solving for the future's pricing function:

$$D_0 - F_0 - d\frac{F_0 - \bar{c}}{c} + \left(1 - F_\theta - d\frac{F_\theta}{c}\right)\theta + \left[\left(1-2\lambda\right)\alpha\sigma_\eta^2 - F_s - d\frac{F_s}{c}\right] s = 0$$

or, after using (A.8),

$$
\begin{aligned}
F_0 &= \frac{cD_0 + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta\bar{c}}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
F_\theta &= \frac{c}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
F_s &= \frac{c\,(1-2\lambda)\,\alpha\sigma_\eta^2}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta},
\end{aligned} \tag{A.10}
$$

Substituting from (A.10) and (A.8) into (A.9), we have

$$
q_0 = \frac{D_0 - \bar{c}}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta},\; q_\theta = \frac{1}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta},\; q_s = \frac{(1-2\lambda)\,\sigma_\eta^2\alpha}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta}. \tag{A.11}
$$

We now develop the equilibrium price function. Substituting the equilibrium q function from (A.11) into (A.5), we obtain

$$
P = P_0 + P_\theta\theta + P_s s + \varepsilon, \tag{A.12}
$$

where

$$
\begin{aligned}
P_0 &= \frac{D_0\,(c+d) - D_q\,(D_0 - \bar{c})}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
P_\theta &= \frac{c + \lambda\alpha\sigma_\varepsilon^2/\delta}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
P_s &= -\frac{D_q\,(1-2\lambda)\,\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\alpha\sigma_\varepsilon^2}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta},
\end{aligned} \tag{A.13}
$$

where d is defined in (A.7). Also,

$$
R = EP - F = R_0 + R_\theta\theta + R_s s. \tag{A.14}
$$

where

$$
\begin{aligned}
R_0 &= \frac{(D_0 - \bar{c})\,\lambda\alpha\sigma_\varepsilon^2/\delta}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
R_\theta &= \frac{\lambda\alpha\sigma_\varepsilon^2/\delta}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} \\
R_s &= -\frac{(D_q + c)\,(1-2\lambda)\,\alpha\sigma_\eta^2}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta}.
\end{aligned} \tag{A.15}
$$

We now provide a proof of Lemma 1, which we restate here for readability,

The equilibrium return and production rules given in (16) and (17) satisfy: $R_\theta > 0$, $R_s < 0$, and $q_\theta, q_s > 0$.

Proof: Substituting for d from (A.8) into (A.15), we have that

$$R_\theta = \frac{\lambda\alpha\sigma_\varepsilon^2/\delta}{c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta} > 0.$$

Inspection of (A.15) shows that $R_s < 0$.

Substituting from (A.10) into (A.9), we obtain,

$$q_0 = \frac{D_0 - \bar{c}}{c+d},\ q_\theta = \frac{1}{c+d},\ q_s = \frac{(1-2\lambda)\left(1 - \frac{\sigma_\nu^2}{\sigma_\varepsilon^2}\right)\alpha\sigma_\varepsilon^2}{c+d}.$$

Clearly, $q_\theta, q_s > 0$. Q.E.D.

A3. Proof of Proposition1

Proof:

Under the assumption $H^b, H^o > 0$, we have, from (18),

$$oi = \lambda H^b + n,$$

so that

$$cov\,(oi, R) = \lambda cov\left(H^b, R\right) + cov\,(n, R)\,.$$

Thus, $cov\,(oi, R) > cov\,(n, R)$ if and only if $cov\left(H^b, R\right) > 0$.

Using (21) and the equilibrium rule for q, (17),

$$H^b = H_0^b + q_\theta\left(1 - \frac{1}{\delta} + \frac{\lambda}{\delta}\right)\theta + q_s\left(1 - \frac{1}{\delta} + \frac{\lambda}{\delta}\right)s - (1-2\lambda)\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}s,$$

where H_0^b is a constant. Using (16), the covariance between H^b and$R$ is

$$cov\left(H^b, R\right) = q_\theta\left(1 - \frac{1}{\delta} + \frac{\lambda}{\delta}\right)R_\theta\sigma_\theta^2 + \left[q_s\left(1 - \frac{1}{\delta} + \frac{\lambda}{\delta}\right) - (1-2\lambda)\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right]R_s\sigma_s^2.$$

From Lemma 1, q_θ,$R_\theta > 0$, so that under (23) it follows that the first term in the covariance is positive. Also from Lemma 1 $R_s < 0$, so that $cov\left(H^b, R\right) > 0$ if the expression in square brackets is negative. To show that the expression in square brackets is negative, substitute for q_s from (A.11) to obtain

$$q_s\left(1 - \frac{1}{\delta} + \frac{\lambda}{\delta}\right) - (1-2\lambda)\frac{\sigma_\eta^2}{\sigma_\varepsilon^2} = -(1-2\lambda)\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\left[\frac{\alpha\sigma_\varepsilon^2(1-\delta) + \delta(c + D_q)}{\delta(c + D_q) + \alpha\lambda\sigma_\varepsilon^2}\right] < 0,$$

so that $cov\left(H^b, R\right) > 0$.

To show that a value for σ_s^2 exists that sets $cov(n,R) = (1-2\lambda)\, cov(H^o,R) = 0$, substitute for equilibrium production, q, from (17) into (22) to obtain

$$H^o = H_0^o + \left[\lambda q_\theta \frac{1}{\delta}\theta + \lambda q_s \frac{1}{\delta} s + 2\lambda \frac{\sigma_\eta^2}{\sigma_\varepsilon^2} s\right], \tag{A.16}$$

where H_0^o is a constant. Using (A.14) and (A.16), we have

$$cov(H^o,R) = \lambda q_\theta \frac{1}{\delta} R_\theta \sigma_\theta^2 + \left[\lambda q_s \frac{1}{\delta} + 2\lambda \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right] R_s \sigma_s^2. \tag{A.17}$$

Since $q_\theta, R_\theta, q_s > 0$ and $R_s < 0$, it follows immediately that there exists a value for σ_s^2 that sets $cov(n,R) = 0$. Q.E.D.

A4. Proof of Lemma 3

For readability, we reproduce the statement of Lemma 3 here.

Lemma 3. Holding the measure of participating outsiders, $1-2\lambda$, fixed, the surplus from participation, $U^P - U^{np}$, is increasing in σ_θ^2 and σ_s^2. Furthermore, if $\bar{s}$ is sufficiently large and λ is sufficiently close to $1/2$, then the surplus is decreasing in σ_ε^2.

Proof. We begin by proving the first part of the lemma. Consider $var_0(H^o) : R_\theta = \frac{\lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}$

$$\begin{aligned}
var_0(H^o) &= var_0\left(\frac{ER}{\alpha\sigma_\varepsilon^2} + s\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right) \\
&= \left(\frac{R_\theta}{\alpha\sigma_\varepsilon^2}\right)^2 \sigma_\theta^2 + \left[\frac{R_s}{\alpha\sigma_\varepsilon^2} + \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right]^2 \sigma_s^2 \\
&= \left(\frac{\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)^2 \sigma_\theta^2 + \left[-\frac{(D_q+c)(1-2\lambda)\,\sigma_\eta^2/\sigma_\varepsilon^2}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta} + \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right]^2 \sigma_s^2 \\
&= \left(\frac{\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)^2 \sigma_\theta^2 + \left[\frac{-(D_q+c)(1-2\lambda) + c + D_q + \lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right]^2 \left(\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right)^2 \sigma_s^2 \\
&= \left(\frac{\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)^2 \sigma_\theta^2 + \left[\frac{(D_q+c)\,2\lambda + \lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right]^2 \left(\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right)^2 \sigma_s^2
\end{aligned}$$

Substituting from (A.14), we have, after some manipulation,

$$\begin{aligned}
var_0(H^o) &= \left(\frac{R_\theta}{\alpha\sigma_\varepsilon^2}\right)^2 \sigma_\theta^2 + \left[\frac{R_s}{\alpha\sigma_\varepsilon^2} + \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right]^2 \sigma_s^2 \\
&= \left(\frac{R_\theta}{\alpha\sigma_\varepsilon^2}\right)^2 \sigma_\theta^2 + \left[\frac{R_s}{\alpha\sigma_\varepsilon^2} + \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\right]^2 \sigma_s^2
\end{aligned}$$

$$var_0\left(H^o\right)=\left[\frac{2\lambda\left(c+D_q\right)+\lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right]^2\frac{\sigma_\eta^2\sigma_s^2}{\sigma_\varepsilon^2}+\left[\left(\frac{\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)\right]^2\sigma_\theta^2. \tag{A.18}$$

Consider next E_0H^o

$$E_0H^o=\bar{s}\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}+\frac{R_0+R_s\bar{s}}{\alpha\sigma_\varepsilon^2}.$$

Substituting from (A.14), after some manipulation we have

$$E_0H^0=\left[\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}\left(\frac{2\lambda\left(c+D_q\right)+\lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)\right]\bar{s}+\frac{\left(D_0-\bar{c}\right)\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}. \tag{A.19}$$

From (A.18), we see that $var_0(H^o)$ is increasing in σ_θ^2 and σ_s^2. From (A.19) does not dependent on σ_θ^2 and σ_s^2. It follows immediately that surplus is increasing in σ_θ^2 and σ_s^2.

To prove the second part of the lemma, it is convenient to let $h=\sigma_\varepsilon H^o$. We show that the derivative of $\left(E_0h\right)^2$ can be made arbitrarily negative by setting $\bar{s}$ sufficiently large and λ sufficiently close to $1/2$. Note that this derivative is given by

$$2E_0\left(h\right)\times\frac{dE_0\left(h\right)}{d\sigma_\varepsilon^2},$$

where from (A.19) we have

$$E_0h=\left[\frac{\sigma_\eta^2}{\sigma_\varepsilon}\left(\frac{2\lambda\left(c+D_q\right)+\lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right)\right]\bar{s}+\frac{\sigma_\varepsilon\left(D_0-\bar{c}\right)\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}.$$

Clearly, E_0h is increasing in $\bar{s}$. Next, note that if $\lambda=1/2$, then the derivative of Eh can be made arbitrarily negative if$\bar{s}$ is sufficiently large. By continuity this derivative is arbitrarily negative forλ sufficiently close to $1/2$. Thus, the derivative of $\left(E_0h\right)^2$ can be made arbitrarily negative.

Consider next $var_0\left(h\right)$. From (A.18) we have

$$var_0\left(h\right)=\left[\frac{2\lambda\left(c+D_q\right)+\lambda\alpha\sigma_\varepsilon^2/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right]^2\sigma_\eta^2\sigma_s^2+\left[\frac{\sigma_\varepsilon\lambda/\delta}{c+D_q+\lambda\alpha\sigma_\varepsilon^2/\delta}\right]^2\sigma_\theta^2.$$

The derivative of $var_0\left(h^o\right)$ with respect to σ_ε^2 is independent of $\bar{s}$. Thus, it follows that the surplus of participating outsiders is decreasing σ_ε^2 if $\bar{s}$ is sufficiently large and λ is sufficiently close to $1/2$. □