Chapter 3

The Behavior of Households with Markets for Commodities and Credit

In this chapter we move from the world in which Robinson Crusoe is alone on his island to a world of many identical households that interact. To begin, we consider one particular representative household. When we add together the behaviors of many households, we get a macroeconomy.

Whereas in Chapter 2 we looked at Crusoe's choices between consumption and leisure at one point in time, now we consider households' choices of consumption over multiple periods, abstracting from the labor decisions of households. Section 3.1 introduces the basic setup of the chapter. In Section 3.2 we work out a model in which households live for only two periods. Households live indefinitely in the model presented in Section 3.3. Both these models follow Barro fairly closely, but of course in greater mathematical detail. The primary difference is that Barro has households carry around money, while we do not.

3.1 The General Setup

The representative household cares about consumption c_t in each period. This is formalized by some utility function $U(c_1, c_2, c_3, \dots)$. Economists almost always simplify intertemporal problems by assuming that preferences are additively separable. Such preferences look like: $U(c_1, c_2, c_3, \dots) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \cdots$. The $u(\cdot)$ function is called the *period utility*. It satisfies standard properties of utility functions. The variable β is called the *discount factor*. It is just a number, say 0.95. The fact that it is less than 1 means that the household cares a little more about current consumption than it cares about future consumption.

The household gets exogenous income y_t in each period. This income is in terms of consumption goods. We say that it is exogenous because it is independent of anything that the household does. Think of this income as some bequest from God or goods that fall from the sky.

At time t, the household can buy or sell consumption goods c_t at a price of P per unit. (As in Barro, the price level P does not change over time.) For example, if the household sells 4 units of consumption goods to someone else, then the seller receives $\$4P$ for those goods.

The household is able to save money by buying bonds that bear interest. We use b_t to denote the number of dollars of bonds that the household buys at period t, for which it will collect principal and interest in period $t + 1$. If the household invests \$1 this period, then next period it gets back its \$1 of principal plus $\$R$ in interest. Hence, if the household buys b_t in bonds this period, then next period the principal plus interest will be $b_t(1 + R)$. The household comes into the world with no bonds, i.e., $b_0 = 0$.

Since each \$1 of investment in bonds pays $\$R$ of interest, R is the simple rate of interest on the bonds. If the bonds pay R "next period", then whether the interest rate is daily, monthly, annual, etc., is determined by what the length of a "period" is. If the "period" is a year, then the interest rate R is an annual rate.

The household can either borrow or lend, i.e., the household can issue or buy bonds, whatever makes it happier. If b_t is negative, then the household is a net borrower.

At period t the household's resources include its income y_t and any bonds that it carries from last period, with interest. The dollar value of these resources is:

$$Py_t + b_{t-1}(1 + R).$$

At period t the household allocates its resources to its current consumption and to investment in bonds that it will carry forward to the next period. The dollar cost of these uses is:

$$Pc_t + b_t.$$

Putting these together gives us the household's period-t budget equation:

$$Py_t + b_{t-1}(1 + R) = Pc_t + b_t.$$

In a general setup, we would have one such budget equation for every period, and there could be arbitrarily many periods. For example, if a period were a year, and the household "lived" for 40 years, then we would have forty budget constraints. On the other hand, a period could be a day, and then we would have many more budget constraints.

3.2 A Two-Period Model

We begin this section with a discussion of the choices of a representative household. Then we put a bunch of these households together and discuss the resulting macroeconomic equilibrium.

Choices of the Representative Household

In this model the household lives for two time periods, $t = 1, 2$. In this case, the household's preferences reduce to:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2). \tag{3.1}$$

Given that the household will not be around to enjoy consumption in period 3, we know that it will not be optimal for the household to buy any bonds in period 2, since those bonds would take away from period-2 consumption c_2 and provide income only in period 3, at which time the household will no longer be around. Accordingly, $b_2 = 0$. That leaves only b_1 in this model.

The household's budget constraints simplify as well. In period 1 the household's budget equation is:

$$Py_1 = Pc_1 + b_1, \tag{3.2}$$

and in period $t = 2$ it is:

$$Py_2 + b_1(1 + R) = Pc_2. \tag{3.3}$$

The household's problem is to choose consumptions c_1 and c_2 and first-period bond holdings b_1 so as to maximize utility (3.1) subject to the budget equations (3.2) and (3.3). The household takes the price level P and the interest rate R as given.

We write out the household's problem:

$$\max_{c_1, c_2, b_1} \{u(c_1) + \beta u(c_2)\}, \text{ subject to:} \tag{3.4}$$

$$Py_1 = Pc_1 + b_1, \text{ and:} \tag{3.5}$$

$$Py_2 + b_1(1 + R) = Pc_2. \tag{3.6}$$

We solve this problem by using the method of Lagrange multipliers. The Lagrangean is:

$$\mathcal{L} = u(c_1) + \beta u(c_2) + \lambda_1[Py_1 - Pc_1 - b_1] + \lambda_2[Py_2 + b_1(1 + R) - Pc_2],$$

where λ_1 and λ_2 are our two Lagrange multipliers. The first-order conditions are:

$$u'(c_1^\star) + \lambda_1^\star[-P] = 0; \tag{FOC c_1}$$

$$\beta u'(c_2^\star) + \lambda_2^\star[-P] = 0; \text{ and:} \tag{FOC c_2}$$

$$\lambda_1^\star[-1] + \lambda_2^\star[(1 + R)] = 0. \tag{FOC b_1}$$

(Again, stars denote that only the optimal choices will satisfy these first-order conditions.) We leave off the first-order conditions with respect to the Lagrange multipliers λ_1 and λ_2, since we know that they will give us back the two budget constraints.

Rewriting the first two FOCs gives us:

$$\frac{u'(c_1^\star)}{P} = \lambda_1^\star, \quad \text{and:} \quad \frac{\beta u'(c_2^\star)}{P} = \lambda_2^\star.$$

We can plug these into the FOC with respect to b_1 to get:

$$-\frac{u'(c_1^\star)}{P} + \frac{\beta u'(c_2^\star)}{P}(1+R) = 0,$$

which we can rewrite as:

$$\frac{u'(c_1^\star)}{u'(c_2^\star)} = \beta(1+R). \tag{3.7}$$

Equation (3.7) is called an *Euler equation* (pronounced: OIL-er). It relates the marginal utility of consumption in the two periods. Given a functional form for $u(\cdot)$, we can use this equation and the two budget equations to solve for the household's choices $c_1^\star$, $c_2^\star$, and $b_1^\star$.

It is possible to use the Euler equation to make deductions about these choices even without knowing the particular functional form of the period utility function $u(\cdot)$, but this analysis is much more tractable when the form of $u(\cdot)$ is given. Accordingly, we assume $u(c_t) = \ln(c_t)$. Then $u'(c_t) = 1/c_t$, and equation (3.7) becomes:

$$\frac{c_2^\star}{c_1^\star} = \beta(1+R). \tag{3.8}$$

Before we solve for $c_1^\star$, $c_2^\star$, and $b_1^\star$, let us think about this equation. Recall, preferences are: $u(c_1)+\beta u(c_2)$. Intuitively, if β goes up, then the household cares more about the future than it used to, so we expect the household to consume more c_2 and less c_1.

This is borne out graphically in Barro's Figure 3.4. Larger β corresponds to smaller slopes in the household's indifference curves, which rotate downward, counter-clockwise. Accordingly, the household's choice of c_2 will go up and that of c_1 will go down, like we expect.

We can show the result mathematically as well. An increase in β causes an increase in right-hand side of the Euler equation (3.8), so $c_2^\star$ goes up relative to $c_1^\star$, just like we expect.

Now we consider changes on the budget side. Suppose R goes up. Then the opportunity cost of consumption c_1 in the first period goes up, since the household can forego c_1 and earn a higher return on investing in bonds. By the same reasoning, the opportunity cost of c_2 goes down, since the household can forego less c_1 to get a given amount of c_2. Accordingly, if R goes up, we expect the household to substitute away from c_1 and toward c_2.

Refer to Barro's Figure 3.4. If R goes up, then the budget line rotates clockwise, i.e., it gets steeper. This indicates that the household chooses larger c_2 and smaller c_1 (subject to being on any given indifference curve), just like our intuition suggests.

Mathematically, we refer once again to the Euler equation. If R goes up, then the right-hand side is larger, so $c_2^\star/c_1^\star$ goes up, again confirming our intuition.

Given $u(c_t) = \ln(c_t)$, we can actually solve for the household's optimal choices. The Euler equation and equations (3.2) and (3.3) give us three equations in the three unknowns, $c_1^\star$, $c_2^\star$, and $b_1^\star$. Solving yields:

$$c_1^\star = \frac{y_2 + y_1(1+R)}{(1+\beta)(1+R)},$$
$$c_2^\star = \left[y_2 + y_1(1+R)\right]\left[\frac{\beta}{1+\beta}\right], \text{ and:}$$
$$b_1^\star = Py_1 - \frac{P[y_2 + y_1(1+R)]}{(1+\beta)(1+R)}.$$

You can verify these if you like. Doing so is nothing more than an exercise in algebra.

If we tell the household what the interest rate R is, the household performs its own maximization to get its choices of c_1, c_2, and b_1, as above. We can write these choices as functions of R, i.e., $c_1^\star(R)$, $c_2^\star(R)$, and $b_1^\star(R)$, and we can ask what happens to these choices as the interest rate R changes. Again, this exercise is called "comparative statics". All we do is take the derivative of the choices with respect to R. For example:

$$\frac{\partial c_2^\star}{\partial R} = \frac{y_1\beta}{1+\beta} > 0,$$

so $c_2^\star$ goes up as the interest rate goes up, like our intuition suggests.

Market Equilibrium

So far we have restricted attention to one household. A macroeconomy would be composed of a number of these households, say N of them, so we stick these households together and consider what happens. In this model, that turns out to be trivial, since all households are identical, but the exercise will give you practice for more-difficult settings to come.

The basic exercise is to close our model by having the interest rate R determined endogenously. Recall, we said that households can be either lenders or borrowers, depending on whether b_1 is positive or negative, respectively. Well, the only borrowers and lenders in this economy are the N households, and all of them are alike. If they all want to borrow, there will be no one willing to lend, and there will be an excess demand for loans. On the other hand, if they all want to lend, there will be an excess supply of loans. More formally, we can write the aggregate demand for bonds as: $Nb_1^\star$. Market clearing requires:

$$Nb_1^\star = 0. \tag{3.9}$$

Of course, you can see that this requires that each household neither borrows nor lends, since all households are alike.

Now we turn to a formal definition of equilibrium. In general, a *competitive equilibrium* is a solution for all the variables of the economy such that: (i) all economic actors take prices as given; (ii) subject to those prices, all economic actors behave rationally; and (iii) all markets clear. When asked to define a competitive equilibrium for a specific economy, your task is to translate these three conditions into the specifics of the problem.

For the economy we are considering here, there are two kinds of prices: the price of consumption P and the price of borrowing R. The actors in the economy are the N households. There are two markets that must clear. First, in the goods market, we have:

$$Ny_t = Nc_t^\star, \ t = 1, 2. \tag{3.10}$$

Second, the bond market must clear, as given in equation (3.9) above. With all this written down, we now turn to defining a competitive equilibrium for this economy.

A competitive equilibrium in this setting is: a price of consumption $P^\star$; an interest rate $R^\star$; and values for $c_1^\star$, $c_2^\star$, and $b_1^\star$, such that:

- Taking $P^\star$ and $R^\star$ as given, all N households choose $c_1^\star$, $c_2^\star$, and $b_1^\star$ according to the maximization problem given in equations (3.4)-(3.6);
- Given these choices of $c_t^\star$, the goods market clears in each period, as given in equation (3.10); and
- Given these choices of $b_1^\star$, the bond market clears, as given in equation (3.9).

Economists are often pedantic about all the detail in their definitions of competitive equilibria, but providing the detail makes it very clear how the economy operates.

We now turn to computing the competitive equilibrium, starting with the credit market. Recall, we can write $b_1^\star$ as a function of the interest rate R, since the lending decision of each household hinges on the interest rate. We are interested in finding the interest rate that clears the bond market, i.e., the $R^\star$ such that $b_1^\star(R^\star) = 0$. We had:

$$b_1^\star(R) = Py_1 - \frac{P[y_2 + y_1(1+R)]}{(1+\beta)(1+R)},$$

so we set the left-hand side to zero and solve for $R^\star$:

$$Py_1 = \frac{P[y_2 + y_1(1+R^\star)]}{(1+\beta)(1+R^\star)}. \tag{3.11}$$

After some algebra, we get:

$$R^{\star} = \frac{y_2}{\beta y_1} - 1. \tag{3.12}$$

This equation makes clear that the equilibrium interest rate is determined by the incomes (y_1 and y_2) of the households in each period and by how impatient the households are (β). We can perform comparative statics here just like anywhere else. For example:

$$\frac{\partial R^{\star}}{\partial y_2} = \frac{1}{\beta y_1} > 0,$$

so if second-period income increases, then $R^{\star}$ does too. Conversely, if second-period income decreases, then $R^{\star}$ does too. This makes intuitive sense. If y_2 goes down, households will try to invest first-period income in bonds in order to smooth consumption between the two periods. In equilibrium this cannot happen, since net bond holdings must be zero, so the equilibrium interest rate must fall in order to provide a disincentive to investment, exactly counteracting households' desire to smooth consumption.

You can work through similar comparative statics and intuition to examine how the equilibrium interest rate changes in response to changes in y_1 and β. (See Exercise 3.2.)

Take note that in this model and with these preferences, only relative incomes matter. For example, if both y_1 and y_2 shrink by 50%, then y_2/y_1 does not change, so the equilibrium interest rate does not change. This has testable implications. Namely, we can test the reaction to a temporary versus a permanent decrease in income.

For example, suppose there is a temporary shock to the economy such that y_1 goes down by 10% today but y_2 is unchanged. The comparative statics indicate that the equilibrium interest rate must increase. This means that temporary negative shocks to income induce a higher interest rate. Now suppose that the negative shock is permanent. Then both y_1 and y_2 fall by 10%. This model implies that $R^{\star}$ does not change. This means that permanent shocks to not affect the interest rate.

The other price that is a part of the competitive equilibrium is $P^{\star}$, the price of a unit of consumption. It turns out that this price is not unique, since there is nothing in our economy to pin down what $P^{\star}$ is. The variable P does not even appear in the equations for $c_1^{\star}$ and $c_2^{\star}$. It does appear in the equation for $b_1^{\star}$, but P falls out when we impose the fact that $b_1^{\star} = 0$ in equilibrium; see equation (3.11). The intuition is that raising P has counteracting effects: it raises the value of a household's income but it raises the price of its consumption in exactly the same way, so raising P has no real effect. Since we cannot tack down $P^{\star}$, any number will work, and we have an infinite number of competitive equilibria. This will become clearer in Chapter 5.

3.3 An Infinite-Period Model

The version of the model in which the representative household lives for an infinite number of periods is similar to the two-period model from the previous section. The utility of the household is now:

$$U(c_1, c_2, \dots) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \cdots .$$

In each period t, the household faces a budget constraint:

$$Py_t + b_{t-1}(1+R) = Pc_t + b_t.$$

Since the household lives for all $t = 1, 2, \dots,$ there are infinitely many of these budget constraints. The household chooses c_t and b_t in each period, so there are infinitely many choice variables and infinitely many first-order conditions. This may seem disconcerting, but don't let it intimidate you. It all works out rather nicely. We write out the maximization problem in condensed form as follows:

$$\max_{\{c_t, b_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t), \text{ such that:}$$
$$Py_t + b_{t-1}(1+R) = Pc_t + b_t, \ \forall\, t \in \{1, 2, \dots\}.$$

The "∀" symbol means "for all", so the last part of the constraint line reads as "for all t in the set of positive integers".

To make the Lagrangean, we follow the rules outlined on page 15. In each time period t, the household has a budget constraint that gets a Lagrange multiplier λ_t. The only trick is that we use summation notation to handle all the constraints:

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) + \sum_{t=1}^{\infty} \lambda_t \left[Py_t + b_{t-1}(1+R) - Pc_t - b_t \right].$$

Now we are ready to take first-order conditions. Since there are infinitely many of them, we have no hope of writing them all out one by one. Instead, we just write the FOCs for period-t variables. The c_t FOC is pretty easy:

(FOC c_t) $$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^{t-1} u'(c_t^\star) + \lambda_t^\star[-P] = 0.$$

Again, we use starred variables in first-order conditions because these equations hold only for the optimal values of these variables.

The first-order condition for b_t is harder because there are two terms in the summation that have b_t in them. Consider b_2. It appears in the $t = 2$ budget constraint as b_t, but it also appears in the $t = 3$ budget constraint as b_{t-1}. This leads to the $t + 1$ term below:

(FOC b_t) $$\frac{\partial \mathcal{L}}{\partial b_t} = \lambda_t^\star[-1] + \lambda_{t+1}^\star[(1+R)] = 0.$$

Simple manipulation of this equation leads to:

$$\frac{\lambda_t^\star}{\lambda_{t+1}^\star} = 1 + R. \tag{3.13}$$

Rewriting equation (FOC c_t) gives us:

$$\beta^{t-1} u'(c_t^\star) = \lambda_t^\star P. \tag{3.14}$$

We can rotate this equation forward one period (i.e., replace t with $t+1$) to get the version for the next period:

$$\beta^t u'(c_{t+1}^\star) = \lambda_{t+1}^\star P. \tag{3.15}$$

Dividing equation (3.14) by equation (3.15) yields:

$$\frac{\beta^{t-1} u'(c_t^\star)}{\beta^t u'(c_{t+1}^\star)} = \frac{\lambda_t^\star P}{\lambda_{t+1}^\star P}, \text{ or:}$$
$$\frac{u'(c_t^\star)}{\beta u'(c_{t+1}^\star)} = \frac{\lambda_t^\star}{\lambda_{t+1}^\star}.$$

Finally, we multiply both sides by β and use equation (3.13) to get rid of the lambda terms on the right-hand side:

$$\frac{u'(c_t^\star)}{u'(c_{t+1}^\star)} = \beta(1 + R). \tag{3.16}$$

If you compare equation (3.16) to equation (3.7), you will find the Euler equations are the same in the two-period and infinite-period models. This is because the intertemporal trade-offs faced by the household are the same in the two models.

Just like in the previous model, we can analyze consumption patterns using the Euler equation. For example, if $\beta = 1/(1 + R)$, then the household's impatience exactly cancels with the incentives to invest, and consumption is constant over time. If the interest rate R is relatively high, then the right-hand side of equation (3.16) will be greater than one, and consumption will be rising over time.

A Present-Value Budget Constraint

Now we turn to a slightly different formulation of the model with the infinitely-lived representative household. Instead of forcing the household to balance its budget each period, now the household must merely balance the present value of all its budgets. (See Barro's page 71 for a discussion of present values.) We compute the present value of all the household's income:

$$\sum_{t=1}^{\infty} \frac{P y_t}{(1 + R)^{t-1}}.$$

This gives us the amount of dollars that the household could get in period 1 if it sold the rights to all its future income. On the other side, the present value of all the household's consumption is:

$$\sum_{t=1}^{\infty} \frac{Pc_t}{(1+R)^{t-1}}.$$

Putting these two present values together gives us the household's single present-value budget constraint. The household's maximization problem is:

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t), \text{ such that:}$$
$$\sum_{t=1}^{\infty} \frac{P(y_t - c_t)}{(1+R)^{t-1}} = 0.$$

We use λ as the multiplier on the constraint, so the Lagrangean is:

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) + \lambda \left[\sum_{t=1}^{\infty} \frac{P(y_t - c_t)}{(1+R)^{t-1}} \right].$$

The first-order condition with respect to c_t is:

(FOC c_t) $$\beta^{t-1} u'(c_t^\star) + \lambda^\star \left[\frac{P(-1)}{(1+R)^{t-1}} \right] = 0.$$

Rotating this forward and dividing the c_t FOC by the c_{t+1} FOC yields:

$$\frac{\beta^{t-1} u'(c_t^\star)}{\beta^t u'(c_{t+1}^\star)} = \frac{\lambda^\star \left[\frac{P}{(1+R)^{t-1}} \right]}{\lambda^\star \left[\frac{P}{(1+R)^{t}} \right]},$$

which reduces to:

$$\frac{u'(c_t^\star)}{u'(c_{t+1}^\star)} = \beta(1+R),$$

so we get the same Euler equation once again. It turns out that the problem faced by the household under the present-value budget constraint is equivalent to that in which there is a constraint for each period. Hidden in the present-value version are implied bond holdings. We could deduce these holdings by looking at the sequence of incomes y_t and chosen consumptions $c_t^\star$.

Exercises

Exercise 3.1 (Hard)
Consider the two-period model from Section 3.2, and suppose the period utility is:

$$u(c_t) = c_t^{\frac{1}{2}}.$$

Variable	Definition
$U(\cdot)$	Overall utility
t	Time
c_t	Consumption at period t
$u(\cdot)$	Period utility
β	Household's discount factor
y_t	Household's income in period t, in units of consumption
P	Cost of a unit of consumption
R	Nominal interest rate
b_t	Number of dollars of bonds bought at period t
λ_t	Lagrange multiplier in period t
N	Number of households

Table 3.1: Notation for Chapter 3

1. Determine the Euler equation in this case.
2. Determine the representative household's optimal choices: $c_1^\star$, $c_2^\star$, and $b_1^\star$.
3. Determine the equilibrium interest rate $R^\star$.
4. Determine the effect on the equilibrium interest rate $R^\star$ of a permanent negative shock to the income of the representative household. (I.e., both y_1 and y_2 go down by an equal amount.) How does this relate to the case in which $u(c_t) = \ln(c_t)$?

Exercise 3.2 (Easy)
Refer to equation (3.12), which gives the equilibrium interest rate $R^\star$ in the two-period model.

1. Suppose the representative household becomes more impatient. Determine the direction of the change in the equilibrium interest rate. (Patience is measured by β. You should use calculus.)
2. Suppose the representative household gets a temporary negative shock to its period-1 income y_1. Determine the direction of the change in the equilibrium interest rate. (Again, use calculus.)

Exercise 3.3 (Moderate)
Maxine lives for two periods. Each period, she receives an endowment of consumption goods: e_1 in the first, e_2 in the second. She doesn't have to work for this output. Her preferences for consumption in the two periods are given by: $u(c_1, c_2) = \ln(c_1) + \beta \ln(c_2)$, where

c_1 and c_2 are her consumptions in periods 1 and 2, respectively, and β is some discount factor between zero and one. She is able to save some of her endowment in period 1 for consumption in period 2. Call the amount she saves s. Maxine's savings get invaded by rats, so if she saves s units of consumption in period 1, she will have only $(1 - \delta)s$ units of consumption saved in period 2, where δ is some number between zero and one.

1. Write down Maxine's maximization problem. (You should show her choice variables, her objective, and her constraints.)

2. Solve Maxine's maximization problem. (This will give you her choices for given values of e_1, e_2, β, and δ.)

3. How do Maxine's choices change if she finds a way reduce the damage done by the rats? (You should use calculus to do comparative statics for changes in δ.)

Exercise 3.4 (Moderate)
An agent lives for five periods and has an edible tree. The agent comes into the world at time $t = 0$, at which time the tree is of size x_0. Let c_t be the agent's consumption at time t. If the agent eats the whole tree at time t, then $c_t = x_t$ and there will be nothing left to eat in subsequent periods. If the agent does not eat the whole tree, then the remainder grows at the simple growth rate α between periods. If at time t the agent saves $100s_t$ percent of the tree for the future, then $x_{t+1} = (1 + \alpha)s_t x_t$. All the agent cares about is consumption during the five periods. Specifically, the agent's preferences are: $U = \sum_{t=0}^{4} \beta^t \ln(c_t)$. The tree is the only resource available to the agent.

Write out the agent's optimization problem.