Formulating and Estimating Monetary Models of the Business Cycle
Overview

• Yesterday, Prof. Eichenbaum Discussed VARs
  – Impulse Response Functions (IRF) and Identification
  – Displayed IRF’s of 10 Variables to 3 Shocks.

• Today and Tomorrow: Use IRF’s to Learn About Dynamic Economic Models.
  – Model Solution Methods.
  – Building a Model Which is Consistent with the Identifying Assumptions Used in VARs.
    * Will Discuss Basic Features of Modern Monetary Models
    * Sticky Prices, Sticky Wages
    * Habit Persistence, Variable Capital Utilization
    * Adjustment Costs in Investment
    * Firm-Specificity of Capital
  – Estimate the Model Using IRF’s.

• Third Lecture: Optimal Policy
Model Solution Methods

1. Example #1: A Simple RBC Model.
   – Define a Model ‘Solution’
   – Motivate the Need to Somehow Approximate Model Solutions
   – Describe Basic Idea Behind Log-Linear Approximations
   – Some Strange Examples to be Prepared For

2. Example #2: Putting the Stochastic RBC Model into General Canonical Form

3. Example #3: Stochastic RBC Model with Hours Worked (Matrix Generalization of Previous Results)

4. Example #4: Example #3 with ‘Exotic’ Information Sets.

8. In this Presentation, Will Review Basic Ideas Only

9. Knowledge of *All* the Technical Details Less Necessary, Since there Is Software Available for Taking Care of Details

   – Software in our Tutorials
   – Perturbation AIM algorithm, Eric Swanson, Gary Anderson, and Andrew Levin,
     http://www.ericswanson.pro/
Example #1: Nonstochastic RBC Model

Maximize \( \left\{ c_t, K_{t+1} \right\} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \),

subject to:

\[ C_t + K_{t+1} - (1 - \delta)K_t = K_t^\alpha, \quad K_0 \text{ given} \]

First order condition:

\[ C_t^{1-\sigma} - \beta C_{t+1}^{1-\sigma} \left[ \alpha K_{t+1}^{\alpha-1} + (1 - \delta) \right], \]

or, after substituting out resource constraint:

\[ v(K_t, K_{t+1}, K_{t+2}) = 0, \quad t = 0, 1, \ldots, \text{ with } K_0 \text{ given.} \]
Example #1: Nonstochastic RBC Model ...

- ‘Solution’: a function, $K_{t+1} = g(K_t)$, such that

$$v(K_t, g(K_t), g[g(K_t)]) = 0, \text{ for all } K_t.$$ 

- Problem:

This is an Infinite Number of Equations
(one for each possible $K_t$)
in an Infinite Number of Unknowns
(a value for $g$ for each possible $K_t$)

- With Only a Few Rare Exceptions this is Very Hard to Solve Exactly
  - Easy cases:
    * If $\sigma = 1, \delta = 1 \Rightarrow g(K_t) = \alpha \beta K_t^\alpha$.
    * If $v$ is linear in $K_t, K_{t+1}, K_{t+1}$.
Approximation Method Based on Linearization

• Three Steps
  – Compute the Steady State
  – Do a Linear Expansion About Steady State
  – Solve the Resulting Linearized System

• Step 1: Compute Steady State -
  – Steady State Value of $K, K^*$ -

\[
C^{-\sigma} - \beta C^{-\sigma} \left[ \alpha K^{\alpha-1} + (1 - \delta) \right] = 0,
\]
\[
\Rightarrow \alpha K^{\alpha-1} + (1 - \delta) = \frac{1}{\beta}
\]
\[
\Rightarrow K^* = \left[ \frac{\alpha}{\frac{1}{\beta} - (1 - \delta)} \right]^\frac{1}{1-\alpha}.
\]

– $K^*$ satisfies:

\[
v(K^*, K^*, K^*) = 0.
\]
Approximation Method Based on Linearization ...

• Step 2:
  – Replace $v$ by First Order Taylor Series Expansion About Steady State:

\[
v(K_t, K_{t+1}, K_{t+2}) \approx \tilde{v}(K_t, K_{t+1}, K_{t+2})
\]

\[
\tilde{v}(K_t, K_{t+1}, K_{t+2}) \equiv v(K^*, K^*, K^*) + v_1(K_t - K^*)
\]

\[
+ v_2(K_{t+1} - K^*) + v_3(K_{t+2} - K^*)
\]

– Here,

\[
v_1 = \frac{dv_u(K_t, K_{t+1}, K_{t+2})}{dK_t}, \text{ at } K_t = K_{t+1} = K_{t+2} = K^*.
\]

– Conventionally, Work With Log-Linear Approximation....
Approximation Method Based on Linearization ...

– Conventionally, work with

\[
\bar{v} \left( \hat{K}_t, \hat{K}_{t+1}, \hat{K}_{t+2} \right) \equiv (v_1 K^*) \left( \frac{K_t - K^*}{K^*} \right) \\
+ (v_2 K^*) \left( \frac{K_{t+1} - K^*}{K^*} \right) + (v_3 K^*) \left( \frac{K_{t+2} - K^*}{K^*} \right)
\]

\[
= \alpha_2 \hat{K}_t + \alpha_1 \hat{K}_{t+1} + \alpha_0 \hat{K}_{t+2}
\]

\[
\hat{K}_t \equiv \frac{K_t - K^*}{K^*}
\]

\[
\alpha_2 = v_1 K, \ \alpha_1 = v_2 K, \ \alpha_0 = v_3 K
\]

Note:

\[
\frac{K_t}{K^*} = \hat{K}_t + 1 \Rightarrow \log \left( \frac{K_t}{K^*} \right) \approx \hat{K}_t
\]
Approximation Method Based on Linearization ...

- Step 3: Find Policy Rule

  - Instead of Looking for $g(K_t)$ that Solves

    $$v(K_t, g(K_t), g[g(K_t)]) = 0, \text{ for all } K_t$$

  - We Solve (Easier Problem): Find $\tilde{g}(\hat{K}_t)$ That Solves:

    $$\tilde{v}(\hat{K}_t, \tilde{g}(\hat{K}_t), \tilde{g}[\tilde{g}(\hat{K}_t)]) = 0, \text{ for all } \hat{K}_t$$

  - The Following Functional Form Works:

    $$\hat{K}_{t+1} = \tilde{g}(\hat{K}_t) = A\hat{K}_t,$$

    Where $A$ is to be Determined.
Approximation Method Based on Linearization ...

– Posit the Following Policy Rule:

\[ \hat{K}_{t+1} = A\hat{K}_t, \]

Where \( A \) is to be Determined.

– Compute \( A \):

\[ \alpha_2\hat{K}_t + \alpha_1A\hat{K}_t + \alpha_0A^2\hat{K}_t = 0, \]

or

\[ \alpha_2 + \alpha_1A + \alpha_0A^2 = 0. \]

– \( A \) is the Eigenvalue of Polynomial

• In General: Two Eigenvalues.
  – Can Show: In RBC Example, One Eigenvalue is Explosive. The Other Not.
  – There Exist Theorems (see Stokey-Lucas, chap. 6) That Say You Should Ignore the Explosive \( A \).
Some Strange Examples to be Prepared For

- Other Examples Are Possible:
  - Both Eigenvalues Explosive
  - Both Eigenvalues Non-Explosive
  - What Do These Things Mean?
Some Strange Examples to be Prepared For ...

• Example With Two Explosive Eigenvalues

• Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^\gamma}{\gamma}, \gamma < 1.$$ 

• Technology:

  – Production of Consumption Goods

  $$C_t = k_t^{\alpha} n_t^{1-\alpha}$$

  – Production of Capital Goods

  $$k_{t+1} = 1 - n_t.$$
Some Strange Examples to be Prepared For ...

• Planning Problem:

$$\max \sum_{t=0}^{\infty} \beta^t \left[ k_t^\alpha (1 - k_{t+1})^{1-\alpha} \right]^{\gamma}$$

• Euler Equation:

$$v(k_t, k_{t+1}, k_{t+2}) = -(1 - \alpha) k_t^{\alpha \gamma} (1 - k_{t+1})^{(1-\alpha)\gamma - 1} + \beta \alpha k_{t+1}^{(\alpha \gamma - 1)} (1 - k_{t+2})^{(1-\alpha)\gamma}$$

$$= 0,$$

$$t = 0, 1, ...$$

• Steady State:

$$k^* = \frac{\alpha \beta}{1 - \alpha + \alpha \beta}.$$
Some Strange Examples to be Prepared For ...

- Log-linearize Euler Equation:

\[ \alpha_0 \hat{k}_{t+2} + \alpha_1 \hat{k}_{t+1} + \alpha_2 \hat{k}_t = 0 \]

- With \( \beta = 0.58, \gamma = 0.99, \alpha = 0.6 \), Both Roots of Euler Equation are both explosive:

\[ -1.6734, -1.0303 \]

- Other Properties:

  - Steady State:

\[ 0.4652 \]

  - Two-Period Cycle:

\[ 0.8882, 0.0870 \]
Some Strange Examples to be Prepared For ...

• Meaning of Stokey-Lucas Example
  – Illustrates the Possibility of All Explosive Roots
  – Economics:
    * If Somehow You Start At Single Steady State, Stay There
    * If You are Away from Single Steady State, Go Somewhere Else
  – If Linearized Euler Equation Around Particular Steady State Has Only Explosive Roots
    * All Possible Equilibria Involve Leaving that Steady State
    * Linear Approximation Not Useful, Since it Ceases to be Valid Outside a Neighborhood of Steady State
  – Could Linearize About Two-Period Cycle (That’s Another Story...)
  – The Example Suggests That Maybe All Explosive Root Case is Unlikely
Some Strange Examples to be Prepared For ...

• Another Possibility:
  – Both Roots Stable
  – Many Paths Converge Into Steady State: Multiple Equilibria
  – Can Happen For Many Reasons
    ✴ Strategic Complementarities Among Different Agents In Private Economy
    ✴ Certain Types of Government Policy
  – This is a More Likely Possibility
  – Avoid Being Surprised by It By Always Thinking Through Economics of Model.
Example #2: RBC Model With Uncertainty

- Model

Maximize $E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$,

subject to

$C_t + K_{t+1} - (1 - \delta)K_t = K_t^\alpha \varepsilon_t$,

where $\varepsilon_t$ is a stochastic process with $E\varepsilon_t = \varepsilon$, say. Let

$\hat{\varepsilon}_t = \frac{\varepsilon_t - \varepsilon}{\varepsilon}$,

and suppose

$\hat{\varepsilon}_t = \rho \hat{\varepsilon}_{t-1} + e_t, e_t \sim N(0, \sigma_e^2)$.

- First Order Condition:

$E_t \{ C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[ \alpha K_{t+1}^{\alpha-1} \varepsilon_{t+1} + 1 - \delta \right] \} = 0$. 
Example #2: RBC Model With Uncertainty ...

- First Order Condition:
  \[ E_t v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0, \]
  where
  \[ v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \]
  \[ = (K_t^\alpha \varepsilon_t + (1 - \delta)K_t - K_{t+1})^{-\sigma} \]
  \[ - \beta (K_{t+1}^\alpha \varepsilon_{t+1} + (1 - \delta)K_{t+1} - K_{t+2})^{-\sigma} \]
  \[ \times [\alpha K_{t+1}^{\alpha-1} \varepsilon_{t+1} + 1 - \delta]. \]

- Solution: a \( g(K_t, \varepsilon_t) \), Such That
  \[ E_t v (g(g(K_t, \varepsilon_t), \varepsilon_{t+1}), g(K_t, \varepsilon_t), K_t, \varepsilon_{t+1}, \varepsilon_t) = 0, \]
  For All \( K_t, \varepsilon_t \).

- Hard to Find \( g \), Except in Special Cases
  - One Special Case: \( v \) is Linear.
Example #2: RBC Model With Uncertainty ...

- Linearization Strategy:
  - Step 1: Compute Steady State of $K_t$ when $\theta_t$ is Replaced by $E\theta_t$
  - Step 2: Replace $v$ By its Taylor Series Expansion About Steady State.
  - Step 3: Solve Resulting Linearized System.

- Logic: If Actual Stochastic System Remains in a Neighborhood of Steady State, Linear Approximation Good
Example #2: RBC Model With Uncertainty ...

- Step 1: Steady State:

\[ K^* = \left[ \frac{\alpha \varepsilon}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\alpha}}. \]

- Step 2: Linearize -

\[ v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \]

\[ \approx v_1 (K_{t+2} - K^*) + v_2 (K_{t+1} - K^*) + v_3 (K_t - K^*) \]
\[ + v_3 (\varepsilon_{t+1} - \varepsilon) + v_4 (\varepsilon_t - \varepsilon) \]

\[ = v_1 K^* \left( \frac{K_{t+2} - K^*}{K^*} \right) + v_2 K^* \left( \frac{K_{t+1} - K^*}{K^*} \right) + v_3 K^* \left( \frac{K_t - K^*}{K^*} \right) \]
\[ + v_3 \varepsilon \left( \frac{\varepsilon_{t+1} - \varepsilon}{\varepsilon} \right) + v_4 \varepsilon \left( \frac{\varepsilon_t - \varepsilon}{\varepsilon} \right) \]
\[ = \alpha_0 \hat{K}_{t+2} + \alpha_1 \hat{K}_{t+1} + \alpha_2 \hat{K}_t + \beta_0 \hat{\varepsilon}_{t+1} + \beta_1 \hat{\varepsilon}_t. \]
• Step 3: Solve Linearized System
  – Posit:
  \[ \hat{K}_{t+1} = A\hat{K}_t + B\hat{\varepsilon}_t. \]
  – Pin Down \( A \) and \( B \) By Condition that log-linearized Euler Equation Must Be Satisfied.
    * Note:
    \[ \hat{K}_{t+2} = A\hat{K}_{t+1} + B\hat{\varepsilon}_{t+1} \]
    \[ = A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}. \]

    * Substitute Posited Policy Rule into Linearized Euler Equation:
    \[ E_t \left[ \alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t \right] = 0, \]
    so must have:
    \[ E_t \{ \alpha_0 \left[ A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1} \right] \]
    \[ + \alpha_1 \left[ A\hat{K}_t + B\hat{\varepsilon}_t \right] + \alpha_2\hat{K}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0e_{t+1} + \beta_1\hat{\varepsilon}_t \} = 0 \]
Example #2: RBC Model With Uncertainty ...

* Then,

\[
E_t \left[ \alpha_0 \hat{K}_{t+2} + \alpha_1 \hat{K}_{t+1} + \alpha_2 \hat{K}_t + \beta_0 \hat{\varepsilon}_{t+1} + \beta_1 \hat{\varepsilon}_t \right] 
\]

\[
= E_t \{ \alpha_0 \left[ A^2 \hat{K}_t + AB \hat{\varepsilon}_t + B \rho \hat{\varepsilon}_t + Be_{t+1} \right] + \alpha_1 \left[ A \hat{K}_t + B \hat{\varepsilon}_t \right] + \alpha_2 \hat{K}_t + \beta_0 \rho \hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1 \hat{\varepsilon}_t \} 
\]

\[
= \alpha(A) \hat{K}_t + F \hat{\varepsilon}_t = 0
\]

where

\[
\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2, \\
F = \alpha_0 AB + \alpha_0 B \rho + \alpha_1 B + \beta_0 \rho + \beta_1
\]

* Find \( A \) and \( B \) that Satisfy:

\[
\alpha(A) = 0, \quad F = 0.
\]
Example #3 RBC Model With Hours Worked and Uncertainty

- Maximize

\[
E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)
\]

subject to

\[
C_t + K_{t+1} - (1 - \delta)K_t = f(K_t, N_t, \varepsilon_t)
\]

and

\[
E\varepsilon_t = \varepsilon,
\]

\[
\hat{\varepsilon}_t = \rho\hat{\varepsilon}_{t-1} + e_t, \quad e_t \sim N(0, \sigma_e^2)
\]

\[
\hat{\varepsilon}_t = \frac{\varepsilon_t - \varepsilon}{\varepsilon}.
\]
Example #3 RBC Model With Hours Worked and Uncertainty ...

- First Order Conditions:

\[ E_t v_K(K_{t+2}, N_{t+1}, K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0 \]

and

\[ v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0. \]

where

\[ v_K(K_{t+2}, N_{t+1}, K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = U_c (f(K_t, N_t, \varepsilon_t) + (1 - \delta)K_t - K_{t+1}, N_t) \]

\[ - \beta U_c (f(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + (1 - \delta)K_{t+1} - K_{t+2}, N_{t+1}) \]

\[ \times [f_K(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + 1 - \delta] \]

and,

\[ v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = U_N (f(K_t, N_t, \varepsilon_t) + (1 - \delta)K_t - K_{t+1}, N_t) \]

\[ + U_c (f(K_t, N_t, \varepsilon_t) + (1 - \delta)K_t - K_{t+1}, N_t) \]

\[ \times f_N(K_t, N_t, \varepsilon_t). \]

- Steady state \( K^* \) and \( N^* \) such that Equilibrium Conditions Hold with \( \varepsilon_t \equiv \varepsilon. \)
Example #3 RBC Model With Hours Worked and Uncertainty ...

- Representation Log-linearized Equilibrium Conditions
  - Let
    \[ z_t = \left( \begin{array}{c} \hat{K}_{t+1} \\ \hat{N}_t \end{array} \right), \ s_t = \hat{\varepsilon}_t, \ \epsilon_t = e_t. \]
  - Then, the linearized Euler equation is:
    \[ E_t [\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0, \]
    \[ s_t = Ps_{t-1} + \epsilon_t, \ \epsilon_t \sim N(0, \sigma_e^2), \ P = \rho. \]

- Here,
  \[ \alpha_0 = \begin{bmatrix} v_{K,1}K^* & v_{K,2}N^* \\ 0 & 0 \end{bmatrix}, \ \alpha_1 = \begin{bmatrix} v_{K,3}K^* & v_{K,4}N^* \\ v_{N,1}K^* & v_{N,2}N^* \end{bmatrix}, \]
  \[ \alpha_2 = \begin{bmatrix} v_{K,5}K^* & 0 \\ v_{N,3}K^* & 0 \end{bmatrix}, \]
  \[ \beta_0 = \begin{pmatrix} v_{K,6}\varepsilon \\ 0 \end{pmatrix}, \ \beta_1 = \begin{pmatrix} v_{K,7}\varepsilon \\ v_{N,4}\varepsilon \end{pmatrix}. \]

- Previous is a Canonical Representation That Essentially All Linearized Models Can be Fit Into (See Christiano (2002).)
Example #3 RBC Model With Hours Worked and Uncertainty ...

- Again, Look for Solution

\[ z_t = A z_{t-1} + B s_t, \]

where \( A \) and \( B \) are pinned down by log-linearized Equilibrium Conditions.

- Now, \( A \) is Matrix Eigenvalue of Matrix Polynomial:

\[ \alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0. \]

- Also, \( B \) Satisfies Same System of Linear Equations as Before:

\[ F = (\beta_0 + \alpha_0 B) P + [\beta_1 + (\alpha_0 A + \alpha_1) B] = 0. \]

- Go for the 2 Free Elements of \( B \) Using 2 Equations Given by

\[ F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]
Finding Eigenvalue of Polynomial Equation, $\alpha(A) = 0$, is a Solved Problem. See Anderson, Gary S. and George Moore, 1985, ‘A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models,’ *Economic Letters*, 17, 247-52 or Articles in Computational Economics, October, 2002.

Solving for $B$

– Given $A$, $F = 0$ Represents Linear System of Equations in the Unknown Elements of $B$.

– To See this, Use

\[
\text{vec}(A_1A_2A_3) = (A_3' \otimes A_1) \text{vec}(A_2),
\]

to Convert $F = 0$ Into

\[
\text{vec}(F') = d + q\delta = 0,
\]

where $\delta = \text{vec}(B')$.

– Find $B$ By First Solving:

\[
\delta = -q^{-1}d.
\]
Example #4: Example #3 With ‘Exotic’ Information Set

- Suppose the Date $t$ Investment Decision is Made Before the Current Realization of the Technology Shock, While the Hours Decision is Made Afterward.

- Now, Canonical Form Must Be Written Differently:

$$\mathcal{E}_t [\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0,$$

where

$$\mathcal{E}_t X_t = \begin{bmatrix} E[X_{1t}|\hat{\epsilon}_{t-1}] \\ E[X_{2t}|\hat{\epsilon}_t] \end{bmatrix}.$$

- Convenient to Change $s_t$:

$$s_t = \begin{pmatrix} \hat{\epsilon}_t \\ \hat{\epsilon}_{t-1} \end{pmatrix}, \quad P = \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix}, \quad \epsilon_t = \begin{pmatrix} e_t \\ 0 \end{pmatrix}.$$

- Adjust $\beta_i$'s:

$$\beta_0 = \begin{pmatrix} v_{K,6}\epsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} v_{K,7}\epsilon & 0 \\ v_{N,4}\epsilon & 0 \end{pmatrix},$$
Example #4: Example #3 With ‘Exotic’ Information Set ...

• Posit Following Solution:
  
  \[ z_t = A z_{t-1} + B s_t. \]

• Substitute Into Canonical Form:
  
  \[ \mathcal{E}_t \left[ \alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t \right] = \alpha(A) z_{t-1} + \mathcal{E}_t F s_t = \alpha(A) z_{t-1} + \mathcal{E}_t F s_t = 0, \]

• Then,
  
  \[ \mathcal{E}_t F s_t = \mathcal{E}_t \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} s_t \end{bmatrix} = \mathcal{E}_t \begin{bmatrix} F_{11} \hat{e}_t + F_{12} \hat{e}_{t-1} \\ F_{21} \hat{e}_t + F_{22} \hat{e}_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & F_{12} + \rho F_{11} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} s_t \end{bmatrix} = \tilde{F} s_t. \]

• Equations to be solved:
  
  \[ \alpha(A) = 0, \quad \tilde{F} = 0. \]

• \( \tilde{F} \) Only Has Three Equations How Can We Solve for the Four Elements of \( B \)?

• Answer: Only Three Unknowns in \( B \) Because \( B \) Must Also Obey Information Structure:
  
  \[ B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \]
Summary

- Solving Models By Linear Approximation Involves Three Steps
  a. Compute Steady State
  b. Log-Linearize Equilibrium Conditions
  c. Solve Linearized Equations.

- Step 3 Requires Finding $A$ and $B$ in:
  \[
  z_t = A z_{t-1} + B s_t, 
  \]
  to Satisfy Log-Linearized Equilibrium Conditions:
  \[
  \mathcal{E}_t \left[ \alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t \right] 
  \]
  \[
  s_t = P s_{t-1} + \epsilon_t, \; \epsilon_t \sim \text{iid} 
  \]

- We are Led to Choose $A$ and $B$ so that:
  \[
  \alpha(A) = 0, 
  \]
  (standard information set) $F = 0$, 
  (exotic information set) $\tilde{F} = 0$
  and Eigenvalues of $A$ are Less Than Unity In Absolute Value.
Computing Impulse Response Functions For Model

• Impulse Response Function (IRF):
  – Suppose System is in Steady State
  – IRF Is Response of System to a One-Time Innovation in Exogenous Variables, Relative to What Trajectory Would have Been, Absent a Shock.
• One-Time Shock: $\epsilon_1 \neq 0$, $\epsilon_t = 0$ for $t > 1$.
• Simulate Exogenous Variables:
  $$s_t = Ps_{t-1} + \epsilon_t, \quad t = 1, 2, ..., T, \quad s_0 = 0.$$  
• Simulate Endogenous Variables:
  $$z_t = Az_{t-1} + Bs_t, \quad t = 1, 2, ..., T, \quad z_0 = 0.$$  
• Note:
  $$z_t = \left( \frac{K_{t+1} - K^*}{N_t - N^*} \right),$$  
  Which are Percentage Deviations From Unshocked, Steady State Path.
Computing Impulse Response Functions For Model ...

- Other Endogenous Variables Not Included in $z_t$

  – Suppose We Also Would Like Output, $Y_t$

    $$Y_t = \varepsilon_t K_t^\alpha N_t^{1-\alpha},$$

    so that

    $$\hat{Y}_t = \hat{\varepsilon}_t + \alpha \hat{K}_t + (1 - \alpha) \hat{N}_t,$$

    a Linear Function of $z_t$ and $s_t$.

  – Can Obtain Other Variables, Say $Z_t$, In the Same Way:

    $$Z_t = \alpha_z z_t + \alpha_s s_t.$$

  – Use This Expression and Previous Results to Simulate the Impulse Response to $Z_t$