Firm-Specific Capital and Aggregate Inflation Dynamics in Woodford's Model[1]

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This note exposits Woodford's (2004) strategy for solving the model with capital in Woodford (2003)'s chapter 5. In this model, the capital used by firms is completely firm-specific, and so the standard assumption that there are aggregate rental markets in capital services is ruled out. Similarly, there is no economy-wide market for labor. The assumption that factors are firm-specific only complicates the computation of the parameters in the reduced form equation characterizing aggregate inflation dynamics. All the other equilibrium conditions coincide with the standard ones that occur in the version of Woodford's model in which capital and labor are homogeneous and are traded in economy-wide factor markets. This is why this note focuses specifically on the equilibrium equation characterizing aggregate inflation dynamics.

Woodford's (2004) strategy for computing the parameters of the reduced form inflation equation as a function of model parameters is based on undetermined coefficients. MATLAB programs for doing these calculations are available on the author's web site.[2]

The details of the equation governing aggregate inflation dynamics depend on the details of the model. It turns out that the undetermined coefficient method discussed here is quite flexible. It has been extended to incorporate Kimball (1995)'s specification of demand by Eichenbaum and Fisher (2004). It has been extended to a model with higher-order adjustment costs in investment and sticky wages in Altig, Christiano, Eichenbaum, and Linde (2004). For a discussion of the motivation for analyzing models with firm-specific factors of production, see the recent papers by Altig, Christiano, Eichenbaum and Linde (2004), Coenen and Levin (2004), de Walque, Smets and Wouters (2004) and Sveen and Weinke (2004a,b).

1 The Model

The preferences of households are as follows:

$$\sum_{t=0}^{\infty} \beta_t \left[u(C_t, M_t/P_t) - \int_0^1 v(H_t(j))dj \right],$$

where u is increasing and concave in its first argument, v is increasing and convex, and

$$\beta_t = \frac{1}{(1+r_0^n)(1+r_1^n)\cdots(1+r_{t-1}^n)},$$

for $t = 1, 2, ...$. Also, $\beta_0 \equiv 1$ and

$$\frac{\beta_{t+1}}{\beta_t} = \frac{1}{1+r_t^n}.$$

[1] In preparing this note I have benefitted greatly from discussions with Jonas Fisher and Roc Armenter.

[2] The url for the software is
http://www.faculty.econ.northwestern.edu/faculty/christiano/research/firmspecific/computegamma.zip

Each household supplies every type of labor, $j \in (0,1)$. Here, C_t denotes consumption and M_t denotes the household's end-of-period t stock of money. Finally, P_t denotes the price of the consumption good. The household's flow budget constraint is:

$$P_t C_t + M_t + B_{t+1} \leq M_{t-1} + B_t(1 + i_{t+1}) + \int_0^1 P_t w_t(j) H_t(j) dj + T_t, \tag{1}$$

where B_t denotes the beginning-of-period t stock of bonds, purchased in period $t-1$. Also, $w_t(j)$ denotes the real wage rate paid to type j labor, and T_t denotes lump sum profits and transfers from the government. We suppose there is a lower bound constraint on B_t. Households are competitive in goods and labor markets.

Final goods are produced using intermediate goods by a representative, competitive firm using the following Dixit-Stiglitz production function:

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \ \theta > 1.$$

The first order condition for profit maximization by the final good firm is:

$$\left(\frac{Y_t}{y_t(j)} \right)^{\frac{1}{\theta}} = \frac{P_t(j)}{P_t}. \tag{2}$$

The i^{th} intermediate good is produced by a monopolist using the following technology:

$$y_t(i) = K_t(i) f\left(\frac{h_t(i)}{K_t(i)} \right). \tag{3}$$

Here, $K_t(i)$ is the capital owned by the monopolist, and $h_t(i)$ is the quantity of labor hired. The firm is competitive in the market for type i labor, and takes the wage rate, $w_t(i)$, as given.[3] Investment by the i^{th} monopolist produces new capital in the next period according to the following adjustment cost function:

$$I_t(i) = I\left(\frac{k_{t+1}(i)}{k_t(i)} \right) k_t(i). \tag{4}$$

Here, investment, $I_t(i)$, corresponds to purchases of the final good. Also, $I(1) = \delta$, $I'(1) = 1$, $I''(1) = \epsilon_\psi > 0$.

[3] At first glance, this may seem odd, because according to the formalism in the text, the j^{th} intermediate good producer is the only employer of type j labor. This suggests that the producer must be a monopsonist. We follow an alternative interpretation suggested by Woodford (2003), which rationalizes competitive labor markets. Think of the j^{th} intermediate good producer as being a member of an industry composed of intermediate goods producers with indeces lying in a small neighborhood, J, of j. Suppose there is a finite, but large, number of such industries that do not intersect, but whose union is the unit interval. Imagine that instead of there being a continuum of labor types in the household utility function, there is a discrete number as in the Riemann approximation to the integral of the household utility function. Each of these labor types works in one of the industries. With this setup, there is a continuum of suppliers and demanders in the labor market corresponding to each industry, so that competition makes sense. This is the case even for intervals, J, whose length is very small. This is how we interpret the model. For further discussion, see Woodford (2003, pp. 148-149).

The i^{th} intermediate good firm faces frictions in the setting of its price, $P_t(i)$. With probability $1-\alpha$ it may set its price optimally, and with probability α is must set

$$P_t(i) = P_{t-1}(i)\pi_{t-1}^{\varrho},\ 0 \leq \varrho \leq 1, \tag{5}$$

where ϱ controls the degree of inflation indexation and $\pi_t = P_t/P_{t-1}$.

The present discounted value of profits of the intermediate good firm are:

$$E_t \sum_{j=0}^{\infty} \beta_{t+j}\Lambda_{t+j}\left\{(1+\tau)P_{t+j}(i)y_{t+j}(i) - P_{t+j}w_{t+j}(i)h_{t+j}(i) - P_{t+j}I_{t+j}(i)\right\}. \tag{6}$$

Here, Λ_t denotes the shadow value of a dollar to the household, the owner of the intermediate good firm. It is the multiplier on (1) in the Lagrangian representation of the household's problem. Also, the subsidy, τ, is designed to eliminate the distorting effects of monopoly power in the model. I assume

$$1+\tau = \frac{\theta}{\theta-1}.$$

The i^{th} intermediate good firm chooses $P_{t+j}(i)$, $y_{t+j}(i)$, $h_{t+j}(i)$, $I_{t+j}(i)$ to maximize profits, subject to (2), (3), (4), as well as its price-setting constraints. The firm takes P_{t+j}, Y_{t+j}, τ and $w_{t+j}(i)$ as given.

The resource constraint is:

$$C_t + I_t + G_t = Y_t,$$

where

$$I_t = \int_0^1 I_t(i)di.$$

To fully close the model requires specifying how the monetary authority controls i_t. This could be done in a variety of ways. However, they do not impact on the equation that characterizes inflation dynamics, which is what interests us here. For for our purposes it is enough to simply specify that it take on some value on steady state.

When the curvature on investment adjustment costs, ϵ_ψ, is large, then the stock of capital is a constant. The elastic investment case corresponds to smaller values of ϵ_ψ.

2 Equilibrium Conditions

This section develops the equations that characterize equilibrium for the model. The first order necessary conditions for household optimization include a transversality condition and:

$$\begin{aligned}
\frac{v'(H_t(j))}{u'_t} &= w_t(j) \\
u'_t &= \frac{1}{1+r_t^n}u'_{t+1}\frac{1+i_t}{\pi_{t+1}} \\
\frac{u_{m,t}}{u'_t} &= \frac{i_t}{1+i_t}
\end{aligned}$$

Here, u_t' denotes the marginal utility of consumption. The first equation is the intratemporal Euler equation for labor. The second equation is the intertemporal Euler equation associated with the household saving decision. The third equation is the Euler equation associated with real balances. From here on, I assume u_t' is not a function of M_t/P_t, and I ignore the last first order condition.

The first order necessary condition associated with the optimal choice of capital by the i^{th} firm is:

$$I'\left(\frac{k_{t+1}(i)}{k_t(i)}\right) = \frac{1}{(1+i_t)/\pi_{t+1}}\left\{\rho_{t+1}(i) - I\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right) + I'\left(\frac{k_{t+2}(i)}{k_{t+1}(i)}\right)\frac{k_{t+2}(i)}{k_{t+1}(i)}\right\}. \tag{7}$$

Here, I have made use of the household's intertemporal Euler equation and the fact, $\Lambda_t = u_{c,t}/P_t$. With some algebra it can be shown that $\rho_{t+1}(i)$ in (7) can be written:

$$\rho_{t+1}(i) = w_{t+1}(i)\frac{f(\bar{h}_{t+1}(i)) - \bar{h}_{t+1}(i)f'(\bar{h}_{t+1}(i))}{f'(\bar{h}_{t+1}(i))}. \tag{8}$$

One interpretation of $\rho_{t+1}(i)$ is that it is the real rental rate of capital that would rationalize the amount of capital used by the intermediate good firm in $t+1$, if there were a competitive capital rental market.[4]

If the i^{th} firm has the opportunity to reoptimize its price, then it does so, taking into account that it must satisfy its demand curve and that it will not be able to reoptimize again, with probability α. The firm's first order condition is, after some algebra,

$$\sum_{j=0}^{\infty}\alpha^j\beta_{t+j}\lambda_{t+j}P_{t+j}^{\theta}Y_{t+j}\left\{p_{t+j}(i) - \frac{\theta}{(\theta-1)(1+\tau)}s_{t+j}(i)\right\} = 0, \tag{9}$$

where

$$p_{t+j}(i) = \frac{P_t(i)}{P_{t+j}},$$

where $P_t(i)$ is the price set in period t. Also, $s_{t+j}(i)$ is the marginal cost of producing an extra unit of output:

$$s_t(i) = \frac{w_t(i)}{f'\left(\frac{h_t(i)}{K_t(i)}\right)}. \tag{10}$$

According to (9), the firm sets its price to an after-tax markup over marginal cost on average.

3 Log-Linearized Equilibrium Conditions

I first develop formulas for the model's steady state. I then log-linearize the equilibrium conditions about the steady state.

[4] See Woodford (2003, p. 355) for an alternative interpretation.

3.1 Steady State

We first compute the capital-to-output ratio, k, and consumption-to-output ratio, c. In steady state, (7) reduces to:

$$\frac{1}{\beta} = \rho - \delta + 1,$$

where ρ is the steady state value of the first object in braces in (7). Then, $\rho = (1/\beta) + \delta - 1$, which, when combined with (8) evaluated in steady state, yields:

$$\frac{1}{\beta} + \delta - 1 = w\frac{f - \bar{h}f'}{f'}.$$

In steady state, all prices are equal, so that $p_t(i) = 1$. This, and the fact that the object in braces in (9) is zero in steady state imply

$$s = \frac{(\theta - 1)(1 + \tau)}{\theta}.$$

Combining this with (10), we obtain:

$$w = f'\frac{(\theta - 1)(1 + \tau)}{\theta},$$

which says that the real wage is an after tax markup over the marginal product of labor. Combining these results and rearranging, we obtain:

$$k = \frac{(\theta - 1)(1 + \tau)}{\theta}\left(\frac{\phi - 1}{\phi}\right)\left(\frac{1}{\frac{1}{\beta} + \delta - 1}\right).$$

where $k \equiv K/Y$. Finally, note that in steady state:

$$Y = C + \delta K + G,$$

so that

$$c \equiv \frac{C}{Y} = 1 - \delta k - g,$$

where

$$g = \frac{G}{Y}.$$

3.2 Log-Linear Expansions

Aggregate Goods-Market Clearing

Consider the national income identity:

$$Y_t = C_t + I_t + G_t,$$

where G_t denotes the exogenous level of government spending. Then,

$$\hat{Y}_t = \hat{C}_t + \hat{I}_t + \hat{G}_t.$$

Unless otherwise noted, a hat over a variable indicates deviation from steady state, expressed as a fraction of steady state. That is, $\hat{z}_t \equiv dz_t/z$, where z is the steady state value of z_t and dz_t is a small deviation, $z_t - z$ (I refer to $\hat{z}_t$ as the log deviation of z_t from steady state, or, simply, as the 'log deviation'.) However, in the case of the aggregate quantities appearing in the national income identity, a hat indicates deviation from steady state, expressed as a fraction of steady state aggregate output:

$$\hat{I}_t = \frac{dI_t}{Y},\ \hat{Y}_t = \frac{dY_t}{Y},\ \hat{C}_t = \frac{dC_t}{Y},\ \hat{G}_t = \frac{dG_t}{Y}.$$

Household Intertemporal Condition

Log-linearizing the household's intertemporal Euler equation:

$$\hat{u}_{c,t} = \hat{u}_{c,t+1} - \hat{r}_t^n + \hat{\imath}_t - \hat{\pi}_{t+1}, \tag{11}$$

where

$$\hat{\imath}_t \equiv \frac{i_t - i}{1+i},\ \hat{r}_t^n \equiv \frac{r_t^n - r^n}{1+r^n}.$$

It is useful to develop an expression for $\hat{u}_{c,t}$. We have

$$\begin{aligned} \hat{u}_{c,t} &\equiv \frac{du_{c,t}}{u_c} = \frac{u_{cc}Y}{u_c}\hat{C}_t \\ &= \frac{u_{cc}C}{u_c}\frac{Y}{C}\hat{C}_t \\ &= -\sigma^{-1}\hat{C}_t, \end{aligned}$$

where

$$\begin{aligned} \sigma &= \sigma_u^{-1}c > 0 \\ \sigma_u &= -\frac{u_{cc}C}{u_c} \end{aligned}$$

Then,

$$\hat{u}_{ct} = -\sigma^{-1}\left[\hat{Y}_t - \hat{I}_t - \hat{G}_t\right]. \tag{12}$$

Firm Investment

Loglinearizing the i^{th} intermediate good firm's Euler equation for investment, (7), about steady state, we obtain:

$$\begin{aligned} \hat{u}_{c,t} + \epsilon_\psi\left[\hat{k}_{t+1}(i) - \hat{k}_t(i)\right] &= (1-\delta)\beta\hat{u}_{c,t+1} - \hat{r}_t^n \\ &\quad + [1-(1-\delta)\beta]\left[\rho_y\hat{y}_{t+1}(i) - \rho_k\hat{k}_{t+1}(i)\right] \\ &\quad + \beta\epsilon_\psi\left[\hat{k}_{t+2}(i) - \hat{k}_{t+1}(i)\right], \end{aligned} \tag{13}$$

where

$$\rho_y = \sigma_v \phi + \frac{\omega_p \phi}{\phi - 1}, \ \rho_k = \rho_y - \sigma_v.$$

Exploiting the symmetry of the steady state equilibrium, we have $\hat{K}_t = \int_0^1 \hat{k}_t(i)di$ and $\hat{Y}_t = \int_0^1 \hat{y}_t(i)di$. Integrating (13) over all $i \in (0,1)$, we then obtain:[5]

$$\begin{aligned} \hat{u}_{c,t} + \epsilon_\psi \left[\hat{K}_{t+1} - \hat{K}_t\right] &= (1-\delta)\beta \hat{u}_{c,t+1} - \hat{r}_t^n \\ &\quad + [1-(1-\delta)\beta]\left[\rho_y \hat{Y}_{t+1} - \rho_k \hat{K}_{t+1}\right] \\ &\quad + \beta \epsilon_\psi \left[\hat{K}_{t+2} - \hat{K}_{t+1}\right]. \end{aligned} \tag{14}$$

Note that (14) implies $\hat{K}_{t+1} - \hat{K}_t = (1/\beta)^t \left[\hat{K}_1 - \hat{K}_0\right]$ when $\epsilon_\psi = \infty$. So, the only $\hat{K}_1 - \hat{K}_0$ which implies a non-explosive path for the capital stock is one in which $\hat{K}_1 - \hat{K}_0 = 0$, in which case the capital stock is constant. This is what one expects when adjustment costs are infinite.

Loglinearizing the investment equation, (4):

$$\hat{I}_t = k\left[\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right], \tag{15}$$

where

$$k = \frac{K}{Y}.$$

4 Aggregate Price Dynamics

This section builds on the results in the previous section to derive a simple expression for the dynamics of inflation in the model. Fundamentally, the strategy for accomplishing this coincides with the three-step strategy used in the existing literature on Calvo pricing (see, for example, Yun, 1996). In the first step, one loglinearizes the equilibrium condition relating P_t to the intermediate good prices, to obtain a relationship between the prices of price-optimizing firms and the aggregate inflation rate. In the second step, one uses a loglinearized version of the first order condition, (9), to obtain an expression for the prices chosen by price-optimizing firms in terms of aggregate inflation and other aggregate variables. In the third step, the equations obtained in the first two steps are used to eliminate the prices of price-optimizing firms, to obtain an expression for aggregate inflation dynamics. The details are somewhat more complicated here, owing to the fact that price-optimizing firms choose a different price, depending on how much beginning-of-period capital they own. This is a complication that is absent in the usual Calvo setting, where all price-optimizing firms choose the same price. It turns out that, following the logic laid out in Woodford (2004), the complication can be handled with just a little extra algebra. This logic, suitably adapted to accommodate inflation indexation, establishes that there is an equilibrium in which the inflation process is:

$$\Delta_\varrho \hat{\pi}_t = \gamma \hat{s}_t + \beta \Delta_\varrho \hat{\pi}_{t+1}, \tag{16}$$

[5] Expression (14) corresponds to Woodford (2003, equation 3.7, page 356).

where

$$\Delta_\varrho \hat{\pi}_t \equiv \hat{\pi}_t - \varrho \hat{\pi}_{t-1}.$$

In (16), γ is a function of the structural parameters, and $\hat{s}_t$ is marginal cost, averaged across all firms. I now describe the equations used to compute γ. It is easily verified that the presence of aggregate uncertainty does not change the basic logic below, and requires only replacing $\Delta_\varrho \hat{\pi}_{t+1}$ with $E_t \Delta_\varrho \hat{\pi}_{t+1}$, where E_t is the conditional expectation operator.

Step 1 begins by integrating (2) over $j \in (0,1)$ and imposing the final goods firm technology,

$$\begin{aligned} P_t &= \left[\int P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \\ &= \left[\int_I P_t^*(i)^{1-\theta} di + \int_J P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}, \end{aligned} \tag{17}$$

where $i \in I$ corresponds to the firms which have the opportunity to reoptimize in period t and $j \in J$ corresponds to the firms which do not. Dividing both sides of (17) by P_t, making use of (5), and rearranging,

$$1 = \int_I p_t^*(i)^{1-\theta} di + \alpha \left(\frac{(\pi_{t-1})^\varrho}{\pi_t} \right)^{1-\theta},$$

where $p_t^*(i) = P_t^*(i)/P_t$. Taking into account that $p_t^*(i) = 1$ for all i in steady state, one finds, after loglinearly expanding the last expression:

$$0 = \int_I \hat{p}_t^*(i) di - \alpha \Delta_\varrho \hat{\pi}_t. \tag{18}$$

Following Woodford (2004), I posit (and later verify) that

$$\hat{p}_t^*(i) = \hat{p}_t^* - \psi \tilde{k}_t(i), \ \tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t, \tag{19}$$

where ψ is a scalar and $\hat{p}_t^*$ is a function of aggregate variables only. Both ψ and $\hat{p}_t^*$ are unknowns that are to be determined. Symmetry of the steady state implies $\int_0^1 \hat{k}_t(i) di = \hat{K}_t$. Since the firms which reoptimize their price are chosen at random, it follows that $\int_I \hat{k}_t(i) di = \hat{K}_t$ too, and

$$\int_I \tilde{k}_t(i) di = 0. \tag{20}$$

Substituting (19) and (20) into (18),

$$\hat{p}_t^* = \frac{\alpha}{1-\alpha} \Delta_\varrho \hat{\pi}_t. \tag{21}$$

This completes the discussion of step 1.

To describe step 2, it is useful to first display two simple relationships. For any firm with relative price $\hat{p}_t(i)$ at time t, which it cannot reoptimize in any period up to, and including, time $t+j$, its relative price at time $t+j$ is

$$\hat{p}_{t+j}(i) = \hat{p}_t(i) - \Delta_\varrho \hat{\pi}_{t+j} - ... - \Delta_\varrho \hat{\pi}_{t+1},\ j \geq 1. \tag{22}$$

Log-linearizing (10), one obtains (see Woodford (2003, page 358)) the following expression for the i^{th} firm's marginal cost:

$$\hat{s}_t(i) = \hat{s}_t - \theta\omega\hat{p}_t(i) - (\omega - \sigma_v)\,\tilde{k}_t(i). \tag{23}$$

Here, aggregate marginal cost, $\hat{s}_t$, is given by:$-\sigma^{-1}\left[\hat{Y}_t - \hat{I}_t - \hat{G}_t\right]$

$$\hat{s}_t = \omega\left(\hat{Y}_t - \hat{K}_t\right) + \sigma_v\hat{K}_t - \hat{u}_{c,t}. \tag{24}$$

Substituting (22) and (23) into the loglinearized version of (9) and rearranging, one obtains:

$$\hat{p}_t^*(i) = \sum_{j=1}^{\infty}(\alpha\beta)^j\Delta_\varrho\hat{\pi}_{t+j} + \frac{1-\alpha\beta}{1+\theta\omega}\sum_{j=0}^{\infty}(\alpha\beta)^j\,\hat{s}_{t+j} - (\omega - \sigma_\nu)\frac{1-\alpha\beta}{1+\theta\omega}\hat{E}_t^i\sum_{j=0}^{\infty}(\alpha\beta)^j\,\tilde{k}_{t+j}(i). \tag{25}$$

Here, $\hat{E}_t^i X_{t+k}(i)$ denotes the expectation of the random variable $X_{t+k}(i)$, conditional on date t information and on the event that the i^{th} firm optimizes its price in period t, but does not do so in any period up to and including $t+k$. In principle, the optimally chosen $\tilde{k}_{t+1}(i)$ is a function of prices in all continuation histories after period t for firm i. These histories include periods in which the i^{th} firm is permitted to reoptimize its price. A frontal assault on this expectation is computationally overwhelming. However, Woodford (2004)'s insight that $\tilde{k}_{t+1}(i)$ can be represented as a linear function of $\tilde{k}_t(i)$ and $\hat{p}_t(i)$ simplifies the expectation drastically. In this case, the expectation in (25) only involves prices in future periods when the firm is not permitted to reoptimize. Consistent with this, I posit (and then verify):

$$\tilde{k}_{t+1}(i) = \kappa_1\tilde{k}_t(i) + \kappa_2\hat{p}_t(i), \tag{26}$$

where κ_1, κ_2 are to be determined. For the linear approximation strategy to be reliable, we will require $|\kappa_1| < 1$. (This turns out to be the case, for the numerical examples I have considered.)

To determine the values of the unknown parameters, ψ, κ_1, and κ_2, and the unknown function, $\hat{p}_t^*$, I use the fact that (29) and (25) must be satisfied. Ultimately, γ is the object that we seek, and we shall see that this is a function of ψ, κ_1, κ_2.

Recursive substitution of (26) yields,

$$\hat{E}_t^i\tilde{k}_{t+k}(i) = \kappa_1^k\tilde{k}_t(i) + \frac{1-\kappa_1^k}{1-\kappa_1}\kappa_2\hat{p}_t^*(i) - \kappa_2\left(\Delta_\varrho\pi_{t+k-1} + (1+\kappa_1)\,\Delta_\varrho\pi_{t+k-2} + ... + \frac{1-\kappa_1^{k-1}}{1-\kappa_1}\Delta_\varrho\pi_{t+1}\right).$$

Using this to evaluate the last expectation in (25), and solving for $\hat{p}_t^*(i)$, I obtain (19) with

$$\hat{p}_t^* = \sum_{j=1}^{\infty}(\alpha\beta)^j E_t\Delta_\varrho\hat{\pi}_{t+j} + \frac{(1-\alpha\beta\kappa_1)(1-\alpha\beta)}{(1+\theta\omega)(1-\alpha\beta\kappa_1) + (\omega-\sigma_\nu)\,\alpha\beta\kappa_2}\sum_{j=0}^{\infty}(\alpha\beta)^j\,\hat{s}_{t+j} \tag{27}$$

and

$$\psi = \frac{(\omega - \sigma_\nu)(1 - \alpha\beta)}{(1 + \theta\omega)(1 - \alpha\beta\kappa_1) + (\omega - \sigma_\nu)\alpha\beta\kappa_2} \equiv \psi(\kappa_1, \kappa_2). \tag{28}$$

This derivation assumes $|\alpha\beta\kappa_1| < 1$.

The parameters, κ_1 and κ_2, must also satisfy the intermediate good firms' optimality condition for investment. Subtract (14) from (13), use (2) and rearrange, to obtain:

$$E_t\left[Q(L)\tilde{k}_{t+2}(i)\right] = \Xi E_t \hat{p}_{t+1}(i), \tag{29}$$

where E_t is the expectation operator, conditional on information dated t and earlier, for firm i. Equation (29) must hold, for all intermediate good firms, whether or not they have the opportunity to reoptimize their price. In (29), $Q(L)$ and Ξ are defined as follows:

$$\begin{aligned} Q(L) &= \beta - \phi L + L^2, \\ \phi &= 1 + \beta + (1 - \beta(1 - \delta))\rho_k \epsilon_\psi^{-1} \\ \Xi &= (1 - \beta(1 - \delta))\,\rho_y \theta \epsilon_\psi^{-1}. \end{aligned}$$

It is useful to first simplify the conditional expectation to the right of (29). Following the lead in Woodford (2004), I use (19), (21) and (22), to obtain, after rearranging:

$$E_t \hat{p}_{t+1}(i) = \alpha \hat{p}_t(i) - (1 - \alpha)\,\psi \tilde{k}_{t+1}(i). \tag{30}$$

Substitute (26) and (30) into (29), to obtain, after rearranging,

$$\begin{aligned} &\left[1 - \phi\kappa_1 + \beta\kappa_1^2 - (1 - \alpha)\,\psi\,(\beta\kappa_2 - \Xi)\,\kappa_1\right]\tilde{k}_t(i) \\ &+ \left[-\phi\kappa_1 + \beta\kappa_1^2 + \beta\kappa_1\alpha - \Xi\alpha\frac{\kappa_1}{\kappa_2} - (1 - \alpha)\,\psi\,(\beta\kappa_2 - \Xi)\,\kappa_1\right]\frac{\kappa_2}{\kappa_1}\hat{p}_t(i) = 0. \end{aligned}$$

For this to be satisfied for all possible $\tilde{k}_t(i)$ and $\hat{p}_t(i)$ requires that the coefficients on $\tilde{k}_t(i)$ and $\hat{p}_t(i)$ be zero:

$$\begin{aligned} 1 - \phi\kappa_1 + \beta\kappa_1^2 &= (1 - \alpha)\,\psi\,(\beta\kappa_2 - \Xi)\,\kappa_1 \\ -\phi\kappa_1 + \beta\kappa_1^2 + \beta\kappa_1\alpha - \Xi\alpha\frac{\kappa_1}{\kappa_2} &= (1 - \alpha)\,\psi\,(\beta\kappa_2 - \Xi)\,\kappa_1 \end{aligned}$$

Subtracting and rearranging,

$$\kappa_2 = \frac{\Xi\alpha\kappa_1}{\beta\kappa_1\alpha - 1} \equiv \kappa_2(\kappa_1). \tag{31}$$

I conclude that equation (31), together with

$$f(\kappa_1, \kappa_2, \psi) = 0, \tag{32}$$

where

$$f(\kappa_1, \kappa_2, \psi) \equiv -\phi\kappa_2 + \beta\kappa_1\kappa_2 + \beta\kappa_2\alpha - \beta\,(1 - \alpha)\,\kappa_2\psi\kappa_2 - \Xi\alpha + (1 - \alpha)\,\Xi\psi\kappa_2.$$

are required for (29) to be satisfied for all possible $\tilde{k}_t(i)$ and $\hat{p}_t(i)$ (the argument assumes $\kappa_1 \neq 0$, $\kappa_2 \neq 0$). This completes the discussion of step 2.

Turning to step 3, use (21) to substitute out for $\hat{p}_t^*$ in (27) and suitably difference the result, to obtain (16) with

$$\gamma = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{(1-\alpha\beta\kappa_1)}{(1+\theta\omega)(1-\alpha\beta\kappa_1) + (\omega - \sigma_\nu)\alpha\beta\kappa_2}. \tag{33}$$

So, to find γ in (16) one must first find κ_1, κ_2, ψ that satisfy the three equations composed of (31), (32) and (28), subject to $|\alpha\beta\kappa_1| < 1$. These equations can be solved by a one-dimensional nonlinear search for a value of κ_1 with the property, $g(\kappa_1) = 0$, where

$$g(\kappa_1) \equiv f(\kappa_1, \kappa_2(\kappa_1), \psi(\kappa_1, \kappa_2(\kappa_1))). \tag{34}$$

Here, $f(\kappa_1, \kappa_2, \psi)$, $\kappa_2(\kappa_1)$ and $\psi(\kappa_1, \kappa_2)$ are defined in (32), (31) and (28), respectively. I summarize these results in the form of a proposition:

Proposition A.1. Suppose κ_1 satisfies $g(\kappa_1) = 0$, for g defined in (34) and $|\beta\alpha\kappa_1| < 1$. Let $\kappa_2 = \kappa_2(\kappa_1)$ for $\kappa_2(\cdot)$ defined in (31) and $\psi = \psi(\kappa_1, \kappa_2)$, for $\psi(\cdot,\cdot)$ defined in (28). If there is an equilibrium in the linearized economy, then aggregate inflation in that equilibrium satisfies (16), with γ given in (33).

5 Closing Remarks

The linearization strategy pursued here has simplified the problem of computing equilibrium even more than is usually the case with linearizations. Because the capital stock is a state variable for an individual firm, in principle the distribution of capital across firms matters for determining aggregate equilibrium outcomes. In addition, in principle the evolution of that distribution over time is a part of the equilibrium that has to be computed. Because of the symmetry properties of the linearized equilibrium, only the aggregate capital stock matters in our linear approximation of the equilibrium. So, the linearization strategy in effect is a device for avoiding the computationally burdensome problem of computing a distribution and its law of motion.[6] Whether this is of interest ultimately depends on whether the linearization strategy is accurate. Whether this is so deserves to be investigated further. If the strategy does turn out to have good accuracy properties, it would be interesting to investigate wether the linearization strategy investigated here works well in other models where distributions matter in principle.

[6] Jeff Campbell analyzed an example in which linearization simplifies the computation of equliibrium in a model in which distributions matter (for a brief description, see Christiano (1992).) However, in his example the distribution still has to be at least approximated. In the example in this paper, the linearization strategy allows one to dispense with the problem of computing distributions altogether.

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