Technical Appendix for 'Modeling Money', Christiano, Eichenbaum and Evans, 11/24/96 (with later corrections)

Following is a detailed discussion of a model with capital investment in which the liquidity effects are strong. A monetary shock produces an immediate, long-lasting effect on output and the price response is delayed. Relative to the simplest liquidity effects model, this property of the model reflects the incorporation of two assumptions. First, the income effects on leisure are assumed to be zero. According to some labor economists, this is a defensible assumption. However, Altonji has voiced some skepticism and so we need to pursue this further. The second assumption is that we follow the CE AER papers and proceedings paper by placing an adjustment cost on adjusting the cash spent on consumption and investment. The effect of this assumption is to keep the financial intermediary sector relatively liquid for a period of time extending beyond the time of the cash injection. This has the effect of reproducing dynamically, the effects that occur in the impact period of a shock. The real world interpretation of this assumption is tenuous. It was severely criticized in a comment by Rotemberg on the paper by CCE in the JMCB. An interpretation of this assumption that avoids the Rotemberg criticism and needs to be explored empirically is the following. Basically, the assumption has the implication that most of the extra cash entering the system via an open market operation is sent on by the household to the financial intermediary in the beginning of the next period, and relatively little of the increment is allocated by households to consumption. A possible real world interpretation of the 'adjustment cost' that gives rise to this is that it reflects that many of the firms owned by households are actually owned indirectly via claims on pension funds and other similar illiquid assets. In this case, households do not have immediate access to the extra cash flow received by such firms. Households in effect, are forced to 'return' such cash immediately to the firms in the form of 'retained earnings'. One way to test this idea is to see how retained earnings respond to a monetary shock. Another would be to explore the idea that most households hold most of their assets in illiquid form like pension funds. On the face of it, housing is also very illiquid, although perhaps home equity loans are a way around this. This would have to be thought through.

One thing is clear, the sort of things that leads society to make the funds set aside for retirement hard to adjust is not in our model. It probably reflects a rational response to the type of time inconsistent preferences that Laibson talks about.

Empirically, we find that the data support the view that the Fed implements monetary policy actions with a delay, although M2 reacts right away. In reality, why does this result in a rise in the interest rate right away? Probably because banks are forced to go into the discount window, and to induce them to do so voluntarily requires a rise in the funds rate. This requires a rise in the funds rate because by going into the discount window they give up the option to go into the window again soon.

1 Household Problem

At date t, the household selects $\{C_{t+j}, L_{t+j}, Q_{t+j}, M_{t++j1}, K_{t+j+1}\}_{j=0}^{\infty}$ to optimize discounted utility subject to a particular cash constraint (which has multiplier ν_t below) and an asset evolution constraint (with multiplier μ_t). The household takes as given rates of return on money and capital, R_t and r_t, and the price of labor and consumption goods, W_t and P_t. The household's choice of $C_t, L_t, M_{t+1}, K_{t+1}$are constrained to be functions of an information set Ω_t, while the household's choice of Q_t is constrained to be a function of Ω_t^Q. The information set Ω_t contains all variables dated t and earlier and

$$\Omega_t \supseteq \Omega_t^Q \supseteq \Omega_{t-1}. \tag{1}$$

Our information assumptions would be more clear if we adopted the Lucas and Stokey s^t notation. However, we avoid this to simplify the notation and in the hope that this does not generate too much confusion.

The Lagrangian representation of the household's problem is:

$$\begin{gathered}\max_{\{C_t, P_t, Q_t, M_{t+1}, K_{t+1}\}_{t=0}^{\infty}} E_{\Omega_0^Q} \sum_{t=0}^{\infty} \beta^t \{U(C_t, L_t, H_t) \\ +\nu_t \left[Q_t + W_t L_t - P_t \left(C_t + (1+S_t)(K_{t+1} - (1-\delta)K_t)\right)\right] \\ +\mu_t [R_t (M_t - Q_t + X_t) + D_t + r_t K_t \\ +Q_t + W_t L_t - P_t \left(C_t + (1+S_t)(K_{t+1} - (1-\delta)K_t)\right) - M_{t+1}]\end{gathered} \tag{2}$$

where

$$S_t = S\left(\frac{K_{t+1}}{K_t}\right) = g_1 \left\{ \exp\left[g_2 \left(\frac{K_{t+1}}{K_t} - 1\right)\right] + \exp\left[-g_2\left(\frac{K_{t+1}}{K_t} - 1\right)\right] - 2\right\}.$$

Here,

$$S_t' = g_1 g_2 \left\{ \exp\left[g_2 \left(\frac{K_{t+1}}{K_t} - 1 \right) \right] - \exp\left[-g_2 \left(\frac{K_{t+1}}{K_t} - 1 \right) \right] \right\},$$

so that $S_t = S_t' = 0$ when $K_{t+1} = K_t$. Thus, with this functional form (say, as opposed to with quadratic adjustment costs), the nonstochastic steady state is invarient to values of g_1, g_2. The variable, S_t, represents an adjustment cost in translating investment expenditures into new capital:

$$I_t = (1 + S_t)(K_{t+1} - (1 - \delta)K_t).$$

We think of this as a technology for converting installed capital, K_t, and investment goods, I_t, into new capital for use in production in the following period, K_{t+1}. This is a linear homogeneous technology in which the dollar price of the inputs is P_t for investment goods and $P_t P_t^k$ for installed capital. The dollar price of the output, new capital, is $P_t P_t^q$. The object, P_t^q, is sometimes referred to as 'Tobin's q'. Although we could adopt a decentralization in which the activity of converting installed capital and investment goods into new capital is undertaken by a 'capital goods firm', for simplicity we instead treat this activity as though it were taken care of by the household.

In (2),

$$H_t = H\left(\frac{Q_t}{Q_{t-1}}\right) = d \left\{ \exp\left[c \left(\frac{Q_t}{Q_{t-1}} - 1 - x \right) \right] + \exp\left[-c \left(\frac{Q_t}{Q_{t-1}} - 1 - x \right) \right] - 2 \right\}.$$

Here,

$$H_t' = cd \left\{ \exp\left[c \left(\frac{Q_t}{Q_{t-1}} - 1 - x \right) \right] - \exp\left[-c \left(\frac{Q_t}{Q_{t-1}} - 1 - x \right) \right] \right\}.$$

In steady state, $Q_t/Q_{t-1} = 1 + x$, when $H_t = H_t' = 0$, so that the steady state is invarient to the values of c, d.

The first order conditions associated with the household problem are as follows, with $t = 0, 1, 2, \ldots.$ For Q_t:

$$E\left\{ U_{H,t} H_t' \frac{1}{Q_{t-1}} - \beta U_{H,t+1} H_{t+1}' \frac{Q_{t+1}}{Q_t^2} + \nu_t + \mu_t (1 - R_t) |\text{-}\, {}_t^Q \right\} = 0. \tag{3}$$

For L_t:

$$U_{L,t} + (\nu_t + \mu_t) W_t = 0. \tag{4}$$

For C_t:

$$U_{C,t} = (\nu_t + \mu_t)P_t. \tag{5}$$

For K_{t+1}:

$$(\nu_t + \mu_t)P_t \left[1 + S_t + S_t' \frac{1}{K_t}\left(K_{t+1} - (1-\delta)K_t\right)\right] = E\{\beta \mu_{t+1} r_{t+1} \tag{6}$$

$$+\beta(\nu_{t+1}+\mu_{t+1})P_{t+1}\left[(1-\delta)(1+S_{t+1}) + S_{t+1}' \frac{K_{t+2}}{K_{t+1}^2}(K_{t+2} - (1-\delta)K_{t+1})\right] |-_t\}.$$

For M_{t+1}:

$$\mu_t = \beta E\{\mu_{t+1} R_{t+1} |-_t\}. \tag{7}$$

We proceed now to eliminate the multipliers from these equations, and to rewrite them in an economically interpretable form. To understand (6), it is useful to distinguish between the period t cost of a unit of new capital, K_{t+1}, and the period t value of a unit of installed capital, K_t. We denote the consumption-value cost of a marginal unit of new, end-of-period t capital by P_t^q:

$$P_t^q = 1 + S_t + S_t' \frac{1}{K_t}\left(K_{t+1} - (1-\delta)K_t\right). \tag{8}$$

The ex-rental value of a marginal unit of beginning-of-period t installed capital is denoted by P_t^k:

$$P_t^k = (1-\delta)(1+S_t) + S_t' \frac{K_{t+1}}{K_t^2}(K_{t+1} - (1-\delta)K_t). \tag{9}$$

This is the consumption value of the period t reduction in investment expenditures that would leave K_{t+1} unchanged, given a marginal increase in K_t. Substituting (8)-(9) and (5) into (6), we get:

$$U_{C,t} P_t^q = \beta E\left\{\mu_{t+1} r_{t+1} |-_t\right\} + E\left\{U_{C,t+1} P_{t+1}^k |-_t\right\}. \tag{10}$$

Below, we further simplify this equation, by eliminating μ_t.

Define $\Lambda_t = \mu_t R_t$, so that, by (7),

$$\Lambda_t = \beta R_t E\{\Lambda_{t+1} |-_t\} = \beta E\{R_t \Lambda_{t+1} |-_t\}. \tag{11}$$

Equation (3) implies:

$$E\{\Lambda_t|\Omega_t^Q\} = E\left\{\frac{U_{C,t}}{P_t} + U_{H,t}H_t'\frac{1}{Q_{t-1}} - \beta U_{H,t+1}H_{t+1}'\frac{Q_{t+1}}{Q_t^2}|\Omega_t^Q\right\}. \tag{12}$$

According to (12), $E\{\Lambda_t|\Omega_t^Q\}$ is the period t anticipated (relative to information set Ω_t^Q) value of a marginal increase in M_t. To see this, suppose the increase is spent on an equal increase in Q_t. The first term after the equality is the period t value of a \$1 increase in spending on consumption goods in period t, while the other two terms reflect the effects of the change in Q_t on adjustment costs. Take the expectation of both sides of (11), conditional on Ω_t^Q, and make use of $\Omega_t \supseteq \Omega_t^Q$ and the law of iterated mathematical expectations to get:

$$E\left\{\tilde{\Lambda}_t - \beta\tilde{\Lambda}_{t+1}R_t|\Omega_t^Q\right\} = 0, \tag{13}$$

where

$$\tilde{\Lambda}_t = \frac{U_{C,t}}{P_t} + U_{H,t}H_t'\frac{1}{Q_{t-1}} - \beta U_{H,t+1}H_{t+1}'\frac{Q_{t+1}}{Q_t^2}. \tag{14}$$

For an intuitive derivation of (13), see CE ('The Smoothing Paper').

We proceed now to simplify (10). First, note that, using (7):

$$\begin{aligned} E[\mu_{t+1}r_{t+1}|\Omega_t] &= E\{\beta E[\Lambda_{t+2}|\Omega_{t+1}]r_{t+1}|\Omega_t\} = E\left\{\beta E[\Lambda_{t+2}|\Omega_{t+2}^Q]r_{t+1}|\Omega_t\right\} \\ &= E\left\{\beta E[\tilde{\Lambda}_{t+2}|\Omega_{t+2}^Q]r_{t+1}|\Omega_t\right\} = E\left\{\beta E[\tilde{\Lambda}_{t+2}|\Omega_{t+1}]r_{t+1}|\Omega_t\right\} \\ &= E\left\{\beta E[\tilde{\Lambda}_{t+2}r_{t+1}|\Omega_{t+1}]|\Omega_t\right\} = E\left\{\beta\tilde{\Lambda}_{t+2}r_{t+1}|\Omega_t\right\}. \end{aligned}$$

The first equality substitutes out for μ_{t+1} using (7) and the second applies the LIME. The third equality applies (12). The fourth and sixth equalities apply the LIME and the fact that $r_{t+1} \in \Omega_{t+1}$. Substituting the last result into (10):

$$E\left\{U_{C,t}P_t^q - \beta\left[\beta\tilde{\Lambda}_{t+2}r_{t+1} + U_{C,t+1}P_{t+1}^k\right]|\Omega_t\right\} = 0. \tag{15}$$

According to (15), the utility value of an increase in spending on new capital available for use in the next period ($U_{C,t}P_t^q$) must equal the present value of the extra dollars generated by next period's increase in rent, plus the present value of the ex-rental value of more beginning-of-next-period capital. Note the different dating on the weights attached to r_{t+1} and P_{t+1}^k. This reflects that the former cannot be spent on consumption goods until the subsequent period. In addition, the extra spending on the former involves passing through the adjustment cost function. That is why the weighting function on r_{t+1} is different from the one on P^q_{t+1}. The extra installed capital next period permits one to shift, in the same period, from investment to consumption expenditures, and this does not involve adjusting Q_t.

The labor Euler equation is, after substituting out for the multipliers:

$$U_{L,t} + U_{C,t}\frac{W_t}{P_t} = 0. \tag{16}$$

2 Final Goods Firms

At time t, a final consumption good, Y_t, is produced by a perfectly competitive firm. It does so by combining a continuum of intermediate goods, indexed by $i \in (0,1)$, using the technology:

$$Y_t = \left[\int_0^1 Y_{it}{}^{\frac{1}{\mu}} di\right]^{\mu}, \tag{17}$$

where $1 \leq \mu < \infty$ and Y_{it} denotes the time t input of intermediate good i. Let P_t and P_{it} denote the time t price of the consumption good and intermediate good i, respectively. Profit maximization implies the Euler equation:

$$\left(\frac{P_t}{P_{it}}\right)^{\frac{\mu}{\mu-1}} = \frac{Y_{it}}{Y_t}. \tag{18}$$

According to (18), the demand for intermediate good i is a decreasing function of the relative price of that good, and an increasing function of aggregate output, Y_t. Integrating (18) and imposing (17), we obtain the following relationship between the price of the final good and the price of the intermediate goods:

$$P_t = \left[\int_0^1 P_{it}^{\frac{1}{1-\mu}} di\right]^{(1-\mu)}. \tag{19}$$

3 Intermediate Good Firms

Intermediate good i is produced by a monopolist who uses the following technology:

$$Y_{it} = \begin{cases} K_{it}^{\alpha} L_{it}^{1-\alpha} - \phi & \text{if } K_{it}^{\alpha} L_{it}^{1-\alpha} \geq \phi \\ 0 & \text{otherwise} \end{cases}, \tag{20}$$

where $0 < \alpha < 1$. Here, L_{it} and K_{it} denote time t labor and capital used to produce the i^{th} intermediate good. The parameter ϕ denotes a fixed cost of production. We rule out entry and exit into the production of intermediate good i.

Intermediate firms rent capital and labor in perfectly competitive factor markets. Economic profits are distributed to the firms' owner, the representative household, at the beginning of time period $t+1$. Workers must be paid in advance of production. As a result, firms need to borrow their wage bill, $W_t L_{it}$, from the financial intermediary at the beginning of the period. Repayment occurs at the end of time period t, at the gross interest rate, R_t. Consequently, the firm's total time t costs are given by $R_t W_t L_{it} + r_t K_{it}$. Their cost function is given by

$$C(r_t, R_t W_t, Y_{it}) = A(r_t)^{\alpha}(W_t R_t)^{1-\alpha}(Y_{it} + \phi), \tag{21}$$

where $A = \left(\frac{1}{1-\alpha}\right)^{(1-\alpha)} \left(\frac{1}{\alpha}\right)^{\alpha}$, so that the time t marginal cost of producing additional output, given $Y_{it} > 0$, is $MC(r_t, R_t W_t) = A r_t^{\alpha}(W_t R_t)^{1-\alpha}$. A convenient representation of this expression, which holds in equilibrium, is given by:

$$MC(r_t, R_t W_t) = \frac{1}{1-\alpha}\left(\frac{L_t}{K_t}\right)^{\alpha} W_t R_t = \frac{1}{\alpha}\left(\frac{K_t}{L_t}\right)^{(1-\alpha)} r, \tag{22}$$

where L_t denotes aggregate employment, and we have taken into account our assumption that the aggregate stock of capital is a constant, unity.[1]

In the version of the model where prices are set flexibly, profit maximization leads the intermediate good firm to set its price equal to a constant markup over marginal cost:

$$P_{it} = \mu MC_t,$$

[1]This expression is obtained by noting that an efficiency condition of firm $i \in (0,1)$ is $r_t/(W_t R_t) = [\alpha/(1-\alpha)]\, N_{it}/K_{it}$. The implied equality of firm labor-capital ratios implies that this expression holds for the aggregate ratio of labor to capital, N_t/K_t. To obtain (22), we substituted this expression with the aggregate variables into $A r_t^{\alpha}(W_t R_t)^{1-\alpha}$.

or

$$\frac{W_t R_t}{P_t} = \frac{f_{L,t}}{\mu},\ \frac{r_t}{P_t} = \frac{f_{K,t}}{\mu} \tag{23}$$

where $f_{L,t} = (1-\alpha)\,(K_{it}/L_{it})^{\alpha}$ is the marginal product of capital and $f_{K,t} = \alpha\,(L_{it}/K_{it})^{(1-\alpha)}$ is the marginal product of labor; and we have imposed the equilibrium condition, $P_{it} = P_t$ for all i.

3.1 Financial Intermediary

At time t, a perfectly competitive financial intermediary receives deposits, $M_t - Q_t$, from the household, and lump sum cash injections, X_t, from the monetary authority. These funds are supplied to the loan market at the gross interest rate R_t. Demand in the loan market comes from the intermediary good producers, who seek to finance their wage bill, $W_t L_t$. Clearing in the loan market requires:

$$W_t L_t = M_t - Q_t + X_t. \tag{24}$$

At the end of the period the intermediary pays $R_t(M_t - Q_t)$ to households in return for their deposits, and distributes $R_t X_t$ to households in the form of profits.

4 Equilibrium

The Euler equations in equilibrium are:

$$E\left\{U_{C,t}P_t^q - \beta\left[\beta\tilde{\Lambda}_{t+2}\frac{f_{K,t+1}}{\mu}P_{t+1} + U_{c,t+1}P_{t+1}^k\right] \vert\text{-}\ {}_t^k\right\} = 0,$$

$$U_{L,t} + U_{C,t}\frac{W_t}{P_t} = 0,$$

$$E\left\{\tilde{\Lambda}_t - \beta\tilde{\Lambda}_{t+1}R_t \vert\text{-}\ {}_t^Q\right\} = 0,$$

$$\frac{W_t R_t}{P_t} = \frac{f_{L,t}}{\mu},\ \frac{r_t}{P_t} = \frac{f_{K,t}}{\mu}.$$

We will assume the following parametric utility function:

$$U(C,L,H) = \left[C - \psi_0\frac{(L+H)^{(1+\psi)}}{1+\psi}\right]^{(1-\sigma)} /(1-\sigma). \tag{25}$$

It is convenient to adopt the following scaling of variables:

$$\lambda_t = \tilde{\Lambda}_t M_t, q_t = Q_t/M_t,\ p_t = P_t/M_t,\ w_t = W_t/M_t, 1+x_t = M_{t+1}/M_t,\ r_t^k = r_t/P_t, \tag{26}$$

so that:

$$\lambda_t = \frac{U_{C,t}}{p_t} + U_{H,t}H_t' \frac{1+x_{t-1}}{q_{t-1}} - \beta U_{H,t+1}H_{t+1}' \frac{q_{t+1}}{q_t^2}(1+x_t).$$

Using this notation, the Euler equations can be rewritten as follows:

$$E\left\{U_{C,t}P_t^q - \beta\left[\beta\lambda_{t+2}\frac{f_{K,t+1}}{\mu(1+x_{t+1})}p_{t+1} + U_{C,t+1}P_{t+1}^k\right]|-_t\right\} = 0, \tag{27}$$

$$\psi_0\left(L_t + H_t\right)^{\psi} = \frac{w_t}{p_t}, \tag{28}$$

$$E\left\{\lambda_t - \beta\lambda_{t+1}\frac{R_t}{1+x_t}|-_t^Q\right\} = 0, \tag{29}$$

$$\frac{w_t R_t}{p_t} = \frac{f_{L,t}}{\mu},\ r_t^k = \frac{f_{K,t}}{\mu}. \tag{30}$$

The resource constraint is:

$$C_t + (1+S_t)(K_{t+1} - (1-\delta)K_t) = K_t^{\alpha}L_t^{1-\alpha} - \phi. \tag{31}$$

The household cash constraint, combined with the loan market clearing condition, and taking scaling into account, is:

$$p_t\left(C_t + (1+S_t)(K_{t+1} - (1-\delta)K_t)\right) = 1 + x_t, \tag{32}$$

and the loan market clearing condition is:

$$w_t L_t = 1 - q_t + x_t. \tag{33}$$

4.1 Analytic Dynamics

To get a feel for the dynamic properties of this model, take the ratio of (33) to (32), using (31):

$$\frac{w_t L_t}{p_t\left(K_t^{\alpha}L_t^{(1-\alpha)} - \phi\right)} = \Gamma_t,$$

where

$$\Gamma_t = \frac{1 + x_t - q_t}{1 + x_t}. \tag{34}$$

The response of Γ_t to an innovation in x_t, given that q_t is unresponsive to this, is $d\Gamma_t/dx_t = -q_t/(1+x_t)^2$, or, in steady state:

$$\Gamma_x = \frac{q}{(1+x)^2}.$$

Now, taking into account (28) and (25), we get:

$$\frac{\psi_0 \left(L_t + H_t\right)^{\psi} L_t^{\alpha}}{K_t^{\alpha} - \phi L_t^{(\alpha-1)}} = \Gamma_t. \tag{35}$$

Evaluating the derivative of L_t with respect to x_t, K_t and q_t in steady state where $L = 1$ (see below), we get:

$$\begin{aligned} a_x &= \frac{dL}{dx} = \frac{\Gamma_x(K^{\alpha} - \phi)}{\psi_0(\psi + \alpha) - \phi(1-\alpha)\Gamma},\ a_K = \frac{dL}{dK} = \frac{\alpha \Gamma K^{\alpha-1}}{\psi_0(\psi + \alpha) - \phi(1-\alpha)\Gamma}, \\ a_q &= \frac{dL}{dq} = \frac{-(K^{\alpha} - \phi)/(1+x)}{\psi_0(\psi + \alpha) - \phi(1-\alpha)\Gamma}. \end{aligned}$$

Interestingly, $dL_t/dx_{t-1} = dL_t/dq_{t-1} = 0$ in steady state. That is, the presence of H_t in this expression plays no role in the linear expansion of (35). This reflects that both H and its derivative are zero in steady-state. This has an important implications. Namely, the presence of adjustment costs does not act as a drag dynamically on employment, apart from its action via q_t. The linear approximation of the policy rule is:

$$L_t = L(K_t, q_t, x_t) = 1 + a_x(x_t - x) + a_K(K_t - K) + a_q(q_t - q). \tag{36}$$

Let $L_x = d\log L/d\log(1+x)$:

$$L_x = \frac{(1+x)\Gamma_x(K^{\alpha} - \phi)}{\psi_0(\psi + \alpha) - \phi(1-\alpha)\Gamma}.$$

Given that q_t is predetermined relative to a monetary shock and $0 < q_t < 1$, a jump in x_t raises Γ_t. Hence, if ϕ is sufficiently small, then a rise in x_t

necessarily increases L_t. The resulting impact on the equilibrium interest rate may be seen by substituting labor supply into (30) and solving for R_t :

$$R_t = \frac{1-\alpha}{\mu} \frac{(K_t/L_t)^\alpha}{w_t/p_t} = \frac{1-\alpha}{\mu\psi_0} K_t^\alpha L_t^{-\alpha} (L_t + H_t)^{-\psi},$$

where the first equality reflects labor demand, and the second substitutes out for the real wage using labor supply. The expression establishes that if (and *only* if) a money injection raises equilibrium employment, the equilibrium rate of interest falls. The steady state response of the interest rate to a money shock, $R_x = dR/d\log(1+x)$, is:

$$R_x = -(\alpha+\psi)\frac{1-\alpha}{\mu\psi_0} K^\alpha L_x.$$

The percent response of output to a monetary shock, Y_x, is:

$$Y_x = (1-\alpha)K^\alpha L_x.$$

Substituting the production function into (32), it is easy to see that

$$P_x + Y_x = 1,$$

where $P_x = d\log(P)/d\log(1+x)$. Hence, if the employment effect of a money shock is sufficiently large, then the price effect is small or zero.

4.2 Nonstochastic Steady State

Taking into account that, in steady state, $U_{c,t}$ and p_t are constant, and

$$\frac{M_{t+1}}{M_t} = 1 + x,$$

we find that (27) reduces to:

$$1 = \beta \left[\frac{\beta}{\mu(1+x)} f_K + 1 - \delta \right].$$

Note that money growth enters as a distortion to the steady state capital-labor ratio and that this can be eliminated by resorting to the Friedman rule, $\beta = 1 + x$. Monetary policy cannot also eliminate the monopoly power distortion because formally this would require $1 + x = \beta/\mu < \beta$, and this implies a negative nominal rate of interest, as long as $\mu > 1$. Solving the above expression, we get:

$$\frac{K}{L} = \left\{ \frac{\alpha\beta}{\mu(1+x)} \frac{1}{\frac{1}{\beta} + \delta - 1} \right\}^{\frac{1}{1-\alpha}}.$$

From (29),

$$R = \frac{1+x}{\beta}.$$

Combining this with (28) and (30), we get:

$$L = \left\{ \frac{1}{\psi_0} \frac{\beta(1-\alpha)}{(1+x)\mu} \left(\frac{K}{L}\right)^{\alpha} \right\}^{\frac{1}{\psi}}.$$

Then, from (35),

$$\Gamma = \frac{\psi_0 L^{\psi+\alpha}}{K^{\alpha} - \phi L^{(\alpha-1)}},$$

so that, from (34),

$$q = (1+x)\left[1 - \Gamma\right].$$

With two exceptions, we consider the 'benchmark' parameter values used in the CEE 'Sticky Price and Limited Participation' paper. The exceptions are μ and ψ. Here, we set $\mu = 1.3$, which splits the difference between the values used by Horsntein ($\mu =$ which we pick to guarantee that $P_x = 0$. Thus, we set ψ_0 so that $L = 1$, or:

$$\psi_0 = \frac{\beta(1-\alpha)}{\mu(1+x)} \left\{ \frac{\alpha\beta}{\mu(1+x)} \frac{1}{\frac{1}{\beta} + \delta - 1} \right\}^{\frac{\alpha}{1-\alpha}}.$$

In CEE, it is shown that if pure profits are zero in steady state, then

$$\mu = 1 + \frac{\phi}{Y} = 1 + \frac{\phi}{K^{\alpha} - \phi},$$

since $Y = (K/L)^{\alpha} L - \phi$. Then, solving for ϕ given μ and K :

$$\phi = \left(\frac{\mu - 1}{\mu}\right) K^{\alpha}.$$

In addition,

$$\alpha = 0.36,\ \beta = 1.03^{-.25},\ \psi = 0.57,\ \mu = 1.30,\ x = 0.20,\ \delta = 0.02.$$

In this case,

$$1/\psi = 1.75,\ \phi = 0.76,\ \psi_0 = 1.34,\ \Gamma_x = 0.39,\ L_x = 1.20,\ R_x = -1.35,\ Y_x = 1.00,\ P_x = 0.00,$$

and

$$K = 27.58,\ Y = 2.54,\ K/Y = 10.86,\ C = 1.99,\ C/Y = 0.78,\ I/Y = 0.22,\ q = 0.56.$$

The labor supply elasticity, 1.75, is high relative to the micro literature. When a more plausible value of 1 is used, then $P_x = 0.37$. Interestingly, even in this case, the output effect of a money shock exceeds the price effect. When $\mu = 1$, but $\psi = 0.57$ then $P_x = 0.39$.

5 Dynamics

We now describe details of how the dynamic properties of the model are obtained. For this, it is useful to drop the t notation, and instead use '$'$' to denote one period ahead, '$''$' to denote two periods ahead, etc. In addition, the subscript, -1, signifies last period.

5.1 Reduction to Two Euler Equations

To write out the Euler equations in a way that is useful for computational purposes, it is convenient to first define several functions. Let

$$L(K, x, q)$$

denote the function defined in (36). Also,

$$H(q_{-1}, q, x_{-1}) = d\left\{\exp\left[c\left(\frac{q}{q_{-1}}(1 + x_{-1}) - 1 - x\right)\right] + \exp\left[-c\left(\frac{q}{q_{-1}}(1 + x_{-1}) - 1 - x\right)\right] - 2\right\}$$

$$dH(q_{-1}, q, x_{-1}) = dc\left\{\exp\left[c\left(\frac{q}{q_{-1}}(1 + x_{-1}) - 1 - x\right)\right] - \exp\left[-c\left(\frac{q}{q_{-1}}(1 + x_{-1}) - 1 - x\right)\right]\right\}$$

$$S(K,K') = g_1\left\{\exp\left[g_2\left(\frac{K'}{K}-1\right)\right] + \exp\left[-g_2\left(\frac{K'}{K}-1\right)\right] - 2\right\}$$

$$dS(K,K') = g_1 g_2\left\{\exp\left[g_2\left(\frac{K'}{K}-1\right)\right] - \exp\left[-g_2\left(\frac{K'}{K}-1\right)\right]\right\}$$

$$C(K,K',x,q) = K^{\alpha}L(K,x,q)^{(1-\alpha)} - \phi - (1+S(K,K'))\left(K'-(1-\delta)K\right)$$

$$\begin{aligned} & U_C(K,K',q_{-1},q,x_{-1},x) \\ = \quad & \left[C\left(K,K',x,q\right) - \psi_0\left(L(K,x,q)+H(q_{-1},q,x_{-1})\right)^{(1+\psi)}/(1+\psi)\right]^{-\sigma} \end{aligned}$$

$$U_H(K,K',q_{-1},q,x_{-1},x) = -\psi_0 U_C\left(K,K',q_{-1},q,x_{-1},x\right)\left[L(K,x,q)+H(q_{-1},q,x_{-1})\right]^{\psi}$$

$$p(K,x,q) = \frac{1+x}{K^{\alpha}L(K,x,q)^{(1-\alpha)} - \phi}$$

$$R(K,x_{-1},x,q_{-1},q) = \frac{1-\alpha}{\mu\psi_0}K^{\alpha}\left[L(K,x,q)\right]^{-\alpha}\left[L(K,x,q)+H(q_{-1},q,x_{-1})\right]^{-\psi}$$

$$P^q(K,K') = 1 + S(K,K') + dS(K,K')\frac{1}{K}\left(K'-(1-\delta)K\right)$$

$$P^k(K,K') = (1-\delta)(1+S(K,K')) + dS(K,K')\frac{K'}{K^2}(K'-(1-\delta)K)$$

$$\begin{aligned} \lambda(K,K',K'',q_{-1},q,q',x_{-1},x,x') \quad = \quad & \frac{U_C\left(K,K',q_{-1},q,x_{-1},x\right)}{p\left(K,x,q\right)} \\ & + U_H\left(K,K',q_{-1},q,x_{-1},x\right)dH(q_{-1},q,x_{-1})\frac{1+x_{-1}}{q_{-1}} \\ & - \beta U_H\left(K',K'',q,q',x,x'\right)dH(q,q',x)\frac{q'}{q^2}(1+x) \end{aligned}$$

It is easily verified that, because $dH = 0$ in steady state,

$$\frac{d\lambda}{dK''} = \frac{d\lambda}{dx'} = 0. \tag{37}$$

With these functions in hand, there are only two endogenous variables to be solved for, K and q, and there are two Euler equations to use for this: (27) and (29). Let:

$$z_{-1} = \begin{pmatrix} K \\ q_{-1} \end{pmatrix}.$$

Define the following function:

$$\begin{aligned} & h_K(z_{-1}, z, z', z'', z''', x_{-1}, x, x', x'', x''') \\ = \; & U_C\left(K, K', q_{-1}, q, x_{-1}, x\right) P^q(K, K') \\ & -\beta[\beta\lambda(K'', K''', K'''', q', q'', q''', x', x'', x''') \frac{\alpha\left(L(K', x', q')/K'\right)^{(1-\alpha)}}{\mu(1+x')} p(K', x', q') \\ & +U_C\left(K', K'', q, q', x, x'\right) P^k(K', K'')]. \end{aligned}$$

It is easily verified that, because $dH = H = 0$ in steady state,

$$\frac{dh_K}{dK''''} = \frac{dh_K}{dq_{-1}} = \frac{dh_K}{dx_{-1}} = \frac{dh_K}{dx'''} = 0.$$

Then, (27) corresponds to:

$$E\left[h_K(z_{-1}, z, z', z'', z''', x_{-1}, x, x', x'', x''')|\text{-}\ \right] = 0.$$

Also,

$$\begin{aligned} & h_q(z_{-1}, z, z', z'', x_{-1}, x, x', x'') \\ = \; & \lambda(K, K', K'', q_{-1}, q, q', x_{-1}, x, x') - \beta\lambda(K', K'', K''', q, q', q'', x, x', x'') \frac{R(K, x_{-1}, x, q_{-1}, q)}{1+x}, \end{aligned}$$

It is easily verified that, because of (37), the derivatives of h_q with respect to K''', x'' are zero, when evaluated in steady state. Then, (29) corresponds to:

$$E\left[h_q(z_{-1}, z, z', z'', x_{-1}, x, x', x'')|\text{-}\ ^Q\right] = 0.$$

Then, the Euler equations can be written in compact form as follows:

$$\mathcal{E}h(z_{-1}, z, z', z'', z''', x_{-1}, x, x', x'', x''') = 0, \tag{38}$$

where

$$h = \begin{pmatrix} h_K \\ h_q \end{pmatrix}, \; \mathcal{E}\left[\cdot\right] = \begin{pmatrix} E\left[\cdot|\text{-}\ \right] \\ E\left[\cdot|\text{-}\ ^Q\right] \end{pmatrix}.$$

5.2 Uncertainty

We consider the following time series representation for money growth, x_t :

$$x_t - x = \rho(x_{t-1} - x) + \theta_0\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3} + \theta_4\varepsilon_{t-4}.$$

We express this in a first order autoregressive form as follows. Let

$$s_t = \begin{pmatrix} x_t - x \\ v_t \\ x_{t-1} - x \\ v_{t-1} \end{pmatrix}, \; P = \begin{bmatrix} \rho & (\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) & 0 & 0_{1\times 4} \\ 0_{4\times 1} & F & 0_{4\times 1} & 0_{4\times 4} \\ 1 & 0_{1\times 4} & 0 & 0_{1\times 4} \\ 0_{4\times 1} & I_4 & 0_{4\times 4} & 0_{1\times 4} \end{bmatrix},$$

$$\varsigma_t = \begin{pmatrix} \theta_0\varepsilon_t \\ \varepsilon_t[1\ 0\ 0\ 0]' \\ 0 \\ 0_{4\times 1} \end{pmatrix}, \; v_t = \begin{pmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \varepsilon_{t-3} \end{pmatrix}.$$

Then the time series representation for x_t may be written as follows:

$$s_t = Ps_{t-1} + \varsigma_t. \tag{39}$$

5.3 Linearized Representation of System

Write the first order Taylor series expansion of h in (38) as follows:

$$\begin{aligned} & h(z_{t-1}, z_t, z_{t+1}, z_{t+2}, z_{t+3}, x_{t-1}, x_t, x_{t+1}, x_{t+2}, x_{t+3}) \\ = & \sum_{i=0}^{4} \alpha_i z_{t+3-i} + \sum_{i=0}^{3} \beta_i s_{t+3-i}, \end{aligned}$$

where we now interpret z_t as the deviation of z_t from its steady state value and the α_i's and β_i's are the coefficients in the Taylor series expansion. Here, $\beta_0, \beta_1, \beta_2$ have non-zero elements in the first column, and zeros elsewhere, while β_3 has (potentially) non-zero elements in the first and and sixth columns and zeros elsewhere. Then, our linearized system is composed of (39) and

$$\mathcal{E}_t \left[\sum_{i=0}^{4} \alpha_i z_{t+3-i} + \sum_{i=0}^{3} \beta_i s_{t+3-i} \right] = 0, \tag{40}$$

where the $\mathcal{E}$ operator is defined after (38). A solution to (39)-(40) is a 2×2 matrix, A, and a 2×10 matrix B in:

$$z_t = Az_{t-1} + Bs_t,$$

which satisfies three conditions: (i) (40) is satisfied for all possible z_{t-1}, s_t, (ii) the eigenvalues of A lie inside the unit circle, and (iii) the $2,1$ and $2,2$ elements of B are zero.