Estimation, Solution and Analysis of Equilibrium Monetary Models

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Questions

• Why Did Inflation Take Off in Many Countries in the 1970s?
• Does Low Inflation Monetary Target Expose Economy to Collapse?
  – Claim -
    * Combination: Low Inflation Target and Zero Lower Bound
    * ⇒ Risk: Fall into a Downward Spiral of Deflation and Low Output
  – Does Claim Hold Water In Reasonably Constructed Models of Economy?
• Japan’s ‘Lost Decade’ of 1990s: Why?
• Was the Severity of the Great Depression a Consequence of Bad Monetary Policy?
• What is the Appropriate Monetary Policy in the Wake of a Financial Crisis?
• Should Central Bank Monitor Monetary Aggregates?
• Stock Market...
  – Why Did the Recent Stock Market Boom/Bust Cycle Occur?
  – Should (Could?) Monetary Policy Have Done Something to Prevent The Cycle?
  – Did Monetary Policy Inadvertently Contribute to the Amplitude of the Cycle?
o **Ways to Answer Questions Like These:**
  - Look at Historical Episodes (limited use)
  - Experiment (not an option!)
  - Experiment in Model Economies.

o **Issues to Confront in Analysis of Model Economies**
  a. Empirical: Formulate and Estimate a Model
  b. Analytic
     * Appropriate Equilibrium Concepts for the Issue Studied
     * Relevant Computational Strategies
     * Other Issues.
Questions

Models

Model Selection and Construction

Model Analysis

Answers
Objective

- Provide Practical Exposure to Key Aspects of Formulation, Estimation and Analysis of Equilibrium Models
- Provide Tutorials and MATLAB Software for Analysis of a Range of Models, Which You May Find Useful as Templates for Future Research
- Target Audience:
  - People with Little Exposure to this Material
  - People Currently Already Actively Applying the Material in their Research.
• Part 1: Vector Autoregressions
  – In Widespread Use for Characterizing Key Features of the Data
  – Can Be Used to Help Design Dynamic General Equilibrium Models
  – Two Assignments, In Which We Will Analyze VARs
• Part 2: Introduction to Linearization Strategy for Solving Models
  – Key Elements of Solution Strategy - Not All the Details
  – Three Assignments:
    * Scalar Example - Unit Roots, Exotic Economics of Variable Capital Utilization
    * Vector Example - Exploring the ‘Hours Worked Hypothesis’ About Japanese Economic Slowdown in 1990s.
    * Vector Example - ‘Overinvestment Boom’, An Interpretation of the 1990s ‘Bubble’
• Part 3: A More Elaborate Monetary General Equilibrium Model (ACEL)
  – Role of Various Frictions: Investment Adjustment Costs, Habit Persistence, Variable Capital Utilization
  – The Importance of Firm-Specificity of Capital
  – Assignment: Can Replicate All Aspects of ACEL Analysis, and Explore Robustness.

• Part 4: Evaluating Recent Criticisms of VARs
  – Monte Carlo Evidence on Performance of VARs With Short Run and Long-Run Restrictions

• Part 5: Introducing Financial Frictions and a Banking Sector Into Analysis
  – Analysis of US Great Depression
  – Analysis of the Role of Monetary Policy in ‘Bubbles’ - An Austrian/Neoclassical Perspective
• Part 6: Monetary Policy Rules
  – Operating Characteristics of Taylor Rule
    ✴ Bad Taylor Rule Could Account for High Inflation of 1970s
    ✴ Potential Advantages of Monetary Monitoring.
  – Should a Central Bank Raise or Lower the Interest Rate in the Wake of a Financial Crisis?
    ✴ Analysis of Small, Open Economy
    ✴ The Impact of Collateral Constraints on the Monetary Transmission Mechanism.
  – Consequences of Zero Lower Bound For Monetary Policy
VAR Analysis: General Background Comments

• My Focus Will Be On The Use of VARs to Learn About Construction and Parameterization of Dynamic, General Equilibrium Models of Money.

• A ‘Shock-Based Approach to Estimating Models’.

• Basic Idea:
  – Use Minimal Restrictions From Class of Economic Models Under Consideration to Estimate the Dynamic Impact of Economic Shocks on Economic Variables of Interest
  – Choose Parameters and Functional Forms for a Dynamic General Equilibrium Model to Match These Effects as Closely As Possible.

• Lucas Program for Model Selection:
  ‘Need to test them (models) as useful imitations of reality by subjecting them to shocks for which we are fairly certain how actual economies would react. The more dimensions on which the model mimics the answers actual economies give to simple questions, the more we trust its answers to harder questions.’
VAR Analysis: General Background Comments

• Alternative Strategy (Discussed in Part 5):
  – Maximum Likelihood
  – Must Have *all* the Shocks in and Without Specification Error.
VAR Analysis: General Background Comments

• Advantages of Shock-Based Approach
  – Focus of Analysis is on Objects of Specific Economic Interest:
    ∗ What Accounts for Inertia in Response of Inflation to Policy Shocks
      (Mankiw, Chari-Kehoe-McGrattan)?
      · Need a Measure of That Inertia
    ∗ What Fraction of Variance of Output is Due to Different Kinds of
      Technology Shocks?
      · Need a Measure of Different Shocks
  – May Only Want a Small Number of Shocks in Model - Capturing All the
    Smaller Shocks May Only Generate Specification Error
    ∗ Models are Abstractions: Small Number of Shocks May Be Enough to
      Account for 75-80% of Variation in Data
  – Modelers Who Want All the Shocks Can Use VARs for Diagnostic Purposes
  – Recently, VARs Have Been Criticized. We will Assess the Criticisms.
Outline of VAR Discussion

• Using VARs to Estimate the Dynamic Effects of Shocks
  – The Identification Problem
  – Prior Restrictions as a Way to Solve the Problem: Bivariate Blanchard-Quah Example
  – The Multivariate, Multi-shock Case
    * Shapiro-Watson Approach
    * ACEL Application: Two Technology Shocks, Monetary Policy Shocks
  – Confidence Intervals for Estimated Impulse Response Functions (Bootstrap)
  – Results

• VAR Diagnostics
  – Deciding Whether or Not to First Difference Data (Not Covered Here)
  – Choosing Lag Length of VAR
  – Results

• Variance Decompositions
  – Results
Identification

• Vector Autoregression (VAR):

\[ Y_t = B_1 Y_{t-1} + ... + B_p Y_{t-p} + u_t, \quad E u_t u_t' = V \]

\(B_i\)’s, \(u_t\)’s and \(V\) Obtained by OLS Regressions.

• Fundamental Economic Shocks, \(e_t\):

\[ u_t = C e_t, \quad E e_t e_t' = D, \quad D \sim \text{Diagonal}, \quad C C' = V. \]

• Impulse Responses \((p = 2)\):

\[ Y_t - E_{t-1} Y_t = C e_t, \quad E_t Y_{t+1} - E_{t-1} Y_{t+1} = B_1 C e_t \]
\[ E_t Y_{t+2} - E_{t-1} Y_{t+2} = B_1^2 C e_t + B_2 C e_t \]
Identification ...

• For Impulse Responses: Need $B_i$’s and Columns of $C$.
• Problem: $N^2$ Unknown Elements in $C$,
  a. Only $N(N+1)/2$ Equations in:

$$CC' = V$$

b. Need to Make (Identifying) Assumptions!
Bivariate Blanchard and Quah Example

- Identification Assumption:
  Technology Shock is *Only* Shock that Has Long-Run Impact on (Forecast of) Level of Labor Productivity:

\[
\lim_{j \to \infty} [E_t y_{t+j} - E_{t-1} y_{t+j}] = f(\text{technology shock only})
\]

(sign restriction) \( f' > 0 \)

\[
y_t = \frac{\text{output}}{\text{hour}}
\]

- Blanchard-Quah/Jordi Gali:
  This Assumption Makes it Possible to Estimate Technology Shock, Even Without Direct Observations on Technology
Bivariate Blanchard and Quah Example ...

- Bivariate VAR:

\[ Y_t = B Y_{t-1} + u_t, \quad E u_t u'_t = V \]

\[ u_t = C e_t \]

\[ Y_t = \begin{pmatrix} \Delta y_t \\ x_t \end{pmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad e_t = \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} \]

\[ e_{zt} \sim \text{Technology Shock.} \]

- From Applying OLS To Both Equations in VAR, We Know: \( B, V \)

- Problem: \( CC' = V \) Provides only Three Equations in Four Unknowns in \( C \).

- Result: Assumption that \( e_{2t} \) Has No Long Run Impact on \( y_t \) Supplies the Extra Required Equation
Bivariate Blanchard and Quah Example ...

- Easy to Verify:

\[
\begin{align*}
E_t[\Delta y_{t+1} + \Delta y_t] - E_{t-1}[\Delta y_{t+1} + \Delta y_t] \\
E_t[y_{t+1}] - E_{t-1}[y_{t+1}]
\end{align*}
\]

\[= (1, 0) [B + I] C e_t\]

\[
E_t[y_{t+2}] - E_{t-1}[y_{t+2}] = (1, 0) \left[ B^2 + B + I \right] C e_t
\]

\[
E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = (1, 0) \left[ B^j + B^{j-1} + \ldots + B^2 + B + I \right] C e_t
\]

as \( j \to \infty \):

\[
\lim_{j \to \infty} E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = \\
\lim_{j \to \infty} (1, 0) \left[ \ldots + B^j + B^{j-1} + \ldots + B^2 + B + I \right] C e_t = (1, 0) [I - B]^{-1} C e_t
\]
Bivariate Blanchard and Quah Example ...

- As $j \to \infty$:
  \[
  \lim_{j \to \infty} E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = (1, 0) [I - B]^{-1} C e_t
  \]

- Identification Assumption About Technology:
  \[
  [I - B]^{-1} C = \begin{bmatrix}
  \text{number} & 0 \\
  \text{number} & \text{number}
  \end{bmatrix}
  \]

- Final Result: Solve for $C$ Using
  
  (exclusion restriction) 1, 2 element of $[I - B]^{-1} C$ is zero

  (sign restriction) 1, 1 element of $[I - B]^{-1} C$ is positive
  \[CC'' = V\]

- Conclude: Long-Run Restriction Supplies Extra Equation Needed to Achieve Identification.
Arbitrary Variables, Arbitrary Lags

• More General Case of Arbitrary Number ($N$) of Variables and Lags:

$$X_t = B_1X_{t-1} + B_2X_{t-2} + ... + B_pX_{t-p} + u_t$$

• To Compute Impulse Response to Technology Shock,
  – Require: $B_1, ..., B_p$ and $C_1$, First Column of $C$ in $CC' = V$
  – Can Obtain by OLS: $B_1, ..., B_p$ and $V$
  – Identification Problem: Find $C_1$
• Solution: Use Restriction, as $j \to \infty$:

$$\lim_{j \to \infty} E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = (1, 0, ..., 0) [I - B(1)]^{-1} Ce_t$$

$$B(1) \equiv B_1 + B_2 + ... + B_p.$$
Arbitrary Variables, Arbitrary Lags ... 

- **VAR:**

\[ X_t = B_1 X_{t-1} + B_2 X_{t-2} + ... + B_p X_{t-p} + u_t \]

- **Long-Run Restriction:**

(exclusion restriction) \[ (I - B(1))^{-1} C = \begin{bmatrix} \text{number} & 0, \ldots, 0 \\ \text{numbers} & \text{numbers} \end{bmatrix} \]

(sign restriction) \((1, 1)\) element of \((I - B(1))^{-1} C\) is positive

\[ CC'' = V \]

- **There Are Many** \(C\) **That Satisfy These Constraints. All Have the Same** \(C_1\).
Arbitrary Variables, Arbitrary Lags ...

- Using the Restrictions to Uniquely Pin Down $C_1$
- Let

$$D \equiv [I - B(1)]^{-1} C$$
so,$$ DD' = [I - B(1)]^{-1} V [I - B(1)']^{-1} \equiv S_0 \text{ (Since } CC' = V)$$

- Exclusion Restriction Requires:

$$D = \begin{bmatrix}
d_{11} & 0, \ldots, 0 \\
D_{21} & D_{22}
\end{bmatrix}$$

- So

$$DD' = \begin{bmatrix}
d_{11}^2 & d_{11} D_{21}' \\
D_{21} d_{11} & D_{21} D_{21}' + D_{22} D_{22}'
\end{bmatrix} = \begin{bmatrix}
S_{011} & S_{021}' \\
S_{021} & S_{022}
\end{bmatrix}.$$

- Sign Restriction:

$$d_{11} > 0.$$

- Then, First Column of $D$ Uniquely Pinned Down:

$$d_{11} = \sqrt{S_{011}}, \quad D_{21} = S_{021} / d_{11}$$

- First Column of $C$ Uniquely Pinned Down:

$$C_1 = [I - B(1)]D_1.$$
Arbitrary Variables, Arbitrary Lags ... 

- In the Application We Will Review Here, More Convenient to Follow Variant of Shapiro-Watson Approach.
- Blanchard-Quah will Be Studied Carefully in Part 4.
A Version of the Shapiro-Watson Approach

- **Reduced Form of** $p^{th}$ **Order VAR:**

$$Y_t = B(L)Y_{t-1} + u_t, \quad E u_t u'_t = V$$

$$B(L) = B_1 + B_2L + \ldots + B_pL^{p-1}$$

- **Structural form:**

$$A_0Y_t = A(L)Y_{t-1} + e_t, \quad E e_t e'_t = D$$

diagonal terms on $A_0 \sim \text{unity}$

where

$$C = A_0^{-1}, \quad B(L) = A_0^{-1}A(L).$$
A Version of the Shapiro-Watson Approach ...

• Data ($N = 10$):

$$
Y_t = \begin{pmatrix}
\Delta \ln \text{(relative price of investment}_t \\
\Delta \ln \left( \frac{GDP_t}{\text{Hours}_t} \right) \\
\Delta \ln \left( \text{GDP deflator}_t \right) \\
\text{capacity utilization}_t \\
\ln \left( \text{Hours}_t \right) \\
\ln \left( \frac{GDP_t}{\text{Hours}_t} \right) - \ln \left( \frac{W_t}{P_t} \right) \\
\ln \left( \frac{C_t}{GDP_t} \right) \\
\ln \left( \frac{I_t}{GDP_t} \right) \\
\text{Federal Funds Rate}_t \\
\ln \left( \text{GDP deflator}_t \right) + \ln \left( GDP_t \right) - \ln \left( MZM_t \right)
\end{pmatrix}
= \begin{pmatrix}
\Delta p_{It} \\
\Delta a_{It} \\
Y_{1t} \\
R_{It} \\
Y_{2t}
\end{pmatrix}
$$
Figure 1: data used in the analysis
Three Identified Shocks:

- Two Technology Shocks: *Only* Shocks that Have a Long Run Impact on Labor Productivity
  * Neutral Shock to Technology.
  * Embodied Shock to Technology
    - only shock with long-run effect on investment good prices (Fisher)
- Monetary Policy Shock
  * Disturbance in OLS Regression:

\[ R_t = f(\Omega_t) + \omega e_{Rt}, \]

\[ e_{Rt} \perp \Omega_t \]

* ⇒ Monetary Policy Has No Contemporaneous Impact on Prices and Aggregate Allocations.
* ⇒ Interest Rate Not Significantly Contemporaneously Affected by Money Demand Shocks, Other than Via \( \Omega_t \).
A Version of the Shapiro-Watson Approach ...

What Is A Monetary Policy Shock?

- Shocks to Preferences of Monetary Authority
- Technical Factors Like Measurement Error (Bernanke-Mihov):

\[
x_t(0) = x_t + v_t, \quad x_t(1) = x_t + u_t
\]
\[
S_t = \beta_0 S_{t-1} + \beta_1 x_t(0) + \beta_2 x_{t-1}(1)
\]

or

\[
S_t = \beta_0 S_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + \varepsilon_t
\]
\[
\varepsilon_t = \beta_1 v_t + \beta_2 u_{t-1}.
\]
A Version of the Shapiro-Watson Approach ...

What Is a Monetary Policy Rule?

- Combination of Structural Policy Rule and Other Stuff
  - Example (Clarida-Gertler):
    
    ‘True’ Policy Rule: \[ S_t = \alpha E_t x_{t+1} + \varepsilon_t \]
    \[ = f(\text{all time } t \text{ data used in } E_t x_{t+1}) + \varepsilon_t \]
A Version of the Shapiro-Watson Approach ...

- Restrictions on $A_0$ Implied by Monetary Policy Identification:

$$A_0 Y_t = A(L) Y_{t-1} + e_t$$

$$
\begin{bmatrix}
A_{0}^{1,1} & A_{0}^{1,2} & A_{0}^{1,3} & 0 & 0 \\
1 \times 1 & 1 \times 1 & 6 \times 6 & 1 \times 1 & 1 \times 1 \\
A_{0}^{2,1} & A_{0}^{2,2} & A_{0}^{2,3} & 0 & 0 \\
1 \times 1 & 1 \times 1 & 1 \times 6 & 1 \times 1 & 1 \times 1 \\
A_{0}^{3,1} & A_{0}^{3,2} & A_{0}^{3,2} & 0 & 0 \\
6 \times 1 & 6 \times 1 & 6 \times 6 & 6 \times 1 & 6 \times 1 \\
A_{0}^{4,1} & A_{0}^{4,2} & A_{0}^{4,3} & A_{0}^{4,4} & 0 \\
1 \times 1 & 1 \times 1 & 1 \times 6 & 1 \times 1 & 1 \times 1 \\
A_{0}^{5,1} & A_{0}^{5,2} & A_{0}^{5,3} & A_{0}^{5,4} & A_{0}^{5,5} \\
1 \times 1 & 1 \times 1 & 1 \times 6 & 1 \times 1 & 1 \times 1
\end{bmatrix}
\begin{bmatrix}
\Delta p_{It} \\
1 \times 1 \\
\Delta a_t \\
1 \times 1 \\
Y_{1t} \\
6 \times 1 \\
R_t \\
1 \times 1 \\
Y_{2t} \\
1 \times 1
\end{bmatrix}
= A(L) Y_{t-1}

\Omega_t = \{\Delta a_t, \Delta p_{It}, Y_{1t}, Y_{t-s}, s > 0\}

- $Y_{2t}$ Assumed Excluded in Monetary Policy Rule
- $Y_{2t}, R_t$ Must Not Affect First 8 Variables
A Version of the Shapiro-Watson Approach ...  

- Analogous Restrictions on $A_0$ From Technology Identification Assumptions
- Instead, We Proceed as in Shapiro-Watson.
- Structural Form:

$$A_0 Y_t = A(L)Y_{t-1} + e_t$$

- Scaled First Structural Equation:

$$\Delta p_{It} = a_{11}(L)\Delta p_{It-1} + a_{12}(L)\Delta^2 a_t + a_{13}(L)\Delta Y_{1t} + a_{14}(L)\Delta R_{t-1} + a_{15}(L)\Delta Y_{2,t-1} + \frac{e_{\gamma,t}}{A_{0,1,1}}$$  

$e_{\gamma,t} \sim$ Shock to Price of Investment Goods

$$\Delta \equiv 1 - L$$
A Version of the Shapiro-Watson Approach...

- Scaled First Structural Equation:

\[
\Delta p_{It} = a_{11}(L)\Delta p_{It-1} + a_{12}(L)\Delta^2 a_t + a_{13}(L)\Delta Y_{1t} + a_{14}(L)\Delta R_{t-1} + a_{15}(L)\Delta Y_{2,t-1} + \frac{e_{\Upsilon,t}}{A^{1,1}},
\]

\[e_{\Upsilon,t} \sim \text{Shock to Price of Investment Goods}\]

\[\Delta \equiv 1 - L\]

\[a_{1j}(L) \sim \begin{cases} p - 1 \text{ unknown coefficients, } j = 4, 5 \\ p \text{ unknown coefficients, } j = 1, 2, 3 \end{cases}\]

- Presence of \(\Delta\) Reflects Identification Assumption that Shocks Other than \(e_{\Upsilon,t}\) Have Zero Impact on \(p_{It}\).
- Presence of \(R_{t-1}, Y_{2,t-1}\) Reflects Monetary Policy Identification.
- Ordinary Least Squares Not Consistent - Use Instrumental Variables.
  * Use \(Y_{t-1}, \ldots, Y_{t-p}\) (Have Testable Overidentifying Restrictions Here).
A Version of the Shapiro-Watson Approach ...

• Scaled Second Structural Equation:

\[
\Delta a_t = a_{22}(L)\Delta a_{t-1} + a_{21}(L)\Delta p_{It} \\
+ a_{23}(L)\Delta Y_{1t} + a_{24}(L)\Delta R_{t-1} + a_{25}(L)\Delta Y_{2,t-1} + \frac{e_{zt}}{A_{0,2}^{2,2}},
\]

\[
a_{2j}(L) \sim \begin{cases} 
  p + 1 \text{ unknown coefficients, } j = 1 \\
  p - 1 \text{ unknown coefficients, } j = 4, 5 \\
  p \text{ unknown coefficients, } j = 2, 3
\end{cases}
\]

– Absence of Any Extra $\Delta$ on $\Delta p_{It}$:
  * Reflects that Non-Technology Shocks Already Affect $\Delta p_{It}$ With Unit Moving Average Root.
  * Implies One Extra Parameter to Instrument.

– Presence of $R_{t-1}, Y_{2,t-1}$ Reflects Monetary Policy Identification.

– Ordinary Least Squares Not Consistent - Use Instrumental Variables.
  * Use $\hat{e}_{\Upsilon,t}, Y_{t-1}, \ldots, Y_{t-p}$
  * Note: Use Assumption that $e_{zt}$ and $e_{\Upsilon,t}$ Uncorrelated
A Version of the Shapiro-Watson Approach ...

- Scaled Ninth Structural Equation:

  \[
  R_t + \frac{A_{0}^{4,1}}{A_{0}^{4,4}} \Delta p_{It} + \frac{A_{0}^{4,2}}{A_{0}^{4,4}} \Delta a_t + \frac{A_{0}^{4,3}}{A_{0}^{4,4}} Y_{1t} = c(L)Y_{t-1} + \frac{e_{Rt}}{A_{0}^{4,4}}
  \]

  – OLS is Fine!

- Scaled Tenth Structural Equation:

  \[
  Y_{2t} + \frac{A_{0}^{5,1}}{A_{0}^{5,5}} \Delta a_t + \frac{A_{0}^{5,2}}{A_{0}^{5,5}} \Delta p_{It} + \frac{A_{0}^{5,3}}{A_{0}^{5,5}} Y_{1t} + \frac{A_{0}^{5,4}}{A_{0}^{5,5}} R_t = d(L)Y_{t-1} + \frac{e_{2t}}{A_{0}^{5,5}}
  \]

  – OLS is Fine! ($Y_{2t}$ Does Not Enter Any Other Equation).
A Version of the Shapiro-Watson Approach ...

- Reminder: Structural Form

\[ A_0 Y_t = A(L) Y_{t-1} + e_t \]

- We Have Established that the 1st, 2nd, 9th and 10th Rows of Structural Form Are Identified.

- The Middle Six Rows of Structural Form Are Not Identified:

\[ A_{3,1}^3 \Delta a_t + A_{0}^{32} \Delta p_{It} + A_{0}^{3,3} Y_{1t} = b(L) Y_{t-1} + e_{1t} \]

  - For Any Setting of \( A_0 \) and \( A(L) \), Premultiplication of Middle Six Rows by an Orthonormal Matrix Leaves Reduced Form (And, Hence, Likelihood) Unaffected.
  - Impulse Responses to Policy and Technology Shocks Invariant to Choice of Orthonormal Matrix.
  - Without Loss of Generality, Restrict \( A_{0}^{3,3} \) To Be Lower Triangular.
  - Estimate Coefficients of Middle 6 Equations by Instrumental Variables.
Confidence Intervals and the Bootstrap

• Estimation Produces:

\[ Y_t = B(L)Y_{t-1} + \hat{A}_0^{-1}\hat{e}_t, \]
\[ \hat{e}_t, \ t = 1, \ldots, T, \]

where

\[ B(L) = \hat{A}_0^{-1}\hat{A}(L). \]

• Bootstrap

– Generate \( r = 1, \ldots, R \) Artificial Data Sets, Each of Length \( T \)

* For \( r^{th} \) Dataset:

\[ \lambda_t^r \in Uniform[0, 1], \ t = 1, \ldots, T \]

* Draw Integers:

\[ \tilde{\lambda}_t^r = \text{integer}(\lambda_t^r \times T), \ t = 1, \ldots, T \]

* Draw Shocks:

\[ \hat{e}_{\tilde{\lambda}_1^r}, \ldots, \hat{e}_{\tilde{\lambda}_T^r} \]
Confidence Intervals and the Bootstrap ...

* Generate Artificial Data:
  \[ Y_t^r = B(L)Y_{t-1}^r + \hat{A}_0^{-1}\hat{e}_t^r, \ t = 1, \ldots, T. \]

– Suppose Statistic of Interest is \( \psi \) (could be vector of impulse response functions, serial correlation coefficients, etc.)
  \[ \psi^r = f (Y_1^r, \ldots, Y_T^r), \ r = 1, \ldots, R \]

* Compute
  \[ \sigma_\psi = \left\{ \frac{1}{T} \sum_{t=1}^{T} (\psi_t^r - \bar{\psi})^2 \right\}^{1/2} \]

* Report
  \[ \hat{\psi} \pm 2 \times \sigma_\psi. \]

* Or, p-value
  \[ prob(\psi^r > \hat{\psi}). \]

* Impulse Response Functions, \( \psi = (\psi_1, \ldots, \psi_{600}) \)

* \( \psi \) Measures of Serial Correlation, etc.
Results for Impulse Response Functions

- Show Data, VAR Lag Length: 4, Sample Period 1959Q1-2001Q3.
- Responses to Monetary Policy Shock
  - Impact on Money Growth and Interest Rate Over in 1 Year, Other Variables Keep Going
  - Significant Liquidity Effect
  - Inflation Peaks in Roughly Two Years
  - Output, consumption, investment, hours worked and capacity utilization are hump-shaped.
  - Velocity comoves with the interest rate
- Responses to a Positive, Neutral Technology Shock
  - Output, hours, investment, consumption display strong, positive, significant responses.
  - Strong, immediate drop in inflation
- Responses to a Negative, Embodied Shock to Technology
  - Rise in Output, Hours Worked, Interest Rate, Strong Rise in Investment.
Figure 3: Benchmark model – dynamic response to a monetary policy shock
Figure 4: Benchmark model – dynamic response to a neutral technology shock
Figure 5: Benchmark model – dynamic response to an embodied technology shock
VAR Diagnostics

- Whether or not to First Difference Hours Worked Important
- Choosing VAR Lag Length

Akaike: \[ s(p) = \log(\det \hat{V}_p) + (m + m^2 p) \frac{2}{T} \]

Hannan-Quinn: \[ s(p) = \log(\det \hat{V}_p) + (m + m^2 p) \frac{2 \log(\log(T))}{T} \]

Schwarz: \[ s(p) = \log(\det \hat{V}_p) + (m + m^2 p) \frac{\log(T)}{Y} \]

\( T \) sample size, \( m \); Number of Variables (10); \( p \) Number of Lags

- Choice:
\[ \hat{p} = \min_p s(p). \]
VAR Diagnostics ...

– With $T = 170$ :

$$\frac{2}{T} = 0.0118, \quad \frac{2 \log(\log(T))}{T} = 0.0192, \quad \frac{\log(T)}{Y} = 0.0302$$

– Akaike Penalizes $p$ the Least
  * Known: In Population, Akaike Has Positive Probability of Overshooting True $p$
  * Hannan-Quinn and Schwarz are Consistent.
VAR Diagnostics ...

- Results (see picklag.m): HQ and SC Choose $p = 1$, AIC Chooses $p = 2$

Table: Standard VAR Lag Length Selection Criteria

<table>
<thead>
<tr>
<th>$h$</th>
<th>AIC</th>
<th>HQ</th>
<th>SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-101.24</td>
<td>-100.42</td>
<td>-99.21</td>
</tr>
<tr>
<td>2</td>
<td>-101.42</td>
<td>-99.84</td>
<td>-97.53</td>
</tr>
<tr>
<td>3</td>
<td>-101.28</td>
<td>-98.94</td>
<td>-95.52</td>
</tr>
<tr>
<td>4</td>
<td>-101.23</td>
<td>-98.13</td>
<td>-93.58</td>
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<td>5</td>
<td>-101.02</td>
<td>-97.14</td>
<td>-91.46</td>
</tr>
<tr>
<td>6</td>
<td>-101.04</td>
<td>-96.37</td>
<td>-89.55</td>
</tr>
<tr>
<td>7</td>
<td>-101.02</td>
<td>-95.57</td>
<td>-87.60</td>
</tr>
<tr>
<td>8</td>
<td>-101.12</td>
<td>-94.88</td>
<td>-85.75</td>
</tr>
</tbody>
</table>
VAR Diagnostics ...

- **Multivariate $Q(s)$ Statistic**
  - Measure of Serial Correlation In Fitted Disturbances
  - Null Hypothesis: the First $s$ Autocorrelations Are Zero:

\[
Q(s) = T(T + 2) \sum_{j=1}^{s} \frac{1}{T - j} \text{trace} \left[ C_j C_0^{-1} C_j' C_0^{-1} \right],
\]

where

\[
C_j = \frac{1}{T} \sum_{t=j+1}^{T} \hat{e}_t \hat{e}_{t-j}'.
\]

- In the Scalar Case, It is the Weighted Sum of the Squares of the First $s$ Correlations.
- Null Distribution:

\[
Q(s) \sim \chi_{m^2(s-p)}^2
\]
VAR Diagnostics ...

– Results (see mkqmv.m):

<table>
<thead>
<tr>
<th>s</th>
<th>Q(s)</th>
<th>degrees of freedom</th>
<th>asymptotic p-value</th>
<th>bootstrap p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>166.81</td>
<td>0</td>
<td>NaN</td>
<td>0.89</td>
</tr>
<tr>
<td>6</td>
<td>350.41</td>
<td>200</td>
<td>0.00</td>
<td>0.92</td>
</tr>
<tr>
<td>8</td>
<td>551.56</td>
<td>400</td>
<td>0.00</td>
<td>0.83</td>
</tr>
<tr>
<td>10</td>
<td>795.67</td>
<td>600</td>
<td>0.00</td>
<td>0.39</td>
</tr>
</tbody>
</table>

– Strikingly Different Between Bootstrap and Asymptotics!

• Conclusion of Lag Length Diagnostics: Go With $p = 4$ Lags, But Redo Everything with $p = 6$, To be Safe
Variance Decompositions

- Forecast Error Variance Due to Various Shocks
- Business Cycle Variation Due to Various Shocks
  - In-Sample Measure
  - Population Measure
Variance Decompositions ...

• Forecast Errors

• Model:

\[ Y_t = \mu + B(L)Y_{t-1} + A_0^{-1}e_t, \]

\[ = \mu + BY_{t-1} + A_0^{-1}e_t, \]

for Simplicity (Note: I Slipped the Constant Term Back In.)

• Conventional \( k \)-step Ahead Forecast Error Variance:

\[ Y_{t+k} - E_{t-1}Y_{t+k} = A_0^{-1}e_{t+k} + BA_0^{-1}e_{t+k-1} + B^2A_0^{-1}e_{t+k-2} + \ldots + B^kA_0^{-1}e_t \]

so

\[ Var = Var (Y_{t+k} - E_{t-1}Y_{t+k}) = \sum_{j=0}^{k} B^j A_0^{-1} (A_0^{-1}) (B^j)' \]
Variance Decompositions ...

- Portion of $k$-step Ahead Forecast Error Due to $i^{th}$ Shock:

$$Var(i) = Var_i (Y_{t+k} - E_{t-1} Y_{t+k}) = \sum_{j=0}^{k} B^j A_0^{-1} I_i (A_0^{-1}) (B^j)' ,$$

where $I_i$ is Identity Matrix With All Zeros on Diagonal Except $i^{th}$ Location.

- Obviously:

$$\sum_{i=1}^{m} Var(i) = Var$$

- Percent of Variance Due to $i^{th}$ Shock:

$$100 \times \frac{Var(i)}{Var}$$

- Obviously, Can Easily Bootstrap the Sampling Distribution of Variance Decomposition.
Variance Decompositions ...

- Complication: Variables in $Y_t$ Are Not the Ones We are Necessarily Interested In!
  - Variables in $Y_t$:

$$
Y_t \begin{pmatrix}
\Delta \ln \text{(relative price of investment}_t

\Delta \ln (GDP_t/\text{Hours}_t)

\Delta \ln (GDP \text{ deflator}_t)

\text{capacity utilization}_t

\ln (\text{Hours}_t)

\ln (GDP_t/\text{Hours}_t) - \ln (W_t/P_t)

\ln (C_t/GDP_t)

\ln (I_t/GDP_t)

\text{Federal Funds Rate}_t

\ln (GDP \text{ deflator}_t) + \ln (GDP_t) - \ln (MZM_t)
\end{pmatrix}
\equiv
\begin{pmatrix}
(1 - L)p_t^I

(1 - L)(y_t - h_t)

(1 - L)p_t

u_t

h_t

y_t - h_t - w_t

c_t - y_t

p_t^I + I_t - y_t

R_t

y_t + p_t - m_t
\end{pmatrix}
$$
Variance Decompositions ...

– We Are Interested in:

\[
\tilde{Y}_t = \begin{pmatrix}
 y_t \\
 4(1 - L)m_t \\
 4(1 - L)p_t \\
 R_t \\
 u_t \\
 h_t \\
 w_t \\
 c_t \\
 I_t \\
 p_t^I \\
\end{pmatrix}, \text{ not } Y_t = \begin{pmatrix}
 (1 - L)p_t^I \\
 (1 - L)(y_t - h_t) \\
 (1 - L)p_t \\
 u_t \\
 h_t \\
 y_t - h_t - w_t \\
 c_t - y_t \\
 p_t^I + I_t - y_t \\
 R_t \\
 y_t + p_t - m_t \\
\end{pmatrix}
\]

– Tedious Formulas Make It Possible to Compute the \( k \)–Percent Forecast Error in \( \tilde{Y}_t \).
Variance Decompositions ...

- Business Cycle Variance Decomposition

\[ Y_t = \mu + BY_{t-1} + A_{0}^{-1}e_t \]

- In-Sample Variance Decomposition
  * Simulate \( \tilde{Y}_t \) Using All Historical Shocks, Then Apply HP Filter
  * Simulate \( \tilde{Y}_t \) Using Only \( i^{th} \) Shocks, Then Apply HP Filter
  * Ratio of Variance of Two Filtered Series is Business Variation Due to \( i^{th} \) Shock.
  * Graph All Data
  * To Straighten Out the Scale, Drop \( \mu \) In Simulation.

- Population Variance Decomposition
  * Simulate an Enormous Amount of Data, Not a Sample Same Length As Actual
  * Easily Done in Frequency Domain (see ACEL techn. appendix)

- Business Cycle Variance Decomposition Can Be Bootstrapped to Obtain Sampling Uncertainty.
Figure 6: Historical decomposition – monetary policy shocks only
Figure 7: Historical decomposition – neutral technology shocks only
Figure 8: Historical decomposition – embodied technology shocks only
Figure 9: Historical decomposition – monetary policy and technology shocks
<table>
<thead>
<tr>
<th>Variable</th>
<th>Embodied Technology</th>
<th>Neutral Technology</th>
<th>Monetary Policy</th>
<th>All Three Shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>26 (15)[11]</td>
<td>10 (14)[11]</td>
<td>42 (12)[8]</td>
<td>75 (41)[18]</td>
</tr>
<tr>
<td>Capacity Util.</td>
<td>25 (16)[12]</td>
<td>3 (9)[8]</td>
<td>38 (12)[8]</td>
<td>56 (37)[18]</td>
</tr>
<tr>
<td>Real Wage</td>
<td>2 (10)[8]</td>
<td>1 (15)[14]</td>
<td>3 (5)[4]</td>
<td>3 (29)[15]</td>
</tr>
<tr>
<td>Consumption</td>
<td>25 (16)[13]</td>
<td>17 (18)[12]</td>
<td>50 (12)[8]</td>
<td>95 (45)[21]</td>
</tr>
<tr>
<td>Investment</td>
<td>23 (15)[11]</td>
<td>6 (12)[10]</td>
<td>32 (11)[8]</td>
<td>63 (38)[17]</td>
</tr>
<tr>
<td>Price of Inv.</td>
<td>38 (42)[22]</td>
<td>1 (8)[7]</td>
<td>8 (6)[4]</td>
<td>56 (55)[21]</td>
</tr>
</tbody>
</table>

Notes: Numbers are point estimates, number in parentheses are mean of point estimates across bootstrap simulations; number in square brackets are standard deviation of point estimates across bootstrap simulations.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Embodied Technology</th>
<th>Neutral Technology</th>
<th>Monetary Policy</th>
<th>All Three Shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>22 ( (16)[14] )</td>
<td>9 ( (11)[9] )</td>
<td>34 ( (15)[12] )</td>
<td>45 ( (41)[21] )</td>
</tr>
<tr>
<td>MZM Growth</td>
<td>6 ( (10)[7] )</td>
<td>4 ( (8)[5] )</td>
<td>30 ( (16)[9] )</td>
<td>31 ( (35)[13] )</td>
</tr>
<tr>
<td>Inflation</td>
<td>17 ( (15)[13] )</td>
<td>11 ( (13)[9] )</td>
<td>22 ( (12)[10] )</td>
<td>44 ( (40)[19] )</td>
</tr>
<tr>
<td>Fed Funds</td>
<td>17 ( (13)[12] )</td>
<td>7 ( (9)[8] )</td>
<td>47 ( (20)[14] )</td>
<td>47 ( (42)[21] )</td>
</tr>
<tr>
<td>Capacity Util.</td>
<td>21 ( (15)[13] )</td>
<td>6 ( (9)[8] )</td>
<td>32 ( (14)[12] )</td>
<td>42 ( (36)[20] )</td>
</tr>
<tr>
<td>Avg. Hours</td>
<td>22 ( (15)[14] )</td>
<td>13 ( (11)[9] )</td>
<td>33 ( (14)[12] )</td>
<td>41 ( (39)[22] )</td>
</tr>
<tr>
<td>Real Wage</td>
<td>9 ( (13)[10] )</td>
<td>4 ( (12)[9] )</td>
<td>6 ( (8)[6] )</td>
<td>20 ( (32)[14] )</td>
</tr>
<tr>
<td>Consumption</td>
<td>21 ( (15)[16] )</td>
<td>20 ( (17)[12] )</td>
<td>43 ( (17)[12] )</td>
<td>57 ( (49)[22] )</td>
</tr>
<tr>
<td>Investment</td>
<td>15 ( (15)[13] )</td>
<td>6 ( (10)[7] )</td>
<td>28 ( (14)[12] )</td>
<td>36 ( (38)[19] )</td>
</tr>
<tr>
<td>Velocity</td>
<td>14 ( (12)[12] )</td>
<td>6 ( (9)[8] )</td>
<td>45 ( (18)[13] )</td>
<td>41 ( (37)[20] )</td>
</tr>
<tr>
<td>Price of Inv.</td>
<td>22 ( (22)[17] )</td>
<td>8 ( (13)[8] )</td>
<td>8 ( (9)[6] )</td>
<td>48 ( (51)[16] )</td>
</tr>
</tbody>
</table>

Notes: Numbers are point estimates, number in parentheses are mean of point estimates across bootstrap simulations; number in square brackets are standard deviation of point estimates across bootstrap simulations.
Conclusion from VAR Analysis

- Our Estimates Suggest that the Three Identified Shocks Account for A Substantial Fraction (Around 50%) of Business Cycle Variation.
- We have a set of Estimates of How Economy Responds to Three Types of Shocks
- Will use these to Estimate a DSGE Model.