Expectation Traps and Monetary Policy

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Abstract

We describe a class of monetary economies that generate persistent episodes of high and low inflation. In these economies variations in expectations can lead private agents to take actions which then make it optimal for the monetary authority to validate those expectations. We think these model economies deserve attention because they display several good empirical implications.

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I Introduction

Many countries have gone through prolonged periods of costly, high inflation, as well as prolonged periods of low inflation. The United States and other industrialized countries went through a high inflation episode during the Great Inflation of the 1970s and are now in a low inflation episode. A central question in monetary economics is why high inflation episodes occurred and what can be done to prevent them from occurring again.

In this paper we examine whether standard monetary general equilibrium models with benevolent monetary authorities acting under discretion can generate persistent episodes of high and low inflation. Specifically, we ask whether private agents' expectations of high or low inflation can lead these agents to take actions which then make it optimal for monetary authorities to validate these expectations. Following Chari, Christiano and Eichenbaum (1998), we call such an outcome an expectation trap. Chari, Christiano and Eichenbaum (1998) showed that expectation traps could occur in conventional general equilibrium monetary models. They relied, however, on trigger strategies on the part of the monetary authority to support such outcomes. One criticism of trigger strategies is that because of folk theorem-like reasons, virtually any inflation outcome can be rationalized as an equilibrium. A key finding of this paper is that expectation traps can occur, even in the absence of trigger strategies. We think this result deserves attention because it occurs in a model which has several good empirical implications. As we explain below, the model is potentially able to resolve a variety of puzzles in the money demand literature. In addition, the model is consistent with the relative volatility of financial and other variables observed in low versus high inflation episodes.

We build on Lucas and Stokey's (1983) cash-credit good model. In our model, the benefit of unexpected growth in the money supply is a rise in output and the cost is the misallocation of resources arising from a distortion in relative prices. The monetary authority optimally balances the benefit and costs. We obtain the following three results:

- for a large range of parameter values, there are at least two equilibria.

- the multiplicity of equilibria depends importantly on the elasticity of money demand.
- financial variables are more volatile in the high inflation equilibrium than in the low inflation equilibrium, while real variables display similar volatility in both equilibria.

We now briefly explain the economic mechanisms in our benchmark model and the intuition underlying our results. In the model, goods are produced in monopolistically competitive markets. The monopoly power of firms causes the level of economic activity to be inefficiently low. A subset of monopolists set their prices before the monetary authority selects the money growth rate, while the rest of the monopolists set prices afterward. Because of the preset prices, a monetary expansion greater than expected can raise output. Such a monetary expansion tends to raise welfare because output is inefficiently low. A monetary expansion also has costs. In our model, some goods must be purchased with previously accumulated cash. A monetary expansion, by raising prices, reduces the consumption of cash goods and welfare. In addition, because some prices are preset and others are flexible, a monetary expansion changes relative prices and induces an inefficient allocation of resources. These aspects of the model formalize old ideas with an extensive literature.[1]

We consider two versions of our model. In the first, the fraction of goods which are purchased with cash is held fixed. The intuition underlying our second result is that the marginal cost of unanticipated inflation is non-monotone in the expected inflation rate. It turns out that the marginal cost of unexpected inflation is roughly proportional to rM/P, where r is the net nominal interest rate, and M/P denotes real balances. Real balances are bounded and since the nominal interest rate is increasing in the expected inflation rate, it follows that the marginal cost of unanticipated inflation is low at low expected inflation. A key feature of our model is the behavior of money demand at high nominal interest rate. Specifically, rM/P goes to zero as r goes to infinity. This feature implies that the marginal cost of unexpected inflation is low at high levels of expected inflation. We conjecture that the relationship between the marginal cost of inflation and rM/P lies in the fact that a monetary expansion acts as a distorting tax on real balances. Because the marginal cost of inflation has an inverted 'U' shape, as in a Laffer curve, while the marginal benefit is roughly constant, there is more than one value of expected inflation in which the marginal benefit of unanticipated inflation equals the marginal cost.

As the reasoning in the previous paragraph suggests, we find that the properties of money demand at high levels of inflation are crucial to the question of the multiplicity of equilibria. We confirm this reasoning by developing a model in which rM/P does not go to zero as r goes to infinity. In this model we find that there is a unique equilibrium.

In the second version of our model, households can also take defensive measures to protect themselves against expected inflation. Specifically, they can choose the fraction of goods purchased with cash and the fraction purchased with credit. This choice is made before the monetary authority chooses its policy. Cash purchases are costly because households forego interest, while credit purchases require payment of a cost in labor time which differs depending on the type of good. If households expect high inflation, they choose to purchase most goods with credit and few goods with cash while if they expect low inflation they purchase few goods with credit and most goods with cash.

This aspect of our model implies that if households expect high inflation and have chosen to purchase most goods with credit, the marginal cost of unanticipated inflation is small because relatively few goods are purchased with cash. The monetary authority has a strong incentive to inflate. If households expect low inflation, however, they choose to purchase most goods with cash and the marginal costs of unanticipated inflation are high. The monetary authority does not have a strong incentive to inflate. These arguments suggest that there might well be multiple equilibria in our model and, indeed, we find that for a large range of parameter values there are two equilibria.

The multiplicity of equilibria in our model raises the possibility of expectation traps. If private agents expect the monetary authority to pursue an expansionary monetary policy, they set prices sufficiently high, and choose to purchase so few goods with cash, that it is optimal for the monetary authority to validate their expectations. Conversely, if private agents expect low inflation, then the monetary authority optimally validates those expectations. The possibility of expectation traps in our model is promising because it may help account for the observed prolonged periods of high inflation as well as prolonged periods of low inflation. This possibility depends in a crucial way on the properties of money demand. At an abstract level, it should not be surprising that the behavior of the monetary authority depends in an essential way on the determinants of demand for the object they supply, namely money. But, to our knowledge, this connection has not been made as yet in the literature.

Our model can, we believe, account for a number of key features of the data on money demand. These include the observed trend in velocity, the differences between short and long-run interest elasticities of money demand and persistence of estimated money demand shocks.

The plan of the paper is as follows. The following section summarizes results in the existing literature. Section 3 describes our model. Section 4 analyzes a restricted version of the model, in which the cash-credit good distinction is exogenous. The endogenous case is treated in section 5. The final section concludes.

II The Kydland-Prescott/Barro-Gordon Model

Most of the literature on the inflation bias hypothesis has been conducted in reduced form models. We briefly describe one such model, the KP-BG model to motivate our analysis. The model is static, and the payoff for the monetary authority is given by $a\left(y-x\right)^2 - b\pi^2$, $a, b, x > 0$, where y is a measure of output, $x > 0$ is a measure of 'full employment output' and π is the realized rate of inflation. Output is given by $y = \alpha\left(\pi - \pi^e\right)$, $\alpha > 0$,where π^e is expected inflation. The timing is that private agents choose π^e first and the monetary authority then chooses π. The policymaker's best response, $\pi(\pi^e, x)$ maximizes the monetary authority's payoff given π^e. The model is completed by imposing rational expectations on private agents, namely, $\pi^e = \pi(\pi^e, x)$. The equilibrium inflation rate is given by the solution to this equation, and is denoted by $\pi^*(x)$. Under commitment, the monetary authority chooses π first, and then private agents set $\pi^e = \pi$. The inflation rate implied by this Ramsey-like programming problem is $\pi = 0$. The *inflation bias* is $\pi^*(x)$.

Figure 1 plots the best response function $\pi(\pi^e, x)$ for a fixed x. The equilibrium inflation rate, π^* is the intersection of the best response function with the 45 degree line. The figure shows the two basic results in this literature. First, since the best response function crosses the 45 degree line only once, the equilibrium value of inflation is determined solely by the variable, x, that shifts the best response function. If the best response function crossed the

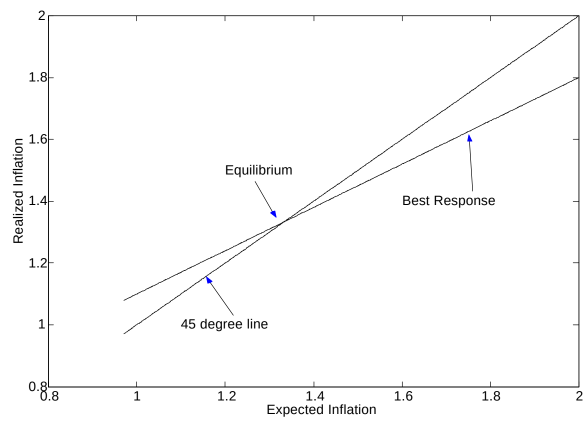

Figure 1: Best Response Function, KP-BG Model

45 degree line more than once, then private sector expectations play an independent role in determining the equilibrium. Chari, Christiano and Eichenbaum (1998) refer to equilibria of this type as 'expectation traps'. The KP-BG model excludes expectation traps. Second, since $a, b, x > 0$ there is always an inflation bias, and it may be substantial.

These two findings hold up in repeated versions of the model as long as attention is restricted to equilibria without reputational mechanisms or trigger strategies. These reputational mechanisms have the unattractive feature that they depend sensitively on the horizon being infinite. In addition, the folk theorem from repeated games implies that the set of equilibria is extremely large and it is difficult to know which of these equilibria are plausible. For these reasons, in our research we focus on stationary Markov equilibria, which rule out such reputational mechanisms.

III A Cash-Credit Goods Model With Financial Intermediation

In this section, we develop a version of Lucas and Stokey's (1983) cash-credit-goods model. We extend the model in a number of ways. Households consume a variety of differentiated goods as in Dixit and Stiglitz. These goods are produced by monopolistically competitive firms. Monetary policy is conducted by a benevolent monetary authority. Furthermore, households can choose whether to purchase each good with cash or with credit. Cash purchases are costly because households forego interest, while credit purchases require payment of a cost which differs depending on the type of good. This aspect of our model is similar to the financial intermediation models of Aiyagari, Braun and Eckstein (1998), Cole and Stockman (1992), Dotsey and Ireland ( ), Freeman and Kydland (1994), Ireland (1994), Lacker and Schreft (1996), Schreft (1992) and others. All of these emphasize the tension between the use of costly transactions technologies and the interest cost of using money to finance transactions. These models and ours have the feature that at low levels of expected inflation, households use cash in a relatively large number of transactions, while at high levels little cash is used.

Our economy is composed of firms, a representative household and a monetary authority. There is a continuum of firms, each of which is a monopolist in the production of some good, $j \in (0,1)$. The sequence of events within a period is as follows. First, a shock, θ, to technology, a shock, g, to government consumption, and a shock, η, to money demand, are realized. Let $s = (\theta, g, \eta)$. These shocks follow a Markov process. Then, a fraction μ of the firms (the 'sticky price firms') set their prices and households choose the fraction, z, of goods to purchase with cash. We let $P^e(s)$ denote the average price set by sticky price firms and $Z(s)$ denote the average value of z. After that, the monetary authority makes its policy decision. Finally, all private decisions are made.

We denote the state of the economy at the time the monetary authority makes its decision as follows:

$$S = (s, P^e, Z), \tag{1}$$

where Z denotes the economy-wide average value of z. We define this state for arbitrary P^e, Z rather than just for their equilibrium values, $P^e(s)$, $Z(s)$. This is not necessary for purposes of defining an equilibrium.[2] We do it because it is useful for purposes of computing and characterizing equilibrium.

Notice also that we do not include the aggregate stock of money in the state. In our economy, all equilibria are neutral in the usual sense that if the initial money stock is doubled, there is an equilibrium in which real allocations and the interest rate are unaffected and all nominal variables are doubled. This consideration leads us to focus on equilibria which are invariant with respect to the initial money stock. We are certainly mindful of the possibility that there can be equilibria which depend on the money stock. For example, if there are multiple equilibria in our sense, it is possible to construct 'trigger strategy-type' equilibria which are functions of the initial money stock. In our analysis we exclude such equilibria and we normalize the aggregate stock of money at the beginning of each period to unity.

The monetary authority makes its money growth decision conditional on S. We denote the gross money growth rate by x and the policy rule by $X(S)$. The state of the economy after the monetary authority makes its decision is (S, x). With these definitions of the economy's state variables, we proceed now to discuss the decisions of firms, households and the monetary authority.

A Firms

There is a continuum of goods denoted by $\omega \in (0, 1)$. Each good is produced by a monopolist using the following production technology:

$$y(\omega) = \theta n(\omega),$$

where $y(\omega)$ denotes output, $n(\omega)$ denotes employment, and θ is the technology shock. The price set by the $1 - \mu$ firms which set their prices after the monetary authority makes is decision ('flexible price firms') is denoted by $\hat{P}(S, x)$. The prices $P^e(s)$ and $\hat{P}(S, x)$ maximize profits subject to a demand curve discussed below. Because of the constant elasticity form of this demand curve, firms set prices as a fixed markup above marginal cost, the wage rate. For the μ sticky price firms and the $1 - \mu$ flexible price firms, this means:

$$\begin{aligned} P^e(s) &= \frac{W(s, P^e(s), Z(s), X(s, P^e(s), Z(s)))}{\theta\rho}, \\ \hat{P}(S, x) &= \frac{W(S, x)}{\theta\rho}, \ 0 < \rho < 1, \end{aligned} \tag{2}$$

where $W(S, x)$ denotes the nominal wage rate.

B Household

The preferences of the representative household are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \tag{3}$$

where

$$c_t = \left[\int_0^1 c_t(\omega)^\rho d\omega \right]^{\frac{1}{\rho}},$$

and $c_t(\omega)$ denotes consumption of type ω good and n_t denotes labor time.

Each consumption good can be purchased either with cash or with credit. The financial intermediation technology is as follows. For goods with $\omega > \bar{z}$, (where $\bar{z}$ is a parameter between 0 and 1), the cost of purchasing with credit is zero. Purchasing goods with $\omega \leq \bar{z}$ on credit requires labor time. The household chooses a fraction, $z \leq \bar{z}$, such that goods with $\omega < z$ are purchased with cash and goods with $\omega > z$ are purchased with credit. The labor time required to purchase fraction, z, of goods with cash is given by $\eta(\bar{z} - z)^{1+\nu}/(1+\nu)$, where $\nu > 0$ is a parameter and $\eta > 0$ is the money demand shock. The household's labor time, including time spent working in the market, n, is

$$l = n + \frac{\eta(\bar{z} - z)^{1+\nu}}{1+\nu}. \tag{4}$$

We adopt the following utility function specification:

$$u(c, l) = \frac{\left[c(1-n)^\psi\right]^{1-\sigma}}{1-\sigma} \tag{5}$$

The sequence of events during the period is as follows. The household begins the period with nominal assets, A. It then chooses z and trades in the asset market where it divides A into money holdings, M, and bonds, B, subject to

$$M + B \leq A. \tag{6}$$

Here, nominal assets, money and bonds are all scaled by the aggregate stock of money. We impose a no-Ponzi constraint of the form $B \leq \bar{B}$, where $\bar{B}$ is a large, finite, upper bound. The household's cash in advance constraint is

$$M - \left[P^e \mu z c_{11} + \hat{P}(S, x)(1-\mu) z c_{12}\right] \geq 0, \tag{7}$$

where c_{11} and c_{12} denote quantities of cash goods purchased from sticky and flexible price firms, respectively.[3] Nominal assets evolve over time as follows:

$$
\begin{aligned}
(8)\ 0 \ \leq\ & W(S,x)n + (1-R(S,x))M - z\left[P^e\mu c_{11} + \hat{P}(S,x)(1-\mu)c_{12}\right] \\
& -(1-z)\left[P^e\mu c_{21} + \hat{P}(S,x)(1-\mu)c_{22}\right] + R(S,x)A + (x-1) + D(S,x) - xA',
\end{aligned}
$$

where c_{21} and c_{22} denote the quantities of credit goods purchased from sticky and flexible price goods, respectively. In (8), M denotes the individual household's beginning of period stock of money, $R(S,x)$ denotes the gross nominal rate of return on bonds, and $D(S,x)$ denotes profits after lump sum taxes. Finally, B has been substituted out in the asset equation using (6). Notice that A' is multiplied by x. This reflects the way we have scaled the stock of nominal assets.

Consider the household's asset, goods and labor market decisions for a given value of z. Given that the household expects the monetary authority to choose policy according to X in the future, prices to be set according to $P^e(\cdot)$ and aggregate cash credit goods decisions to be made according to $Z(\cdot)$, the household solves the following problem:

$$
(9) \qquad v(A,z,S,x) = \max_{n,M,A',c_{ij};i,j=1,2} u(c,l) + \beta E_{s'}[\max_{z'} v(A',z',S',X(S'))|s]
$$

subject to (4), (6), (7), (8), non-negativity on allocations and

$$
(10) \qquad c = \left[z\mu c_{11}^{\rho} + z(1-\mu)c_{12}^{\rho} + (1-z)\mu c_{21}^{\rho} + (1-z)(1-\mu)c_{22}^{\rho}\right]^{\frac{1}{\rho}}.
$$

Also,

$$
S' = (s', P^e(s'), Z(s')).
$$

In (9), v is the household's value function. The solution to (9) yields decision rules of the form $n(A,z,S,x)$, $M(A,z,S,x)$, $A'(A,z,S,x)$, and $c_{ij}(A,z,S,x)$, $i,j=1,2$. To keep our notation simple, we suppress the dependence of these functions on the wage and price functions, and the monetary policy rule. Finally, the household's choice of z, $z(A,s)$, solves the following problem:

$$
z(A,s) = \max_{0\leq z\leq \bar{z}} v(A,z,s,P^e(s),Z(s),X(s,P^e(s),Z(s))).
$$

C Private Sector Equilibrium

We now define an equilibrium for the private sector, taking as given the current monetary action, x, and the rule governing future monetary policy, $X(S)$. This equilibrium requires that households and firms optimize and markets clear.

Definition Given a monetary policy rule, $X(S)$, and a current money growth rate, x, a *Private Sector Equilibrium* is a collection of functions $Z(s)$, $P^e(s)$, $\hat{P}(S_1)$, $W(S_1)$, $v(A, S_1)$, $z(A, s)$, $c_{ij}(A, S_1)$, $n(A, S_1)$, $M(A, S_1)$, $A'(A, S_1)$, $R(S_1)$, where $S_1 = (s, P^e(s), Z(s), x)$, such that:

1. The functions v, z, c_{ij}, n, M, A' solve (9),
2. Firms optimize, i.e., (2) is satisfied,
3. The asset markets clear, i.e., $A'(1, S_1) = 1$ and $M(1, S_1) = 1$,
4. Consistency, $z(1, s) = Z(s)$,
5. The resource constraint is satisfied, i.e., $\theta n(1, S_1) = g + z\left[\mu c_{11} + (1-\mu)c_{12}\right] + (1-z)\left[\mu c_{21} + (1-\mu)c_{22}\right]$,

D Monetary Authority

The monetary authority chooses the money growth rate to maximize the utility of the representative household. Taking future monetary policy and private sector allocation rules as given, the monetary authority chooses the current money growth rate, x, to solve the problem:

$$\max_x v(1, Z, S, x). \tag{11}$$

We restrict x to be at least as large as $\underline{x}$, where $0 < \underline{x} < \beta$.

Definition A *Markov equilibrium* is a private sector equilibrium and a monetary policy rule such that $X(S)$ solves (11).

IV Analysis of Exogenous Cash Credit Good Model

In this section, we consider a version of the model without financial intermediation. We call it the exogenous cash-credit good model because we simply set the fraction of cash goods,

z, to a constant. The qualitative properties of this simpler model are easier to analyze than those of the full-blown model. Also, many of the basic forces acting on inflation are present in this version of the model.

To analyze our model, we decompose the first order condition associated with the monetary authority problem, (11), into two pieces. The two pieces formalize the notion emphasized in the introduction that inflation outcomes reflect the interplay between the marginal costs and marginal benefits of unexpected inflation. The benefits are captured by the piece which we call the monopoly distortion. This exists because, as indicated in the introduction, the presence of monopoly power implies that output is inefficiently low in our model economy. This creates a temptation for the monetary authority to stimulate the economy. The second piece, which we call the inflation distortion, captures the costs associated with increased inflation. These costs, also discussed in the introduction, reflect that inflation leads to a costly reduction in cash good purchases because of the operation of the cash in advance constraint. In addition, surprise inflation also leads to a misallocation of resources because some prices are fixed in advance, while others are not.

To develop the decomposition just described, we exploit the fact that the Markov equilibrium is the fixed point of a particular map. For each fixed s, the fixed point is of the mapping from an expectation of government policy to actual government policy. Let $q \equiv \hat{P}/P^e$. It can be verified that in any Markov equilibrium q is a strictly monotone function of x for given P^e.[4] This feature of the equilibrium implies that it is equivalent to characterize policy as a choice of the relative price, q, rather than the money growth rate, x. The government's decision problem is simplified in our setting because its choice of q has no impact on future allocations. As a result, the government faces a static problem.

The mapping can be defined as follows. For each fixed s, start with an arbitrary value of P^e and map to q in three steps. First, for arbitrary q, calculate the mapping from (s, P^e, q) to the private allocations and prices that are determined after the monetary authority makes its decision. This allows us to define the policy maker's objective function, $U(s, P^e, q)$. Second, optimize this objective with respect to q, and denote the solution, $q = q(s, P^e)$. Finally, adjust P^e until $q = 1$. This value of P^e corresponds to the function, $P^e(s)$.

The first order condition that we study is the derivative of U with respect to q, evaluated in equilibrium, when $q = 1$. This is the object which we represent as the sum of the costs and benefits of unexpected inflation.

A Private Allocations and Prices

We now solve for the private allocations given $(s,\ P^e,\ q)$. The first order necessary and sufficient conditions for household optimization are:

$$
\begin{aligned}
\frac{u_{11}}{u_{12}} &= \frac{u_{21}}{u_{22}} = \frac{\mu}{1-\mu}\frac{1}{q}, \\
\frac{u_{11}}{u_{21}} &= \frac{u_{12}}{u_{22}} = \frac{z}{1-z}R, \\
-u_n &= \frac{\theta\rho u_{22}}{(1-\mu)(1-z)}, \\
\frac{x u_{21}}{P^e\mu(1-z)} &= \beta E_{s'}[v_1(1, z(1,s'), S', X(S'))|s].
\end{aligned}
\tag{12}
$$

Here, u_{ij} denotes the partial derivative of u with respect to c_{ij}, and v_1 denotes the partial derivative of v with respect to its first argument. In the labor Euler equation, we have used the result, $W = \theta\rho\hat{P}$.

In addition, the cash in advance constraint can be written as

$$P^e\mu z c_{11} + qP^e(1-\mu)zc_{12} \leq 1,$$

so that the complementary slackness condition is

$$\{1 - [P^e\mu z c_{11} + qP^e(1-\mu)zc_{12}]\}[R-1] = 0. \tag{13}$$

The resource constraint is:

$$g + z[\mu c_{11} + (1-\mu)c_{12}] + (1-z)[\mu c_{21} + (1-\mu)c_{22}] = \theta n. \tag{14}$$

There are 7 unknowns, c_{ij}, $i,j = 1,2$, n, x, R, and 7 independent equations in (12), (13) and (14). (Of the first four first order conditions in (12), only three are independent.) Notice that x only appears in the last equation in (12), so that the allocations and the interest rate can be computed as functions of P^e, q, s and x can be solved for at the last stage. Denote the allocations defined in this way by:

$$c_{ij}(s, P^e, q),\ i,j = 1,2,\ R(s,P^e,q),\ n(s,P^e,q). \tag{15}$$

We denote the set, (P^e, q), for which there exists a solution to the seven equations by D.[5]

For later reference, it is useful to know that when utility is given by (5), then (12) reduce to:

$$(16) \qquad \begin{aligned} c_{12} &= c_{11} q^{\frac{-1}{1-\rho}},\ c_{22} = c_{21} q^{\frac{-1}{1-\rho}}, \\ R &= \left(\frac{c_{22}}{c_{12}}\right)^{1-\rho},\ \theta(1-n) = \frac{\psi}{\rho} c^{\rho} c_{22}^{1-\rho}. \end{aligned}$$

Incorporating these results into the cash in advance constraint gives:

$$(17) \qquad c_{11}\left\{\mu + (1-\mu) q^{\frac{-\rho}{1-\rho}}\right\} \leq \frac{1}{zP^e}.$$

B Government Problem

We can summarize the previous discussion as providing functions:

$$c = c(s, P^e, q),\ n = n(s, P^e, q),$$

where c is obtained by substituting (15) into (10). These functions can be substituted into the utility function,

$$U(s, P^e, q) = u\left[c(s, P^e, q), n(s, P^e, q)\right].$$

Define

$$q(s, P^e) = \arg\max_{q \in D} U(s, P^e, q),$$

where the set D was defined after (15). The function, $q(s, P^e)$, is the monetary authority's best response, given s, P^e. Equilibrium requires that $q(s, P^e) = 1$. This equilibrium requirement allows us to construct the equilibrium function, $P^e(s)$.

C Qualitative Characteristics of Equilibrium

We are interesting in determining how many equilibrium functions there are. Notice from (16) that $c_{11} = c_{12} = c_1$, say, in an equilibrium. Define c_2 similarly. We proceed in two steps. We first look for equilibria with the property that $c_1/c_2 < 1$, or, equivalently, that $R > 1$.

We then consider the case, $c_1/c_2 = 1$. Our analysis is based on first order conditions which are necessary for equilibrium. In the computational examples we verify sufficient conditions numerically.

When $R > 1$, the cash in advance constraint is binding and the allocations are determined by (16), (17) with equality, and (14). It is easy to see that the interest rate and allocation functions, (15) are differentiable in q.

Monetary authority optimality implies:

$$U_q(P^e, q, s) = u_c c_q(P^e, q, s) + u_n n_q(P^e, q, s) = 0, \tag{18}$$

where U_q is the derivative of U with respect to q. In equilibrium, (18) is satisfied for $q = 1$. From here on we suppress the arguments of functions, where there is no possibility of confusion. It is useful to rewrite the expression for U_q by adding and subtracting $u_c\theta(c/c_2)^{(1-\rho)}n_q$:

$$U_q = u_c c_q - \theta u_c (\frac{c}{c_2})^{1-\rho} n_q + \left[u_n + \theta u_c (\frac{c}{c_2})^{1-\rho}\right] n_q = 0. \tag{19}$$

We now develop formulas for c_q and n_q. Consider first c_q. Differentiating (10), and evaluating the derivatives at $q = 1$, we obtain:

$$c_q = \left(\frac{c}{c_1}\right)^{1-\rho} z\left[\mu c_{11,q} + (1-\mu)c_{12,q}\right] + \left(\frac{c}{c_2}\right)^{1-\rho} (1-z)\left[\mu c_{21,q} + (1-\mu)c_{22,q}\right], \tag{20}$$

where $c_{ij,q}$ is the derivative of c_{ij} with respect to q. From the resource constraint, we obtain:

$$n_q = \frac{z\left[\mu c_{11,q} + (1-\mu)c_{12,q}\right] + (1-z)\left[\mu c_{21,q} + (1-\mu)c_{22,q}\right]}{\theta}. \tag{21}$$

Substituting for c_q and n_q in (19) we obtain

$$U_q = u_c\left[\left(\frac{c}{c_1}\right)^{1-\rho} - \left(\frac{c}{c_2}\right)^{1-\rho}\right] z\left[\mu c_{11,q} + (1-\mu)c_{12,q}\right] + \left[u_n + \theta u_c(\frac{c}{c_2})^{1-\rho}\right] n_q. \tag{22}$$

Using (16), we obtain

$$U_q = u_c\left(\frac{c}{c_2}\right)^{1-\rho} (R-1)\, z\left[\mu c_{11,q} + (1-\mu)c_{12,q}\right] + \left[u_n + \theta u_c(\frac{c}{c_2})^{1-\rho}\right] n_q. \tag{23}$$

From (17) and (16),

$$c_{11,q} = c_1\frac{(1-\mu)\rho}{1-\rho},\ c_{12,q} = -c_1\frac{1-(1-\mu)\rho}{1-\rho}, \tag{24}$$

so that

$$\mu c_{11,q} + (1-\mu)c_{12,q} = -c_1(1-\mu). \tag{25}$$

Substituting this into (23):

$$U_q = -u_c \left(\frac{c}{c_2}\right)^{1-\rho} (R-1)\, z c_1 (1-\mu) + \left[u_n + \theta u_c (\frac{c}{c_2})^{1-\rho}\right] n_q. \tag{26}$$

Notice that if the government could commit itself to monetary policy, it would follow the Friedman rule and set $R = 1$, so that the first term would be zero. In light of this observation, we call this term the inflation distortion. Specifically, we define the inflation distortion as $(R-1)\, z\, (c_1/c_2)\, (1-\mu)$. Since $R = (c_1/c_2)^{\rho-1}$, we can write the inflation distortion as

$$\psi_{ID}\left(\frac{c_1}{c_2}\right) = \left[\left(\frac{c_1}{c_2}\right)^{\rho-1} - 1\right] z \frac{c_1}{c_2}(1-\mu). \tag{27}$$

Then, it is easy to see that the first term in (26) is

$$-u_c \left(\frac{c}{c_2}\right)^{1-\rho} c_2 \psi_{ID}\left(\frac{c_1}{c_2}\right).$$

We call the second term in (26) the monopoly distortion for the following reason. If firms behaved competitively the marginal rate of substitution between credit good consumption and leisure would be set equal to the corresponding marginal rate of transformation, and the second term would be zero. In the appendix we show that n_q is of the form $\psi_{MD}(c_1/c_2)c_2/(\theta(1-\rho))$, where $\psi_{MD}(c_1/c_2)$ is the monopoly distortion. Using the labor first order condition in (16), (26) can be rewritten as

$$U_q = u_c \left(\frac{c}{c_2}\right)^{1-\rho} c_2 \left[-\psi_{ID}\left(\frac{c_1}{c_2}\right) + \psi_{MD}\left(\frac{c_1}{c_2}\right)\right]. \tag{28}$$

Consider the inflation distortion function, ψ_{ID}. From inspection of (27), it is immediate that

$$\psi_{ID}(0) = \psi_{ID}(1) = 0.$$

That is, there is no inflation distortion when expected inflation rates are high or low. This feature of the model plays a central role in generating multiplicity of Markov equilibria.

Let

$$\psi(\frac{c_1}{c_2}) = -\psi_{ID}(\frac{c_1}{c_2}) + \psi_{MD}(\frac{c_1}{c_2}).$$

A necessary condition for equilibrium is $\psi = 0$. In the appendix, we establish that $\psi_{MD}(0) > 0$. Suppose next that $\psi_{MD}(1) > 0$. Then, $\psi(0) > 0$ and $\psi(1) > 0$. With one exception, by continuity of ψ there are at least two values of c_1/c_2 such that $\psi(c_1/c_2) = 0 = U_q$. The exceptional case occurs when the graph of ψ is tangent to the horizontal axis. This case is clearly non-generic. We have established that if $\psi_{MD}(1) > 0$, there are generically two allocations which satisfy the necessary conditions for equilibrium.

Suppose next that $\psi_{MD}(1) < 0$. Then, $\psi(0) > 0$ and $\psi(1) < 0$. By continuity of ψ there is at least one value of $c_1/c_2 < 1$ such that $\psi(c_1/c_2) = 0 = U_q$. This value satisfies the necessary conditions for an equilibrium.

Next, we show that if $\psi_{MD}(1) < 0$, then $c_1/c_2 = 1$ satisfies the necessary conditions for an equilibrium. We do this by examining the behavior of U_q when $c_1/c_2 = 1$. In this case, the allocation functions are not differentiable functions of q. The problem is, as we show in the appendix, that for $q > 1$ the cash in advance constraint is binding, while it is not binding for $q < 1$. The allocation functions are different in the two cases because the equations characterizing a private sector equilibrium are different.[6] When the cash in advance constraint is not binding, we replace it by $R = 1$ in (16).

In the appendix, we establish that the right derivative of U with respect to q, denoted by $U_{q\downarrow 1}$, is identical to (26). We also establish that the left derivative of U, denoted by $U_{q\uparrow 1}$, is strictly positive.

These observations imply that a necessary condition for $R = 1$ to be a Markov equilibrium is that the right derivative of U be non-positive. That is, $\psi_{MD}(1) \leq 0$. To see this, notice that, when the right derivative of U is non-positive, the monetary authority has no incentive to raise q. Since the left derivative is strictly positive, the authority also has no incentive to reduce q.

We have established:

Proposition 1: Suppose the first order conditions for an equilibrium are sufficient. Then, generically, there are at least two Markov equilibria.

Next, we show that, in a Markov equilibrium, the equilibrium interest rate is constant.

Proposition 2: In a stationary Markov Equilibrium, the interest rate, R, does not depend on the realization of the technology shock, θ, and of government consumption, g.

Proof: Notice from (27) that ψ_{ID} does not depend on θ or g. This is true of ψ_{MD} as well (see (30) and (32)). Thus, c_1/c_2 does not depend on θ or g. The result follows because $R = (c_2/c_1)^{1-\rho}$ (see (16)).

In this subsection we have argued that multiple equilibria are possible in our model. The proof of Proposition 1 suggests that the multiplicity arises because of the Laffer Curve shape of the inflation distortion function, ψ_{ID}. This shape in turn depends on $(R-1)M/P$ going to zero as the interest rate R goes to infinity. That is, it depends on money demand being sufficiently elastic with respect to the interest rate. One conjecture is that if money demand is not very elastic, then the equilibrium is unique. In the Appendix, we develop a model in which rM/P does not go to zero as r goes to infinity. In this model we find that there is a unique equilibrium.

D Quantitative Characteristics

To illustrate the multiplicity result in Proposition 1, we constructed a numerical example in which there are no shocks. We obtained parameter values for this example as follows. Some of the parameters are obtained from the money demand relationship in our model. To develop this relationship, denote private purchases of consumption goods by

$$c^p = zc_1 + (1-z)c_2.$$

Notice this is the value of consumption goods purchased in markets by the households in the model since both cash and credit goods sell for the same price. Using (16) and that $zc_1 = M/P$ from the cash in advance constraint, we have that, in a Markov equilibrium, the following relationship must hold:

$$\frac{c^p}{M/P} = 1 + \frac{1-z}{z} R^{\frac{1}{1-\rho}}.$$

This relationship can be interpreted as a money demand equation. We regressed $\log(Pc^p/M - 1)$ on $\log R$ and a constant. Under the appropriate orthogonality condition on the regression, the parameters z and ρ can be obtained from the least squares estimates. Using this

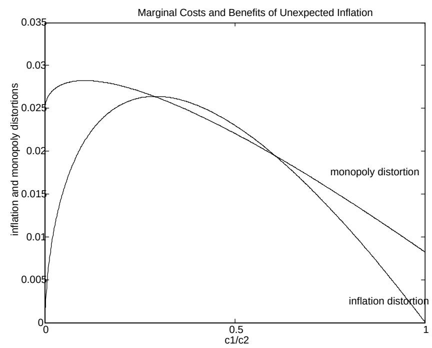

Figure 2: ψ_{ID} and ψ_{MD} for $c_1/c_2 \in (0,1)$

procedure we obtained $z = 0.182$ and $\rho = 0.643$.[7] From Christiano and Eichenbaum (1992) we obtained an estimate of $\psi = 4$. We used a value of $\mu = 0.1$, which is somewhat higher than results reported in Parks.[8] We set $g = 0.05$, so that government consumption, as a fraction of output, is roughly consistent with what it is in the data. Finally, we normalized $\theta = 1$.

Figure 2 displays the monopoly distortion, ψ_{MD}, and the inflation distortion, ψ_{ID}, for $c_1/c_2 \in (0,1)$. An equilibrium value of c_1/c_2 is one for which the two are equal or, $c_1/c_2 = 1$ and $\psi_{MD} \leq 0$. In this example, $R = 1.2$ in the low inflation equilibrium and $R = 1.6$ in the high inflation equilibrium.[9]

We conducted a variety of other numerical experiments with this model and obtained two kinds of interesting results. First, for a surprisingly large range of parameters, it turns out that, in the low inflation Markov equilibrium, $R = 1$. That is, this equilibrium coincides with the equilibrium if the monetary authority could commit to its policies. Second, for some parameter values, the first order conditions turn out sometimes to characterize local maxima but not global maxima.[10]

V Analysis of the Endogenous Cash-Credit Good Model

In this section we analyze the version of our model in which the fraction of goods purchased with cash, z, is endogenous. We first display the results of simulations of this model. We then discuss some of its implications for money demand.

A Simulation Analysis of the Model

In the experiments conducted so far, we find multiple equilibria are very easy to generate. Here, we report on the results of one experiment. In this experiment, there are no shocks. We used the following parameter values for this experiment: $\beta = 1/1.03$, $\eta = .065$, $\nu = 0$, $\psi = 1.64$, $\rho = .83$, $\mu = 0.1$, $\bar{z} = 0.3$, $\sigma = 1.01$, $g = 0$, $\theta = 1$. These parameters are close to those in the literature. This economy has two Markov equilibria: one with a low inflation rate and the other with a high inflation rate.

To develop intuition for this result, in Figure 3, we plot the best response of the monetary authority in terms of its optimal growth rate of the money supply, x, against the corresponding expected growth rate, X.[11] (Note that x is labelled G and X is labelled g in Figure 3.) The points labelled A and C correspond to the two Markov equilibria. Over the range A to B households choose to set $z = \bar{z}$. When X rises above the point corresponding to B, households find it optimal to pay the costs of purchasing some goods on credit, and as X rises further, choose to purchase an increasing fraction of goods using credit. As a result, the marginal cost of unexpected inflation - the implicit tax on cash goods - is actually falling as expected money growth rises, since households react to higher expected money growth by reducing the fraction of goods purchased with cash. This explains why the best response function has a steeper slope after the point B, and why it intersects the 45 degree line at point C. This multiplicity of equilibria implies that it is possible for the economy to get caught in an expectations trap. In such a trap, if private agents, for whatever reason, expect the high money growth associated with point C, they choose to purchase a small fraction of goods with cash and set prices at a high level. The monetary authority optimally chooses to validate these expectations for two reasons. First, the small number goods purchased with cash implies that the cost - in terms of the implicit tax on cash goods - of high money growth

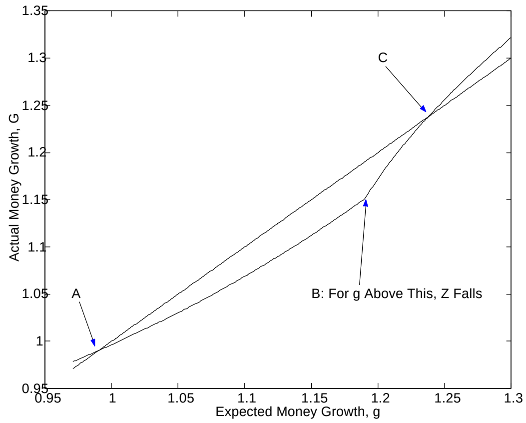

Figure 3: Monetary Authority's Best Response

is small. Second, the fact that agents have posted high prices means that if the monetary authority does not validate expectations, output will be inefficiently low. The existence of multiple, stationary Markov equilibria also raises the possibility that the economy could be subject to excessive volatility. It is easy to show that in this economy there are 'sunspot' equilibria in which inflation rates and output fluctuate erratically over time. For example, suppose that at the beginning of each period, agents observe a random variable which can take on two values, h, l, and is very persistent over time. In this case, there is an equilibrium in which the economy is at A if the random variable takes on the value h and is at B if the random variable is at value l. Then, inflation rates will fluctuate around A for a long period of time, occasionally change by a large amount and then fluctuate around C for a long period of time.

We have also conducted simulations of the stochastic version of our model. We report on results for one such experiment here. The nonstochastic parameters for this experiment are: $\beta = 1/1.03$, $\nu = 0.25$, $\psi = 4$, $\rho = 0.643$, $\mu = 0.1$, $\bar{z} = 0.182$, $\sigma = 1.01$. The shocks are all first order autoregressive, mutually independent. The mean levels of the government consumption shock, the technology shock, and the money demand shock are 0.05, 1, 0.2 respectively. The autocorrelations of the government consumption shock, the technology shock and the money demand shock are 0.9 each, while the standard deviations of the innovations are 0.001, 0.05, 0.0005 respectively. In Table 1, we report the standard deviations of key variables in the low and high inflation equilibria. This Table shows that the volatility of real variables is not much affected by which equilibrium the economy happens to find itself in, but the volatility of inflation and nominal interest rates is quite different in the two equilibria. This aspect of the model seems in accord with much evidence that financial variables are much more volatile in high inflation times than in low inflation times.

Table 1		
	High Inflation	Low Inflation.
σ_y	0.017	0.017
σ_n	0.007	0.007
σ_R	0.001	0.000
σ_π	0.024	0.020
Note: σ_y and σ_n refer to the standard deviation		
of logged, then HP filtered output and employment.		
The financial variables have not been filtered.		

B Money Demand and Extensions

The money demand relationship in this model is:

$$\frac{c}{M/P} = 1 + \frac{1-z}{z} R^{\frac{1}{1-\rho}},$$

where c is total consumption, $c = zc_1 + (1-z)c_2$, and c_1 and c_2 denote consumption of each cash and credit good, respectively. We believe that this money demand function represents an improvement over the one in the exogenous cash credit good model, and studied empirically in Lucas and Stock and Watson. For example, it is likely to be consistent with the well known observation that short run money demand elasticities are smaller than long run money demand elasticities. That is, the immediate impact on real balances of a change in the interest rate is smaller than the longer run impact. To see why, consider a stochastic version of our model in which money shocks cause interest rates to be positively serially correlated. Since the current interest rate is not known at the time that z chosen, then the immediate response of velocity is exactly the same as in the cash credit good model. Dynamically, however, the response to velocity is likely to be stronger, as z falls in response to the rise in R.

Another feature of empirical money demand equations is that shocks tend to be persistent and heteroscedastic. This is likely to be a feature of our model, since money demand shocks are functions of the product of z and R. Finally, an interesting capital-theoretic extension is one in which z in the current period is determined by its past value and by incurring current costs. In such an extension velocity may exhibit a trend, and the data suggests that some empirical measures do too.

We plan to consider stochastic versions of this model. Since the nonstochastic version has two equilibria, it seems likely that in the stochastic version monetary policy will exhibit regime shifts. Clarida, Gali and Gertler (1999) and Taylor (1999) argue that the data seems to show regime shifts. We plan to investigate whether these resemble the regime shifts in the model.

VI Conclusion

The results in this paper show that absence of commitment in monetary policy is in principle capable of rationalizing a high inflation bias, as well as prolonged periods of low and high inflation. The analysis suggests that whether it can do so depends on how high is the elasticity of money demand with respect to the interest rate.

Our analysis of the endogenous cash-credit goods model convinces us that there is a good chance that it is indeed high enough. This model gives people opportunities to substitute out of money (cash goods) and so - in view of the intuition developed here - it is not surprising that that model also has multiple equilibria.

One aspect of the US data is that in the transition between inflation regimes, inflation seems unexpected, in the sense that long term interest rates remained low in the transition to high inflation, and remained high in the transition back down to low inflation. We believe that our model can be consistent with these observations. Suppose that private agents coordinate their beliefs according to a "sunspot". Then, beliefs about long run inflation prospects depend on the persistence properties of the sunspot variable. Suppose, for example, that the sunspot variable follows a regime switching stochastic process with state dependent switching probabilities. Then, when the economy switches to the high inflation regime, people's confidence about the duration of the switch would initially be low and this would be reflected in initially low long term interest rates. Similarly, when the economy switches to the low inflation regime, long term rates would remain high for a while, as people's confidence that the switch is permanent grows. Whether the model in fact is consistent with the behavior of long-term interest rates remains to be investigated.

The particular empirical virtue of this model is that it holds the potential to account quantitatively for a variety of puzzles noted in the empirical literature on money demand. Specifically, the data suggest that short run interest elasticities of money demand are lower than longer run elasticities, that shocks to money demand are persistent, and that there have been secular shifts in velocity. Standard monetary models like the one emphasized in this draft of the paper cannot reproduce these features of the data, even qualitatively. The endogenous cash credit good model can do so qualitatively and seems promising quantitatively. We plan to explore this further.

A Appendix

In this appendix we prove four results used in section C. First, we show that n_q is of the form:

$$n_q = \frac{c_2 \psi_{MD}\left(\frac{c_1}{c_2}\right)}{(1-\rho)\theta}.$$

Second, we show that $\psi_{MD}(0) > 0$. Third, we show that $U_{q\downarrow 1}$ is identical to (26). Fourth, we show that $U_{q\uparrow 1}$ is strictly positive.

A First result

To establish the first result, we begin by differentiating (16) to obtain,

$$c_{22,q} = c_{21,q} - \frac{c_2}{1-\rho}. \tag{29}$$

Combining this and (25) with (21), we obtain:

$$\frac{(1-\rho)\theta}{c_2} n_q = (1-\rho)\left\{-\frac{c_1}{c_2} z(1-\mu) + (1-z)\left[\frac{c_{21,q}}{c_2} - \frac{1-\mu}{1-\rho}\right]\right\} \tag{30}$$

To get $c_{21,q}$ we work with the labor first order condition in (16), after substituting out for θn from (14) and for c using (10) and for c_{12} and c_{22} from (16) to obtain:

$$\begin{aligned}\theta &= g + [zc_{11} + (1-z)c_{21}]\left[\mu + (1-\mu)q^{\frac{-1}{1-\rho}}\right] \\ &\quad + \frac{\psi}{\rho}\left[zc_{11}^{\rho} + (1-z)c_{21}^{\rho}\right]\left[\mu + (1-\mu)q^{\frac{-\rho}{1-\rho}}\right] c_{21}^{1-\rho}\frac{1}{q}.\end{aligned}$$

Totally differentiating this expression, we obtain:

$$c_{21,q} = \frac{-c_{11,q} z\left[1 + \psi\left(\frac{c_2}{c_1}\right)^{1-\rho}\right] + [zc_1 + (1-z)c_2]\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho} c^{\rho} c_2^{1-\rho}\frac{1-\rho\mu}{1-\rho}}{(1-z)(1+\psi) + \frac{\psi}{\rho}(1-\rho)\left(\frac{c}{c_2}\right)^{\rho}} \tag{31}$$

Using (24), this reduces to:

$$\frac{c_{21,q}}{c_2} = \frac{-\frac{c_1}{c_2}\frac{(1-\mu)\rho}{1-\rho} z\left[1 + \psi\left(\frac{c_1}{c_2}\right)^{\rho-1}\right] + \left[z\frac{c_1}{c_2} + 1 - z\right]\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho}\left(\frac{c}{c_2}\right)^{\rho}\frac{1-\rho\mu}{1-\rho}}{(1-z)(1+\psi) + \frac{\psi}{\rho}(1-\rho)\left(\frac{c}{c_2}\right)^{\rho}}. \tag{32}$$

Substituting (32) into (30) it is easily verified that n_q is of the desired form.

B Second Result

To verify $\psi_{MD}(0) > 0$, it is sufficient to establish that the expression in square brackets in (30) is positive when $c_1/c_2 = 0$. Evaluating this, taking into account (32):

$$\begin{aligned}
\frac{c_{21,q}}{c_2} - \frac{1-\mu}{1-\rho} &= \frac{(1-z)\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho}(1-z)\frac{1-\rho\mu}{1-\rho}}{(1-z)(1+\psi) + \frac{\psi}{\rho}(1-\rho)(1-z)} - \frac{1-\mu}{1-\rho} \\
&= \frac{1}{1-\rho}\frac{1-\mu+\frac{\psi}{\rho}(1-\rho\mu)}{1+\psi+\frac{\psi}{\rho}(1-\rho)} - \frac{1-\mu}{1-\rho} \\
&= \frac{1}{1-\rho}\left\{\frac{1-\mu+\frac{\psi}{\rho}(1-\rho\mu) - (1-\mu)(1+\psi) - (1-\mu)\frac{\psi}{\rho}(1-\rho)}{1+\psi+\frac{\psi}{\rho}(1-\rho)}\right\} \\
&= \frac{1}{1-\rho}\frac{\psi}{\rho}\frac{(1-\rho)\mu}{1+\psi+\frac{\psi}{\rho}(1-\rho)} > 0.
\end{aligned}$$

This establishes the desired result.

C Third Result

To establish the third result, it suffices to establish that the interest rate is increasing in q at the point $c_1/c_2 = 1$. That is, since $R = (c_{22}/c_{12})^{1-\rho}$, we need $c_{21,q} \geq c_{11,q}$ at the point $c_{21} = c_{11} = c_{22} = c_{12} = c$. Substituting for $c_{21,q}$ from (31), we need to show that

$$\frac{-c_{11,q}z\left[1+\psi\right] + c\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho}c\frac{1-\rho\mu}{1-\rho}}{(1-z)(1+\psi)+\frac{\psi}{\rho}(1-\rho)} \geq c_{11,q},$$

or

$$\left\{\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho}\frac{1-\rho\mu}{1-\rho}\right\} \geq \frac{c_{11,q}}{c}\left\{z(1+\psi) + (1-z)(1+\psi) + \frac{\psi}{\rho}(1-\rho)\right\},$$

or, substituting for $c_{11,q}$ and simplifying,

$$\left\{\frac{1-\mu}{1-\rho} + \frac{\psi}{\rho}\frac{1-\rho\mu}{1-\rho}\right\} \geq \frac{(1-\mu)\,\rho}{1-\rho}\left\{(1+\psi) + \frac{\psi}{\rho}(1-\rho)\right\}$$

or,

$$1-\mu+\frac{\psi}{\rho}(1-\rho\mu) \geq (1-\mu)\,\rho(1+\psi) + (1-\mu)\,\psi(1-\rho).$$

Dividing through by $1-\mu$, we need to show that

$$1+\frac{\psi}{\rho}\frac{1-\rho\mu}{1-\mu} \geq \rho(1+\psi) + \psi(1-\rho)$$

or

$$1+\frac{\psi}{\rho}\frac{1-\rho\mu}{1-\mu}\geq\rho+\psi.$$

Since $\rho \leq 1$ and $(1-\rho\mu)\,/\,[\rho(1-\mu)] = (1/\rho-\mu)\,/(1-\mu) \geq 1$, we have the desired result.

D Fourth Result

To obtain the fourth result, we can see from (23) that, since the first term is zero and the term in square brackets is positive, the result follows if $n_q > 0$. We establish this result here.

Inspecting (20) and (21), and evaluating the derivatives at $c_1 = c_2 = c$, it follows that

$$c_q = \theta n_q.$$

>From (16), we have

$$c_{12,q} = c_{11,q} - \frac{c}{1-\rho},\ c_{22,q} = c_{21,q} - \frac{c}{1-\rho}$$

Totally differentiating the equation, $R = 1$, i.e., $c_{22} = c_{12}$, we obtain $c_{22,q} = c_{12,q}$. Using this result in the previous equation, we obtain

$$c_{11,q} = c_{21,q}.$$

Using these results in (20), we obtain

$$c_q = \mu c_{11,q} + (1-\mu)c_{12,q},$$

or

(33) $$c_q = c_{12,q} + \frac{\mu c}{1-\rho}.$$

Next, totally differentiating the labor first order condition in (16) and using $c_q = \theta n_q$ and $c_{22,q} = c_{12,q}$, we obtain

(34) $$c_q = -\frac{\psi(1-\rho)}{\rho(1+\psi)}c_{12,q}.$$

Substituting for $c_{12,q}$ from (34) into (33), we obtain

$$c_q = \frac{\mu c}{1-\rho}\frac{1}{1+\frac{\rho(1+\psi)}{\psi(1-\rho)}} > 0.$$

Since $n_q = c_q/\theta$, the desired result follows.

B Model With Inelastic Money Demand

In the previous analysis, we found that the equilibria of our model depend upon the elasticity of substitution in utility between cash and credit goods. In that model, this is the same as the elasticity of demand faced by suppliers. To ensure that their profit function is bounded above, it is necessary that that elasticity be no less than unity. To understand the robustness of our results to situations in which the elasticity of substitution between cash and credit goods is low, we break the link between the elasticity of demand faced by suppliers and the elasticity of substitution between cash and credit goods. We do this by modifying the household's utility function. The market structure of the firm sector, and firm technology remain the same. In addition, the sequence of events in the period is also unchanged. That is, at the beginning of the period a fraction of firms set their prices. Then, the monetary authority selects its action. Finally, the remaining prices and quantities for the period are determined in a private sector equilibrium. We abstract from uncertainty in this section.

A Firms

The firm sector is essentially identical to what it was before. There is a continuum of goods. Each good is produced by a monopolist who faces a demand curve with elasticity denoted here by $1/(1-\lambda)$, where $0 < \lambda < 1$. Some firms ('sticky price firms') set prices before the monetary authority takes it current period action, and other firms ('flexible price firms') set their price afterward. All firms operate competitively in homogeneous factor markets. As before, sticky and flexible price firms set prices as follows:

$$P^e = \frac{W(P^e)}{\lambda}, \ \hat{P}(S,x) = \frac{W(S,x)}{\lambda}. \tag{35}$$

where the state S now consists only of the price set by the sticky price firms, P^e. With one exception, all the notation is the same as before. The exception is that ρ has been replaced by λ.

B Households

Preferences are as in (3), with

$$c = \left[zc_1^\rho + (1-z)\,c_2^\rho\right]^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

and

$$c_1 = \left[\int_0^1 c_1(x)^\lambda dx\right]^{\frac{1}{\lambda}}, \; c_2 = \left[\int_0^1 c_2(x)^\lambda dx\right]^{\frac{1}{\lambda}}. \tag{36}$$

The individual goods, $c_i(x)$, $x \in (0,1)$, $i = 1,2$, are produced by the firms discussed in the previous section.

To purchase cash goods, $c_1(x)$, households must use cash accumulated in advance:

$$M - \left[P^e \mu_1 c_{11} + \hat{P}(S,x)\,(1-\mu_1)\,c_{12}\right] \geq 0, \tag{37}$$

where c_{1i} denotes consumption of the sticky price cash goods when $i = 1$ and of the flexible price cash goods when $i = 2$ (c_{2i} is the corresponding notation for credit goods.)[12] Also, μ_1 denotes the fraction of cash goods whose prices are sticky (μ_2 is the corresponding fraction sticky price credit goods).

As before, the household begins the period with nominal assets, A. It then goes to the asset market where it faces constraint, (6). The household's nominal assets evolve as follows:

$$\begin{aligned} xA' \;\leq\; & W(S,x)n + R(S,x)A + (x-1) + D(S,x) - (R(S,x)-1)M \\ & - \left[P^e \mu_1 c_{11} + \hat{P}(S,x)(1-\mu_1)c_{12}\right] - \left[P^e \mu_2 c_{21} + \hat{P}(S,x)(1-\mu_2)c_{22}\right], \end{aligned} \tag{38}$$

where W, R, D and x are as defined before.

The household problem is formally identical to (9), with (8) replaced by (38), (7) replaced by (37), and (10) replaced by:

$$c = \left[z\left[\mu_1 c_{11}^\lambda + (1-\mu_1)c_{12}^\lambda\right]^{\frac{\rho}{\lambda}} + (1-z)\left[\mu_2 c_{21}^\lambda + (1-\mu_2)c_{22}^\lambda\right]^{\frac{\rho}{\lambda}}\right]^{\frac{1}{\rho}}. \tag{39}$$

As before, the solution to the household's problem yields decision rules of the form, $n(A,S,x)$, $M(A,S,x)$, $A'(A,S,x)$, and $c_{ij}(A,S,x)$, $i,j = 1,2$.

Note how our specification of the household problem disentangles the elasticity of substitution between cash and credit goods, c_1 and c_2, from the elasticity of demand for the individual goods, $c_i(x)$, $i = 1,2$, $x \in (0,1)$.

C Markov Equilibrium

Our definition of a Markov equilibrium coincides with the definition given in the previous section, with the obvious modifications. For example the labor market clearing condition is:

$$n(1, S, x) = \mu_1 c_{11} + (1 - \mu_1)c_{12} + \mu_2 c_{21} + (1 - \mu_2)c_{22},$$

where c_{ij} is as previously defined.

D Characterization

This section displays the qualitative properties of the Markov equilibrium of our economy. We proceed as in section on the benchmark model. In particular, we first derive the equations which characterize a Markov equilibrium. For this, we need to first construct the private sector allocation rule, the mapping from government policies to the prices and quantities that define a private sector equilibrium. We then need to express the first order conditions for the monetary authority, who optimizes subject to the private sector allocation rule. In the second section we use our equations to characterize the set of Markov equilibria for the model.

D.1 Private Allocations and Prices

The monetary authority's action, x, is taken at a time when P^e is known. The private sector prices and quantities to be determined are c_{ij}, $i = 1, 2$, q, n, w, R, where $q = \hat{P}/P^e$. As before, we find it convenient to think of the government's policy variable as q instead of x. So, we compute c_{ij}, $i = 1, 2$, n, w, R as a function of q and P^e.

We proceed now to pin down the seven unknowns, c_{ij}, $i = 1, 2$, n, w, R, conditional on q and P^e. For this, we use 7 equations that characterize the equilibrium. As before, the 7 equations depend upon whether or not the cash in advance constraint is binding.

The resource constraint is:

$$g + \mu_1 c_{11} + (1 - \mu_1)c_{12} + \mu_2 c_{21} + (1 - \mu_2)c_{22} = n. \tag{40}$$

Given our utility function, the first order conditions can be written as follows:

$$c_{12} = c_{11} q^{\frac{-1}{1-\lambda}}, \tag{41}$$

$$c_{22} = c_{21} q^{\frac{-1}{1-\lambda}}, \tag{42}$$

$$R = \frac{z}{1-z}\left(\frac{c_1}{c_2}\right)^{(\rho-\lambda)}\left(\frac{c_{21}}{c_{11}}\right)^{1-\lambda}, \tag{43}$$

$$\lambda = \frac{\psi c^{\rho} c_2^{\lambda-\rho} c_{22}^{1-\lambda}}{(1-n)(1-z)}. \tag{44}$$

The cash in advance constraint is given by:

$$P^e \mu_1 c_{11} + q P^e (1-\mu_1) c_{12} \leq 1. \tag{45}$$

When the cash in advance constraint is not binding, then we impose $R = 1$, i.e.,

$$\frac{z}{1-z}\left(\frac{c_1}{c_2}\right)^{(\rho-\lambda)}\left(\frac{c_{21}}{c_{11}}\right)^{1-\lambda} = 1. \tag{46}$$

As before, we use these equations to define the private sector allocation rules, $c_{ij}(P^e, q)$, $i = 1, 2$, $R(P^e, q)$, $n(P^e, q)$.

D.2 Government Problem

We can summarize the previous discussion as providing functions:

$$c = c(P^e, q), \;\; n = n(P^e, q),$$

where c is obtained by substituting (39) into (10). These functions can be substituted into the utility function,

$$U(P^e, q) = u\left[c(P^e, q), n(P^e, q)\right].$$

Define

$$q(P^e) = \arg\max_{q \in D} U(P^e, q).$$

The function, $q(P^e)$, is the monetary authority's best response, given P^e. Equilibrium requires that $q(P^e) = 1$. This equilibrium requirement allows us to construct the equilibrium price, P^e.

D.3 Qualitative Characteristics of Markov Equilibria

As in section C, we begin by considering equilibria which satisfy $R > 1$. In this case, the allocation rules are differentiable and U_q is given by

$$U_q = u_c c_q + u_n n_q$$

Adding and subtracting $u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho} n_q$, we obtain

$$U_q = u_c\left[c_q - (1-z)\left(\frac{c}{c_2}\right)^{1-\rho} n_q\right] + \left[u_n + u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right] n_q. \tag{47}$$

Differentiating (39) and evaluating the result at $q = 1$,in which case $c_{ij} = c_i$, $i, j = 1, 2$, we obtain:

$$c_q = \left(\frac{c}{c_1}\right)^{1-\rho} z\left[\mu_1 c_{11,q} + (1-\mu_1)c_{12,q}\right] + \left(\frac{c}{c_2}\right)^{1-\rho}(1-z)\left[\mu_2 c_{21,q} + (1-\mu_2)c_{22,q}\right],$$

which is just (20). We now derive n_q, by differentiating (40):

$$n_q = \mu_1 c_{11,q} + (1-\mu_1)c_{12,q} + \mu_2 c_{21,q} + (1-\mu_2)c_{22,q}. \tag{48}$$

Substituting out for c_q and n_q in the first set of square brackets in (47), we obtain:

$$\begin{aligned} U_q &= u_c\left[\mu_1 c_{11,q} + (1-\mu_1)c_{12,q}\right]\left[\left(\frac{c}{c_1}\right)^{1-\rho} z - \left(\frac{c}{c_2}\right)^{1-\rho}(1-z)\right] \\ &\quad + \left[u_n + u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right] n_q. \end{aligned}$$

Using the expression for the interest rate, (43), we obtain:

$$\begin{aligned} U_q &= u_c\left[\mu_1 c_{11,q} + (1-\mu_1)c_{12,q}\right](1-z)\left(\frac{c}{c_2}\right)^{1-\rho}(R-1) \\ &\quad + \left[u_n + u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right] n_q. \end{aligned}$$

Differentiating (45) with respect to q, and evaluating the result at $q = 1$:

$$\mu_1 c_{11,q} + (1-\mu_1)c_{12,q} = -(1-\mu_1)c_1. \tag{49}$$

Substituting this into the preceding expression, we obtain:

$$(50) \qquad U_q = -u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}(R-1)\,c_1(1-\mu_1) + \left[u_n + u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right]n_q.$$

Notice that this expression is essentially the same as the corresponding expression, (26), in the benchmark model. In this case, however, it is no longer true that the first term in (50) is zero when $c_1/c_2 = 0$. To see this, it is convenient to write (50) as

$$(51) \qquad U_q = -\left[u_c c_2(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right]\psi_{ID}\left(\frac{c_1}{c_2}\right) + \left[u_n + u_c(1-z)\left(\frac{c}{c_2}\right)^{1-\rho}\right]n_q,$$

where

$$\psi_{ID}\left(\frac{c_1}{c_2}\right) = \left[\frac{z}{1-z}\left(\frac{c_1}{c_2}\right)^{(\rho-1)} - 1\right]\frac{c_1}{c_2}(1-\mu_1).$$

When $\rho > 0$, it is possible to use exactly the same kind of argument used in the previous section to demonstrate that there are at least two Markov equilibria. When $\rho < 0$, $\psi_{ID}(c_1/c_2) \to \infty$ as $c_1/c_2 \to 0$. When $\rho = 0$, ψ_{ID} converges to a constant as $c_1/c_2 \to 0$. Therefore, in these cases, it is not possible to use the same argument as earlier to demonstrate multiplicity of equilibria.

Indeed, in the log case ($\rho = 0$), for certain values of the model parameters, it is possible an analytical expression for the best response. This expression shows that there is a unique equilibrium, which also yields the outcome, $R = 1$. We have also constructed robust numerical examples in which it appears that the Markov equilibrium is unique.

To summarize, a key result of the two previous sections is that the issue of multiplicity of equilibria turns on the elasticity of money demand at very high rates of inflation. Specifically, we found that the multiplicity issue depends on the behavior of

$$(R-1)\frac{c_1}{c_2}.$$

In both economies, c_1 is proportional to M/P, and when inflation rates are high, c_2 is approximately proportional to aggregate consumption. Thus, the multiplicity of equilibria depends on the behavior of $(R-1)M/(Pc)$ at high inflation rates. This expression is equivalent in a sense to the magnitude of the inflation tax, where the net nominal interest rate is interpreted as the tax rate and the base, of course, is the stock of real balances. One interpretation of our results is that if these inflation tax revenues go to zero, as inflation goes to infinity, there are necessarily multiple equilibria, while if the inflation tax revenues do not go to zero, there are often unique equilibria.

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Notes

[1]Our model formalizes the idea in Kydland and Prescott (1977) and in Barro and Gordon (1983) that an unanticipated monetary expansion raises output and can raise welfare. For models and evidence on the effects of inflation on relative allocations, see Cukierman (1983), Parks (1978), and Vining and Elwertowski (1976).

[2]A more compact definition of the state would include only s and $P^e(s)$, $Z(s)$ where $P^e(s)$ and $Z(s)$ are the equilibrium pricing functions. Effectively, this means that S need only

include s. To see why this more compact definition of a state is adequate in our environment, see Chari and Kehoe (1991).

[3]We have adopted a slight change in the notation. Concavity of the function that aggregates $c(\omega)$ into c in the utility function implies that $c(\omega)$ is optimally chosen to be constant for ω associated with sticky price cash goods, flexible price cash goods, etc. Thus, we let c_{12} denote $c(\omega)$ for those ω such that $\omega \leq z$ and which are produced by flexible price firms.

[4]The result will be presented in a subsequent draft of this paper.

[5]In practice, we obtain the functions in (15) and determine D as follows. For given q, P^e, we first solve (12), (14) and the cash in advance constraint as a strict equality for c_{ij}, $i, j = 1, 2, n, x, R$. If R computed in this way satisfies $R \geq 1$, then q, $P^e \in D$. If R violates $R \geq 1$ we resolve the system, replacing the cash in advance constraint by $R = 1$, i.e., u_{11}/u_{21} $= u_{12}/u_{22} = z/(1-z)$. If the cash in advance constraint is satisfied for the resulting values of c_{ij}, $i, j = 1, 2, n, x$, then q, $P^e \in D$. If not, then q, $P^e \notin D$.

[6]For details, see an earlier footnote.

[7]The estimation period is 1970Q1 to 1997Q1. For our monetary aggregate, we used $M1$ (FM1) and for our output measure we used GDP (GDP) (Citibase Mnemonics appear in parentheses). For the interest rate, we used the three-month Treasury bill rate (FYGM3).

[8]Parks (1978) reports that a one percentage point rise in aggregate inflation is associated with a rise in the variance of the log of relative prices of 0.015. Consider a version of this model in which the money stock is stochastic and growth rates of money are i.i.d. over time. In this version it is easy to show that a 1% rise in inflation relative to its steady state value is associated with an increase in the variance of relative prices of $\mu(1-\mu)$ suggesting a value of μ of roughly 0.017.

[9]To verify that sufficient conditions for an optimum are satisfied, we proceeded as follows.

Corresponding to each of the two candidate equillibria, there is a value of P^e. For each P^e we examined the graph of $U(s, P^e, q)$ for a wide range of values of q. In each case, we verified that $q = 1$ is the global maximum.

[10]For example, when we modify the example in the text by raising z to 0.3, the candidate high inflation equilibrium identified by the intersection of ψ_{MD} and ψ_{ID} is not, in fact, an equilibrium. Although $q = 1$ is a local maximum for the monetary authority, U is higher for a very low value of q. In addition, in the modified example $\psi_{MD}(1) < 1$, and the low inflation equilibrium is a Ramsey equilibrium. Relative to the example emphasized in the text, in this example more goods are subject to the cash in advance constraint. As a result, the distortions from inflation are greater and it is not a surprise that there is less inflation in equilibrium.

[11]The underlying parameter values are $\beta = 1/1.03$, $\eta = .065$, $\psi = 1.64$, $\rho = .83$, $\mu = 0.1$, $\bar{z} = 0.3$, $\sigma = 1.01$.

[12]As in the previous section, concavity of the utility function guarantees that households optimally choose to consume all sticky price cash goods at the same rate, and similarly for the flexible price cash goods and the sticky and flexible price credit goods.