

Estimation, Solution and Analysis of Equilibrium Monetary Models  
 Answers for Assignment 3: Solution and Analysis of a Simple Dynamic General  
 Equilibrium Model: a Tutorial

1. Trivial.
2. To compute the first order conditions of the model, substitute out as many of the linear constraints as possible. Thus, investment is:

$$I_t = I_t^k + I_t^u = k_{t+1} - (1 - \delta)k_t + a(u_t) k_t.$$

Then, use the resource constraint and the definition of  $Y_t$  to express consumption as follows:

$$C_t = \varepsilon_t (u_t k_t)^\alpha - k_{t+1} + (1 - \delta)k_t - a(u_t) k_t.$$

Then, this can be substituted into the utility function:

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) \\ = & u(\varepsilon_0 (u_0 k_0)^\alpha - k_1 + (1 - \delta)k_0 - a(u_0) k_0) \\ & + \beta E_0 u(\varepsilon_1 (u_1 k_1)^\alpha - k_2 + (1 - \delta)k_1 - a(u_1) k_1) \\ & + \beta^2 E_0 u(\varepsilon_2 (u_2 k_2)^\alpha - k_3 + (1 - \delta)k_2 - a(u_2) k_2) \\ & + \dots, \end{aligned}$$

or,

$$E_0 \sum_{t=0}^{\infty} \beta^t u(\varepsilon_t (u_t k_t)^\alpha - k_{t+1} + (1 - \delta)k_t - a(u_t) k_t).$$

Differentiate with respect to  $k_{t+1}$  :

$$\begin{aligned} & E_t \{-u'(\varepsilon_t (u_t k_t)^\alpha - k_{t+1} + (1 - \delta)k_t - a(u_t) k_t) \quad (0.1) \\ & + \beta u'(\varepsilon_{t+1} (u_{t+1} k_{t+1})^\alpha - k_{t+2} + (1 - \delta)k_{t+1} - a(u_{t+1}) k_{t+1}) \left[ \alpha \varepsilon_{t+1} u_{t+1}^\alpha k_{t+1}^{\alpha-1} + 1 - \delta - a(u_{t+1}) \right] \} = 0, \end{aligned}$$

or,

$$E_t v_k(k_t, k_{t+1}, k_{t+2}, u_t, u_{t+1}, \varepsilon_t, \varepsilon_{t+1}) = 0.$$

Now, differentiate with respect to  $u_t$  :

$$\alpha \varepsilon_t u_t^{\alpha-1} k_t^\alpha - a'(u_t) k_t = 0, \quad (0.2)$$

or,

$$v_u(k_t, u_t, \varepsilon_t) = 0.$$

3. Note that  $v_u(k_t, u_t, \varepsilon_t)k_t^{-\alpha}$  is  $\alpha\varepsilon_t u_t^{\alpha-1} - a'(u_t)k_t^{1-\alpha}$ . The properties of marginal benefit are straightforward to see, marginal benefit is proportional to  $u_t^{\alpha-1}$ . This goes to  $\infty$  as  $u_t \rightarrow 0$  and to 0 as  $u_t \rightarrow \infty$ . Similarly,  $a'$  is a straight line with positive slope so that it eventually turns positive and it is finite at  $u = 0$ . To see how  $u_t$  responds to  $\varepsilon_t$  and  $k_t$ , simply draw a graph with the benefits and costs on the vertical axis and utilization on the horizontal axis, and shift the benefits and costs curves. As you increase  $k$ , the cost curve shifts up and this is what leads to a fall in utilization. Basically, the reason for the fall in utilization is that an increase in capital increases costs more than benefits. As you increase  $\varepsilon_t$  the benefit curve shifts up and this is why utilization rises.

4. Note that  $dMP_{k,t}/du_t = \alpha MP_{k,t}/u_t$ . So, the change in the rate of return on capital with respect to a change in  $u_t$  is  $\alpha MP_{k,t}/u_t - a'(u_t)$ . After multiplying (0.2) by  $k_t^{-1}$ , we find that that equation says  $MP_{kt}/u_t = a'(u_t)$ . Thus, the derivative of the rate of return on capital, with respect to  $u_t$ , is:

$$\begin{aligned} \alpha MP_{k,t}/u_t - a'(u_t) &= \alpha MP_{k,t}/u_t - MP_{kt}/u_t \\ &= (\alpha - 1)MP_{kt}/u_t < 0. \end{aligned}$$

Thus, an increase in utilization definitely reduces the current, ex post, rate of return on capital. The intuition is that the effect of an increase in utilization on the marginal product of capital is smaller than the effect on maintenance costs.

5. Following the discussion in the previous question, in steady state,

$$a'(u) = \alpha\varepsilon u^{\alpha-1}k^{\alpha-1},$$

or, given the functional form assumption on  $a'$  :

$$b\sigma_a u + b(1 - \sigma_a) = \alpha\varepsilon u^{\alpha-1}k^{\alpha-1},$$

or, imposing  $u = 1$  :

$$b = \alpha\varepsilon k^{\alpha-1}.$$

Evaluate (0.1) in steady state:

$$\alpha\varepsilon u^\alpha k^{\alpha-1} + 1 - \delta - a(u) = \frac{1}{\beta},$$

or, after imposing  $u = 1$  and our functional form assumption, which implies  $a(1) = 0$ ,

$$\alpha \varepsilon k^{\alpha-1} + 1 - \delta = \frac{1}{\beta},$$

so that

$$k = \left[ \frac{\alpha \varepsilon}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\alpha}}.$$

Note that this is a function of parameters only, and so is computable. We now have an expression of  $b$  in terms of parameters:

$$b = \frac{1}{\beta} - (1 - \delta).$$

The parameter,  $b$ , is the marginal cost of utilization in steady state (i.e.,  $a'$ ) which must equal the marginal product of capital. The latter is  $1/\beta - (1 - \delta)$ , so that the previous formula should not be surprising. It is easy to verify that:

$$\begin{aligned} I^k &= \delta k, \\ C &= \varepsilon k^\alpha - \delta k. \end{aligned}$$

6. (a) The log-linear expansion of  $v_u$  is as follows

$$\alpha \varepsilon k^\alpha [\hat{\varepsilon}_t + (\alpha - 1) \hat{u}_t + \alpha \hat{k}_t] - bk [\sigma_a \hat{u}_t + \hat{k}_t]$$

Here, we have used the fact,  $dx_t = x \hat{x}$ , and the easily verified results,  $\widehat{xy} = \hat{x} + \hat{y}$ ,  $\hat{y} = \alpha \hat{x}$ , when  $y = x^\alpha$ . Collecting terms:

$$\begin{aligned} V_{u,t} &= (\alpha - 1) \alpha \varepsilon k^\alpha \hat{k}_t + [\alpha - 1 - \sigma_a] \alpha \varepsilon k^\alpha \hat{u}_t + \alpha \varepsilon k^\alpha \hat{\varepsilon}_t, \\ &= V_1^u \hat{k}_t + V_2^u \hat{u}_t + V_3^u \hat{\varepsilon}_t, \end{aligned}$$

where

$$V_1^u = (\alpha - 1) \alpha \varepsilon k^\alpha, \quad V_2^u = [\alpha - 1 - \sigma_a] \alpha \varepsilon k^\alpha, \quad V_3^u = \alpha \varepsilon k^\alpha.$$

In terms of an earlier question, it is useful to write our the expression for  $\hat{u}_t$  implied by the equilibrium condition,  $V_{u,t} = 0$ :

$$\begin{aligned} \hat{u}_t &= -\frac{V_1^u}{V_2^u} \hat{k}_t - \frac{V_3^u}{V_2^u} \hat{\varepsilon}_t \\ &= -\frac{1 - \alpha}{1 - \alpha + \sigma_a} \hat{k}_t + \frac{1}{1 - \alpha + \sigma_a} \hat{\varepsilon}_t. \end{aligned}$$

Note that if  $\hat{k}_t > 0$ , so that the capital stock is above its steady state, then utilization is below its steady state. Interestingly, if  $\sigma_a$  is nearly zero (there are no utilization adjustment costs) then if capital is one percent above steady state, utilization is one percent below, so that capital services,  $u_t k_t$ , are constant. If utilization costs are infinite,  $\sigma_a = \infty$ , then utilization does not vary at all.

The MATLAB routine, `derivative.m`, can do the above differentiation numerically. For it to do so, it needs a program that defines the function,  $v_u$ , that needs to be differentiated. Here is the code:

```
function [vu] = Vuanswer(x,sigma_a,alpha,b)
k=x(1);
u=x(2);
epsil=x(3);
ap=b*sigma_a*u+b*(1-sigma_a);
vu=alpha*epsil*(u^(alpha-1))*(k^alpha)-ap*k;
```

For `derivative.m` to log-linearize the intertemporal Euler equation, it needs a program that defines  $v_k$ . The following will work:

```
function [eul] = Vkanswer(x,sigma_a,alpha,b,delta,sigma,beta)
%variables being differentiated
k=x(1);%k(t)
kp=x(2);%k(t+1)
kpp=x(3);%k(t+2)
u=x(4);%u(t)
up=x(5);%u(t+1)
epsil=x(6);%epsilon(t)
epsilp=x(7);%epsilon(t+1)
%the function
[a] = gogeta(u,b,sigma_a);
[ap] = gogeta(up,b,sigma_a);
c=epsil*((u*k)^alpha)-kp+(1-delta)*k-a*k;
cp=epsilp*((up*kp)^alpha)-kpp+(1-delta)*kp-ap*kp;
retp=alpha*epsilp*(up^alpha)*(kp^(alpha-1))+1-delta-ap;
eul=-(c^(-sigma))+beta*(cp^(-sigma))*retp;
```

The program calls `gogeta.m`, which is:

```
function [a] = gogeta(u,b,sigma_a)
a=0.5*b*sigma_a*(u^2)+b*(1-sigma_a)*u+b*((sigma_a/2)-1);
```

6. (b) the derivatives are  $V_1^u = -0.98$ ,  $V_2^u = -1.13$ , and  $V_3^u = 1.53$ , respectively.

6. (c)  $V_1^k = 5.72$ ,  $V_2^k = -11.39$ ,  $V_3^k = 5.67$ ,  $V_4^k = -1 \times 10^{-7}$ ,  $V_5^k = -0.01$ ,  $V_6^k = 0.43$ ,  $V_7^k = -0.42$ .

7.  $\alpha_0 = 5.67$ ,  $\alpha_1 = -11.39$ ,  $\alpha_2 = 5.72$ ,  $\beta_0 = -0.43$ ,  $\beta_1 = 0.43$ ,  $A = 0.99$ ,  $B = 0.06$ .

8. (a) For benchmark parameter values,  $t^*/4 = 69$ . With  $\delta = 0$ ,  $t^*/4 = 391$ ; with  $\alpha = 0.10$ ,  $t^*/4 = 27$ ; with  $\sigma = 4$ ,  $t^*/4 = 184$ . (b) Finally, for  $\sigma_a = 10,000$ ,  $t^*/4 = 21$ . Qualitatively, these results are in line with the intuition described in the tutorial.

9. The resource constraint is:

$$C_t + k_{t+1} - (1 - \delta)k_t + a(u_t)k_t = \varepsilon_t (u_t k_t)^\alpha$$

Log-linearly expanding this,

$$C\hat{C}_t + k\hat{k}_{t+1} - (1 - \delta)k\hat{k}_t + bk\hat{u}_t = \varepsilon k^\alpha [\hat{\varepsilon}_t + \alpha(\hat{u}_t + \hat{k}_t)],$$

using  $a'(1) = b$  and  $a(1) = 0$ . Then,

$$\begin{aligned} \hat{C}_t &= \frac{1}{C} [\varepsilon k^\alpha [\hat{\varepsilon}_t + \alpha(\hat{u}_t + \hat{k}_t)] - k\hat{k}_{t+1} + (1 - \delta)k\hat{k}_t - bk\hat{u}_t] \\ &= \frac{1}{C} [\varepsilon k^\alpha [\hat{\varepsilon}_t + \alpha(\hat{u}_t + \hat{k}_t)] - kA\hat{k}_t - kB\hat{\varepsilon}_t + (1 - \delta)k\hat{k}_t - bk\hat{u}_t] \\ &= \frac{1}{C} [\varepsilon k^\alpha (\hat{\varepsilon}_t + \alpha\hat{k}_t) - kA\hat{k}_t - kB\hat{\varepsilon}_t + (1 - \delta)k\hat{k}_t] \\ &= \frac{1}{C} [\alpha\varepsilon k^\alpha - kA + (1 - \delta)k] \hat{k}_t + \frac{1}{C} [\varepsilon k^\alpha - kB] \hat{\varepsilon}_t \end{aligned}$$

since  $\alpha\varepsilon k^\alpha = bk$ . Then,

$$\begin{aligned} C_k &= \frac{1}{C} [\alpha\varepsilon k^\alpha - kA + (1 - \delta)k] \\ C_\varepsilon &= \frac{1}{C} [\varepsilon k^\alpha - kB] \end{aligned}$$

The expression for investment is:

$$I_t^k = k_{t+1} - (1 - \delta)k_t$$

so that,

$$I^k \hat{I}_t^k = k\hat{k}_{t+1} - (1 - \delta)k\hat{k}_t,$$

so that (taking into account  $I^k = \delta k$ ),

$$\begin{aligned}\hat{I}_t^k &= \frac{1}{\delta} \hat{k}_{t+1} - \frac{1-\delta}{\delta} \hat{k}_t \\ &= \frac{1}{\delta} [A - (1-\delta)] \hat{k}_t + \frac{1}{\delta} B \hat{\varepsilon}_t\end{aligned}$$

Then,

$$I_k = \frac{1}{\delta} [A - (1-\delta)], \quad I_\varepsilon = \frac{1}{\delta} B.$$

Note that if  $A$  is close enough to unity (and  $\sigma_a$  can always be made small enough to make this happen), then the coefficient on  $\hat{k}_t$  is positive. This can happen because of the definition of investment we're using here, *gross* investment. Gross investment is net investment (i.e., the actual change in the capital stock) plus depreciation investment. The stability property of our system simply implies that net investment will be negative when the capital stock is positive and this can be negative even if gross investment is positive. You can easily verify that net investment,  $k_{t+1} - k_t$ , is always negative when the capital stock is above steady state, because  $A < 1$ .

The expression for GNP is:

$$Y_t^{gnp} = Y_t - I_t^u,$$

or,

$$Y_t^{gnp} = \varepsilon_t (u_t k_t)^\alpha - a(u_t) k_t.$$

Log-linearly expanding this:

$$Y^{gnp} \hat{Y}_t^{gnp} = \varepsilon (k)^\alpha [\hat{\varepsilon}_t + \alpha (\hat{u}_t + \hat{k}_t)] - b k \hat{u}_t,$$

taking into account  $u_t = 1$ , in steady state and  $a'(1) = b$ ,  $a(1) = 0$ . Then, (again, using  $a(1) = 0$ , so that  $Y^{gnp} = \varepsilon k^\alpha$ ),

$$\hat{Y}_t^{gnp} = \hat{\varepsilon}_t + \alpha (\hat{u}_t + \hat{k}_t) - \frac{bk}{\varepsilon k^\alpha} \hat{u}_t$$

But,  $bk = \alpha \varepsilon k^\alpha$ , so that

$$\begin{aligned}\hat{Y}_t^{gnp} &= \hat{\varepsilon}_t + \alpha (\hat{u}_t + \hat{k}_t) - \alpha \hat{u}_t \\ &= \hat{\varepsilon}_t + \alpha \hat{k}_t,\end{aligned}$$

and

$$Y_k = \alpha, Y_\varepsilon = 1.$$

When evaluated at the benchmark parameter values, these parameters are:

$$\begin{aligned} I_k &= 0.58, I_\varepsilon = 3.04, \\ C_k &= 0.28, C_\varepsilon = 0.27. \end{aligned}$$

The investment to output ratio in the model is  $I^u/Y = \delta k/Y = 0.26$  and the consumption to output ratio is  $C/Y = 0.74$ . (Here, I use  $Y$  to signify  $Y^{gnp}$ , because the two are the same in steady state.) Let's verify that the response parameters are consistent with each other. So,

$$\hat{Y}_t^{gnp} = \frac{C}{Y} \hat{C}_t + \frac{I^u}{Y} \hat{I}_t^u.$$

Suppose  $\varepsilon_t$  jumps by 1 percent, so that  $\hat{\varepsilon}_t = 0.01$ . Then,  $\hat{Y}_t^{gnp} = 0.01$ ,  $\hat{C}_t = -.0127$ ,  $\hat{I}_t^u = .0304$ . Inserting these numbers into the previous expression,

$$0.74 \times 0.027 + 0.26 \times .0304 = 1.$$

A similar condition holds across  $C_k$ ,  $I_k$  and  $Y_k$ .