1. Introduction

Periodically, there are concerns that the economy has entered an ‘overinvestment boom’. During such a boom, investment and the stock market are high, until a day of reckoning occurs, when it is realized that the hoped-for returns in fact will never be realized. On the day of reckoning, the stock market and investment collapse, and the economy slips into recession. On a small scale, examples are fiber optic cable that was over-produced in the 1990s and now lays around unused. On a larger scale, many observers interpret the 1920s in the US and the 1980s in Japan as examples when entire economies were caught up in overinvestment booms that were terminated in a stock market crash and recession. In recent decades, mainstream economists have been skeptical that the notion of an overinvestment boom could be placed on sound economic foundations. However, journalists and other observers often interpret the 1990s and the stock market crash that occurred in 2000 as another example of an overinvestment boom. This has encouraged economists to revisit the possibility that overinvestment booms may be a real possibility after all.\(^1\) Of particular interest is the possibility that these events pose novel questions about the appropriate conduct of monetary and fiscal policy.

In this tutorial, we will explore one attempt to model an overinvestment boom, based on research I am currently pursuing with Robert Motto and Massimo Rostagno of the European Central Bank. The boom is modelled as being triggered by a signal which indicates to everyone that productivity will rise in the future. When the future actually occurs, however, the signal turns out to be false. In effect the signal turns out to be a mirage. We describe a particular stochastic process for technology that allows us to formalize the mirage idea.

The model economy is a standard real business cycle model, with two additional twists. First, there is curvature in the technology for converting investment goods into installed capital (‘adjustment costs’). This allows the price of capital to change with changing economic circumstances. We work with a simple theory

\(^1\)For an important contribution, see Beaudry and Portier (2000).
of the stock market, according to which equity has value because it represents a
claim on the physical capital of firms (we abstract from other forms of firm capital
on which equity represents a claim, such as ‘goodwill’, patents, etc.). Since move-
ments in the price of equity are an essential part of the overinvestment booms we
study, investment adjustment costs are an important part of our model. As we
will see, to develop a successful theory of an overinvestment boom, the adjustment
costs must be of a particular kind. The adjustment costs in investment are on
the first difference of investment. That is, if investment is to be increased, doing
so quickly is very costly, and less waste is incurred if the increase is done slowly.
This is intended to capture, in a reduced form way, the notion that learning must
occur as investment increases from one level to another.

Second, there is habit persistence in preferences: if consumption has been high
recently, then the marginal utility of current consumption is higher. The particular
adjustment costs we introduce, as well as the assumption of habit persistence, have
been shown to be important ingredients in business cycle models. In this tutorial,
we will see that the adjustment costs and habit persistence are a critical ingredient
in any theory of overinvestment booms that is built on the notion that they are
driven by a mirage.

A surprising outcome of this analysis, is that the model ingredients discussed
so far in fact are not sufficient to generate a rise in asset prices during an overin-
vestment boom. In fact, the theory implies - despite the presence of adjustment
costs in investment and strong investment - that asset prices will be low during
the run up to the boom.

In this tutorial, you will discover these observations for yourself, by simulating
a model under alternative settings of the parameters. You will, in effect, see first-
hand how equilibrium modelling can be used to sharpen up one’s ideas about an
economic phenomenon. You will also find that the mirage theory, as developed
in this tutorial, is incomplete as a theory of overinvestment booms. Of course,
one possibility is that the theory is fundamentally misspecified. But, my work
with Motto and Rostagno raises another possibility, that a critical thing that is
missing from the theory is a monetary sector. When we incorporate money into
the model, and model monetary policy as following a Taylor rule, we find that the
mirage theory offers reasonable theory of overinvestment booms, complete with
stock market boom and collapse. The reason is simple. The Taylor rule specifies
that when anticipated inflation rises, then the authorities should sharply raise
nominal rates of interest. During the overinvestment boom, inflation actually

\textsuperscript{2}See, for example, Christiano, Eichenbaum and Evans (2004).
drifts down. The monetary authority responds by driving the nominal rate of interest down. To do this requires increasing the money supply. When the money supply is increased in this way, this adds extra fuel to investment and raises stock prices as well. So, it may well be that the mirage theory that we develop here is a reasonable basis for a theory of overinvestment after all.

Below, we explore the non-monetary theory of overinvestment booms. Clearly, the theory as developed is missing some crucial elements (probably, monetary factors). Still, it suits our purposes. Basic components of the theory are revealed without the additional complications that monetary factors introduce. In addition, the exercise provides a way for the student to get their ‘feet wet’ in the analysis and simulation of a dynamic, general equilibrium model. Accompanying this tutorial, there is software that can be used to simulate the overinvestment boom in MATLAB, written by Etienne Gagnon, a Northwestern graduate student who will be on the market this year!

2. Model

Suppose the preferences of households are

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left( (C_t - bC_{t-1}) (1 - h_t)^\psi \right) 1^{-\gamma} \]

Here, \( h_t \) is hours worked, \( C_t \) is consumption and the amount of time that is available is unity. When \( b > 0 \) then there is habit persistence in preferences. The resource constraint is

\[ I_t + C_t \leq Y_t, \quad (2.1) \]

where \( K_t \) denotes the beginning of period \( t \) stock of physical capital, \( I_t \) is investment, \( C_t \) is consumption and \( Y_t \) is output of goods.

Output \( Y_t \) is produced using the technology

\[ Y_t = K_t^\alpha \left( \exp(z_t) h_t \right)^{1-\alpha}, \quad (2.2) \]

where \( z_t \) represents a stochastic shock to technology. Its law of motion will be described shortly.

Physical capital is accumulated with the following technology

\[ K_{t+1} = (1 - \delta) K_t + \left( 1 - S \left( \frac{I_t}{I_{t-1}} \right) \right) I_t, \quad (2.3) \]
where the function $S$ captures the notion that there are adjustment costs in changing the level of investment. For convenience, $S(x_t)$ has the functional form

$$S(x_t) = \frac{\chi}{2} \left[ \exp \{x_t - 1\} + \exp \{- (x_t - 1)\} - 2 \right]$$

with $\chi > 0$. Notice that $S'(1) = S''(1) = 0$ and $S'''(1) = \chi$. The first two zeros guarantee that the adjustment costs have no impact on the steady state.

We model an overinvestment boom as follows. Up until period 1, the economy is in a steady state. In period 1, a signal arrives that suggests $z_t$ will be high in period $z_{1+p}$. Then, in period $1 + p$, the expected rise in technology in fact does not occur. A time series representation for $z_t$ which captures this possibility is:

$$z_t = \rho z_{t-1} + \varepsilon_{t-p} + \xi_t, \quad (2.4)$$

where $\varepsilon_t$ and $\xi_t$ are uncorrelated over time and with each other. To see that this setup can capture the mirage idea, suppose $p = 1$. Then, if $\varepsilon_1$ is seen to have a high value, this shifts up the expected value of $z_2$. But, if (by accident!) $\xi_2 = -\varepsilon_1$, the high expected value of $z_2$ does not materialize. In effect, $\varepsilon_1$ turns out to be a mirage. Of course this is not the only possible outcome. The variable, $\varepsilon_1$, is not a mirage if $\xi_t = 0$ for $t = 2, 3, \ldots$. In this case, the high value of $z_2$ signalled by $\varepsilon_1$ actually happens and we get $z_{2+t} = \rho^t z_2$, for $t = 1, 2, 3, \ldots$. In this framework, an overinvestment boom is something that occurs in the event that $\varepsilon_1$ is high, but its effects are later cancelled by $\xi_t$.

For purposes of solving and simulating the model, it is useful to formulate this model of $z_t$ in the canonical form discussed in class. To this end, consider the following formulation:

$$
\begin{bmatrix}
  z_t \\
  \varepsilon_t \\
  \varepsilon_{t-1} \\
  \vdots \\
  \varepsilon_{t-p+1}
\end{bmatrix} =
\begin{bmatrix}
  \rho & 0 & \cdots & 0 & 1 \\
  0 & 0 & \cdots & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  z_{t-1} \\
  \varepsilon_{t-1} \\
  \varepsilon_{t-2} \\
  \vdots \\
  \varepsilon_{t-p}
\end{bmatrix} +
\begin{bmatrix}
  \xi_t \\
  \varepsilon_t \\
  \varepsilon_{t-2} \\
  \vdots \\
  \varepsilon_{t-p}
\end{bmatrix}
$$

The first equation here corresponds to (2.4). The second equation says that $\varepsilon_t = \varepsilon_{t-1}$ (who can argue with that!). The third says $\varepsilon_{t-1} = \varepsilon_{t-1}$, and so on. Let

$$s_t = \begin{bmatrix}
  z_t \\
  \varepsilon_t \\
  \varepsilon_{t-1} \\
  \vdots \\
  \varepsilon_{t-p+1}
\end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix}
  \xi_t \\
  \varepsilon_t \\
  0 \\
  \vdots \\
  0
\end{bmatrix}, \quad P = \begin{bmatrix}
  \rho & 0 & \cdots & 0 & 1 \\
  0 & 0 & \cdots & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 1 & 0
\end{bmatrix}.$$
Then, we have
\[ s_t = P s_{t-1} + \epsilon_t. \tag{2.5} \]
We suppose that at the beginning of period \( t \), \( s_t \) is observed. Then, future values of \( s_t \) are forecasted using \( E_t s_{t+j} = P^j s_t \), for \( j > 0 \).

To see how this ‘works’, consider the following example. Suppose \( p = 2 \). Then
\[
\begin{align*}
    s_t &= \begin{bmatrix} z_t \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}, \\
    P &= \begin{bmatrix} \rho & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
    \epsilon_t &= \begin{bmatrix} \xi_t \\ \epsilon_t \\ 0 \end{bmatrix}
\end{align*}
\]
For additional concreteness, suppose \( \rho = 0.90 \). Then,
\[
P^2 = \begin{bmatrix} 0.81 & 1.0 & 0.9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
so that the forecast, as of date 1, of \( s_3 \) is:
\[
E_1 \begin{bmatrix} z_3 \\ \epsilon_3 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} 0.81 & 1.0 & 0.9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ \epsilon_1 \\ \epsilon_0 \end{bmatrix},
\]
using the fact that \( E_1 \epsilon_3 = 0 \). From this, we can read the impact on \( E_1 z_3 \) of a shock, \( \epsilon_1 \). In order to isolate the effect of \( \epsilon_1 \), suppose \( \epsilon_t = 0 \) for \( t < 1 \) and \( \xi_t = 0 \), \( t \leq 0 \). These assumptions imply that \( z_1 \) is at its unconditional mean of zero. Note that
\[
E_1 z_3 = 0.81 z_1 + \epsilon_1 + 0.9 \epsilon_0
\]
by our assumption that only \( \epsilon_1 \) has moved and that \( E_1 \xi_t = 0 \) for \( t > 1 \). Note in particular that \( \epsilon_1 \) leads to an upward revision in the expectation of \( z_3 \). In period 1, agents will act on that expectation. In period 3, \( z_3 \) is actually realized, but its realization is determined in part by the realization of \( \xi_3 \). It it happens that \( \xi_3 = -\epsilon_1 \), then the expected positive move in \( z_3 \) does not occur. Any investment that occurs based on the expectation of higher \( z_3 \) constitutes ‘overinvestment’, ex post.

This event (it is a combination of two events, a positive value of \( \epsilon_1 \) and \( \xi_3 = -\epsilon_1 \)) can be simulated very simply. First, solve the model to get a representation
of the endogenous variables of the model. Let the endogenous variables of the model, whose values are determined at time $t$ be:

$$Z_t = \begin{bmatrix} \hat{K}_{t+1} \\ \hat{C}_t \\ \hat{I}_t \\ \hat{h}_t \\ \hat{Y}_t \\ \hat{\lambda}_t \\ \hat{P}_{K', t} \end{bmatrix},$$

where $\lambda_t$ is the multiplier on the resource constraint in the Lagrangian representation of the problem (see the next section for a formal discussion). Then: (i) compute the steady state of the model economy; (ii) log-linearize the equilibrium conditions around this steady state (the equilibrium conditions are summarized below); and (iii) solve the resulting linear system to obtain:

$$Z_t = AZ_{t-1} + Bs_t. \tag{2.7}$$

The full system is then composed of (2.5) and (2.7).

We simulate an ‘overinvestment boom’ as the economy’s response to

$$\epsilon_1 = \begin{bmatrix} 0 \\ \varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \epsilon_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \epsilon_{t+p} = \begin{bmatrix} -\varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \epsilon_{t+j} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad j > p,$$

with initial condition:

$$Z_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This means that the economy begins in a non-stochastic steady state for all the variables.
To proceed with the exercise, you will need the software that accompanies it. Consider a benchmark parameterization, with $\beta = 1.03^{-0.25}$, $b = 0.6$, $\chi = 5$, $\delta = 0.02$, $\gamma = 1$, $\psi = 2.3$, $\rho = 0.95$, $p = 8$. As the magnitude of the depreciation discount rates suggest, this model is specified at a quarterly frequency. To simulate the economy with this parameterization, run the MATLAB program, overinvestment.m. Note the parameter settings at the beginning of the program. The program will actually accommodate variable capital utilization. However, for purposes of this tutorial this feature has in effect been shut down by making changes in utilization prohibitively costly (note sigmaw=1000). The interested reader can, of course, experiment with this feature.\footnote{In addition, there is a parameter, $xbar$, which has been set to 1. There is a deterministic term, $\exp(xbar \times t)$, in front of $h_t$ in the production function. An empirically realistic setting for this parameter is $xbar = 1 + 0.015/4$, which corresponds to an annual rate of growth in steady state of 1.5 percent. This is roughly the average growth rate in per capita output for the US economy in the 20th century.}

The key steps in model solution can be seen by running overinvestment.m in ‘debug mode’. Note how the steady state of the model is computed in lines 12-23 of steadystate.m, which is called in line 36 of overinvestment.m. The equilibrium conditions of the model appear in focs_change.m (for the specification of investment adjustment costs appearing in (2.3)) and in focs_level.m (for a different specification of adjustment costs to be specified shortly). The Taylor series expansion of these equations, about steady state, is computed in lines 14-18 of getAB.m. As its name suggests, the function of getAB.m is to find the matrices $A$ and $B$ that characterize the model’s solution, in (2.7). The canonical form of the log-linearized equilibrium conditions is assembled in lines 25-33 (see the notes in lines 61-71). The matrices, $A$ and $B$, are finally solved for in 39-44.

The program, overinvestment.m, does various things that we will not exploit in this tutorial, although the reader may want to experiment with them. For example, line 44 has been commented out. Executing that line causes the program to compute and print various second moment properties of the model. In addition, the reader can print out impulse responses for the model under various alternative assumptions about the shocks (the model can also be used to analyze the dynamic effects of disturbances to the time endowment, which can be used to think about the effects of legislation designed to change the amount of time people spend at work.)

Running overinvestment.m with the default parameter settings produces the following result:
IRFs: Anticipated shock to technology is not realized (Logs)

The horizontal axis displays time, $t = 1, 2, \ldots$. The signal occurs in period 2, about a shock that is expected to occur 8 periods later (the parameter, perant, is set to 8), in period $t = 9$. Note how investment, consumption, output and employment respond positively to the signal about future technology. (Disregard $U_t$, capital utilization, which in any case does not move because utilization costs are set very high in the program.)

Now, redo the same run, changing the value of $\chi$ to zero (this is chi in line 13 of the code). Note how dramatically different the findings are! Employment and output drop, while consumption is high, during the period leading up to $p = 8$. Evidently, the rise in employment and investment in the benchmark specification is due to the investment adjustment costs. This is because, with this specification of adjustment costs, to be in a position to exploit the expected technology shock in the future it is efficient to slowly increase investment as soon as the signal comes in. In this way, the capital stock rises gradually to the point where it is
at an efficient level to take advantage of the expected high level of technology. The response when \( \chi = 0 \) - raise the capital stock quickly at the last moment - generates too many adjustment costs.

We consider an alternative specification of adjustment costs. The traditional specification is in terms of the level of investment, scaled by the stock of capital:

\[
K_{t+1} = (1 - \delta)K_t + (1 - S\left(\frac{I_t}{K_t}\right))I_t.
\]

Note that that \( K_t \) replaces \( I_{t-1} \) in the denominator of (2.3). As before, to ensure that the steady state values of the variables are independent of the adjustment costs, we assume \( S = S' = 0 \) for \( I_t/K_t \) at its steady state value. In addition, we suppose \( S'' > 0 \) in steady state. Although with the linearization the only the properties of the adjustment cost function in steady state are needed, still it is useful to have a particular example for specificity:

\[
S(x) = \frac{\chi}{2} \left[ \exp\{x - \delta\} + \exp\{- (x - \delta)\} - 2 \right],
\]

where \( x = I/K \) and we have taken into account that \( I/K = \delta \) in steady state. Note that, as before, \( S'' = \chi \) in steady state. Return the program back to the benchmark parameter values. Notice that investment now drops when the signal about future productivity arrives. With this form of adjustment costs, the optimal thing to do is to reduce investment now and then increase it when the higher level of productivity actually occurs. To understand this better, it is useful to run the code one more time, removing the \( \% \) in the first column of line 45 in overinvestment.m and inserting a \( \% \) at the start of line 50. With this change, the program displays the response to a signal about future productivity that actually occurs. Evidently, investment adjustment costs are important, and the form they take is important too.

We now consider the role of habit persistence in the benchmark simulation. Go back to the benchmark parameter settings, only set \( b = 0 \). In this case, the employment, output and investment booms occur, but consumption drops. With habit persistence, households do not like this fall in consumption preceding the rise in consumption that is anticipated to occur after the favorable shock is realized.

3. The Price of Capital During the Overinvestment Boom

To understand how the price of capital evolves during the overinvestment boom, it is useful to study the first order condition with respect to investment, in the
model. For convenience, we work with a version of the model in which there is no uncertainty. Let the Lagrangian representation of the planning problem be:

\[
\sum \beta^t \left\{ \frac{(C_t - bC_{t-1})(1 - h_t)^\psi}{1 - \gamma} \right\}^{1-\gamma} + \lambda_t \left[ (K_t)^\alpha (z_t h_t)^{1-\alpha} - C_t - I_t \right] \tag{3.1}
\]

\[
+ \mu_t \left[ (1 - \delta)K_t + (1 - S)\left( \frac{I_t}{I_{t-1}} \right)I_t - K_{t+1} \right],
\]

\[
\psi(C_t - bC_{t-1})^{1-\gamma} (1 - h_t)^{\psi(1-\gamma)-1} = \lambda_t (1 - \alpha) (K_t)^\alpha (z_t h_t)^{-\alpha}
\]

where \( \lambda_t \) and \( \mu_t \) are non-negative multipliers. The first order condition with respect to \( C_t \) is:

\[
\lambda_t = (C_t - bC_{t-1})^{-\gamma} (1 - h_t)^{\psi(1-\gamma)} - \beta b(C_{t+1} - bC_t)^{-\gamma} (1 - h_{t+1})^{\psi(1-\gamma)} . \tag{3.2}
\]

Note that the right side of this expression is the marginal utility of consumption, taking into account habit persistence. So, \( \lambda_t \) is the marginal utility of consumption. The first order condition with respect to \( K_{t+1} \) is:

\[
\mu_t = \beta \left[ \lambda_{t+1} \alpha (K_{t+1})^{\alpha - 1} (z_{t+1} h_{t+1})^{1-\alpha} + \mu_{t+1} (1 - \delta) \right]
\]

Note that the object on the right side of the equality is the marginal utility of an extra unit of \( K_{t+1} \). It is tomorrow’s marginal physical product of capital, converted to marginal utility terms by multiplying by \( \lambda_{t+1} \) plus the value of the undepreciated part of \( K_{t+1} \) that is left over for use in subsequent periods, which is converted into marginal utility terms by \( \mu_{t+1} \). Divide both sides of the first order condition for \( K_{t+1} \) with respect to \( \lambda_t \) and rearrange:

\[
\frac{\mu_t}{\lambda_t} = \beta \left[ \frac{\lambda_{t+1}}{\lambda_t} \alpha (K_{t+1})^{\alpha - 1} (z_{t+1} h_{t+1})^{1-\alpha} + \frac{\mu_{t+1}}{\lambda_{t+1}} (1 - \delta) \right]
\]

Now, recall that \( \mu_t \) is the marginal utility of \( K_{t+1} \), loosely, \( dU/dK_{t+1} \). Similarly, \( \lambda_t \) is the marginal utility of \( C_t \), loosely \( dU/dC_t \). Thus, the ratio is the consumption cost of a unit of \( K_{t+1} \), or Tobin’s \( q \):

\[
\frac{\mu_t}{\lambda_t} = \frac{dU_t}{dK_{t+1}} = \frac{dC_t}{dK_{t+1}}.
\]
In light of this, it is natural to call \( \mu_t/\lambda_t \) the ‘price of capital’, which we denote by \( P_{k',t} \). Substituting this into the first order condition for \( K_{t+1} \), we obtain:

\[
P_{k',t} = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[ \alpha (K_{t+1})^{\alpha-1} (z_{t+1} h_{t+1})^{1-\alpha} + P_{k',t+1} (1 - \delta) \right].
\]

This is the sort of first order condition we would expect if the households purchased capital in a competitive market at price, \( P_{k',t} \). Thus, the left side has the cost of a unit of new capital in period \( t \), \( P_{k',t} \), and the right side has the payoff, discounted properly to the future. If the household had a real bond which paid a net return, \( r_{t+1} \), from \( t \) to \( t+1 \), it would turn out that the household’s first order condition would be:

\[
\beta \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1 + r_{t+1}}.
\]

Thus, the first order condition for \( K_{t+1} \) becomes:

\[
P_{k',t} = \frac{1}{1 + r_{t+1}} \left[ \alpha (K_{t+1})^{\alpha-1} (z_{t+1} h_{t+1})^{1-\alpha} + P_{k',t+1} (1 - \delta) \right]. \tag{3.3}
\]

Finally, if there were a rental market for capital, with rental rate, \( R_k \), that rental rate would be equal to the marginal product of capital. So, we obtain:

\[
P_{k',t} = \frac{1}{1 + r_{t+1}} \left[ R_k + P_{k',t+1} (1 - \delta) \right].
\]

Note that by recursive substitution, this implies the standard formula for the price of capital:

\[
P_{k',t} = \frac{1}{1 + r_{t+1}} \left[ \frac{1}{1 + r_{t+1}} R_k + \frac{(1 - \delta)}{1 + r_{t+1}} P_{k',t+1} \right.
\]

\[
= \frac{1}{1 + r_{t+1}} \left[ \frac{1}{1 + r_{t+1}} R_k + \frac{(1 - \delta)}{1 + r_{t+1}} \left[ \frac{1}{1 + r_{t+2}} R_k + \frac{(1 - \delta)}{1 + r_{t+2}} P_{k',t+2} \right] \right]
\]

\[
= \frac{1}{1 + r_{t+1}} \left[ \frac{1}{1 + r_{t+1}} R_k + \frac{(1 - \delta)}{1 + r_{t+1}} \left[ \frac{1}{1 + r_{t+2}} R_k + \frac{(1 - \delta)}{1 + r_{t+2}} R_k + \frac{(1 - \delta)}{1 + r_{t+2}} P_{k',t+2} \right] \right]
\]

\[
= \ldots
\]

\[
= \sum_{i=1}^{\infty} \left( \prod_{j=1}^{i} \frac{1}{1 + r_{t+j}} \right) (1 - \delta)^{i-1} R_k.
\]
According to this, the price of capital is the present discounted value of its future earnings, where the discounting is done using the real rate of interest, and the fact that capital depreciates over time is taken into account. For our purposes, this formula is of limited use for thinking about $P_{K',t}$, because it has so many endogenous variables. However, since capital is reproducible in this model, there is first order condition corresponding to the construction of capital, and this first order condition (a variant of the ‘Tobin’s q’ condition), is useful for thinking about the price of capital.

The first order condition with respect to $I_t$ is:

$$-\lambda_t + \mu_t(1 - S\left(\frac{I_t}{I_{t-1}}\right)) - \mu_tS'(\frac{I_{t+1}}{I_{t-1}})\frac{I_t}{I_{t-1}} + \beta\mu_{t+1}S'(\frac{I_{t+1}}{I_t})\left(\frac{I_{t+1}}{I_t}\right)^2 = 0.$$ 

Rewriting this, taking into account the definition of the price of capital,

$$P_{K',t} = \frac{1}{1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)} - \left(1 + r_{t+1}\right)\frac{P_{K',t+1}S'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2}{1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}}.$$  

The right side of (3.5) is the marginal cost of an extra unit of capital. This marginal cost is the sum of two pieces. The first term is the usual marginal cost term that occurs in a static environment. It is the ratio of the consumption cost of a unit of investment goods, $dC_t/dI_t$ (which is unity), divided by the marginal productivity (in producing new capital) of an extra investment good, $dK_{t+1}/dI_t$. To see that this is indeed marginal cost, note that this corresponds to

$$\frac{dC_t}{dI_t} = \frac{dC_t}{dK_{t+1}}.$$

i.e., the consumption cost of capital. It is easy to verify, by differentiating (2.3) with respect to $I_t$, that the denominator in the first expression after the equality in (3.5) is indeed the marginal product of investment goods in producing $K_{t+1}$. If we just focus on this first term to the right of the equality, the puzzle about why $P_{K',t}$ drops during an overinvestment boom deepens. This is because, with the growth rate of investment high (see the above figure, which shows that $I_t/I_{t-1}$ is high for several periods), the first term after the equality should unambiguously be high during the boom. Both $S$ and $S'$ rise, and this by itself makes $P_{K',t}$ rise.
Now it is time to look at the other term to the right of the equality in (3.5). This depresses the price of capital. Since $P_{K',t}$ falls, this must be the term that is dominating. What is it? When a technology shock is expected to rise in the future, then it is also expected that high investment will be desirable in the future. But, in this case, the value of investment today rises, because this helps reduce the adjustment costs associated with high future investment. The future terms shows the extra capital that can be produced from increased investment today, due to the reduction in future adjustment costs. This converted into future consumption units by $P_{K',t+1}$, and discounted to the present by $1 + r_{t+1}$. In a sense, although the cost of investment today is actually only one consumption good, the net cost, after counting the future benefits from current investment, is much smaller. Not only is it smaller because of the reduction in future adjustment costs, but it is also smaller because the real interest rate, $1 + r_{t+1}$, is lower. This can be seen from the above figure, which shows a sharp rise in $\lambda_t$ during the boom.

The anticipated future technology shock in effect gives the planner who acquires investment goods today a ‘kickback’, in the form of a service in the future. This means the marginal cost an investment good is actually lower, making the marginal cost of capital lower. It is this reduction in the marginal cost of capital that gives the planner the incentive to increase capital and investment. The fall in $P_{K',t}$ is actually at the heart of the overinvestment boom. It is hard to see how one might contemplate an equilibrium with the property that $P_{K',t}$ rises in this type of environment.

As noted before, a modification of this environment pursued by Motto, Rosstagno and me does produce a rise in $P_{K',t}$ during an overinvestment boom. A very natural monetary policy causes this to happen, by leading the central bank to increase the money supply in the wake of a signal that future technology will be high. This extra money adds additional vigor to the boom in investment, output, employment and consumption. The monetary policy is a Taylor rule in which the interest rate is increased vigorously when anticipated inflation rises.

### 4. Equilibrium Conditions

To complete the discussion of the equations that characterize equilibrium, we derive the first order condition with respect to labor. Differentiate (3.1) with respect to $h_t$ to obtain:

$$\psi(C_t - bC_{t-1})^{1-\gamma} (1 - h_t)^{\psi(1-\gamma)-1} = \lambda_t (1 - \alpha) (K_t)^{\alpha} (z_t h_t)^{-\alpha}$$
This equation, together with (2.1), (2.2), (2.3), (3.2), (3.3), and (3.5), allow us to solve for the equilibrium. These seven equations, in effect, are sufficient to determine the evolution of the seven variables in $Z_t$, in (2.6). The steps in obtaining the linearized solution, $A$ and $B$ in (2.7), involve first computing the steady state values of the variables in $Z_t$. Then take a log-linear expansion of our six equations (the differentiation is done numerically in lines 14-18 of getAB.m). This gives rise (in lines 25-33 of getAB.m) to log-linearized system of Euler equations, and $A$ and $B$ are chosen to solve those (see lines 39-44 in getAB.m).

References
