Preliminaries

Fix a finite type structure $T = (I, (S_i, T_i, \beta_i)_{i \in I})$ and a probability $\mu \in \Delta (S \times T)$. Let $T^\mu = (I, (S_i, T_i^\mu, \beta_i^\mu)_{i \in I})$ be a type structure that admits $\mu$ as a common prior and such that $T_i^\mu \subseteq T_i$ for every $i$.

Fix a player $i$ and a type profile $t^* \in T_i^\mu$. Define

$$E_1(t^*_i) = \{(s, t_{-i}, t^*_i) : \mu (s, t_{-i}, t^*_i | t^*_i) > 0\}$$

Suppose $E_k(t^*_i)$ has been defined for every $1 < k \leq n$ and let

$$E^{n+1}(t^*_i) = \{(s, t) : \exists (s', t') \in E^n, j \in I \text{ s.t. } t_j = t'_j \text{ and } \mu (s, t_{-j}, t'_j | t'_j) > 0\}$$

Let $E(t^*_i) = \bigcup_{n=1}^{\infty} E^n(t^*_i)$ and $E(t^*) = \bigcup_{i \in I} E(t^*_i)$.

**Proposition.** Let $t^*_i$ be in $CP_i(\mu)$ and $\nu = \mu (\cdot | E(t^*_i))$. Define $T^\nu_j = proj_{T_j} E(t^*_i)$ for every $j$ and let $T^\nu = \left( I, \left( S_j, T^\nu_j, \beta^\nu_j \right)_{j \in I} \right)$ be the type structure generated by the common prior $\nu$. Then $t^*_i$ is in $CP_i(\nu)$. In particular, if $\mu$ is minimal for $t^*_i$ then $\nu = \mu$.

**Proof.** We first prove that for all $j$ and all $t_j \in T^\mu_j$,

$$\nu \left( S \times T^\mu_{-j} \times \{t_j\} \right) > 0 \implies \text{marg}_{S \times T^\nu_{-j}} \nu (\cdot | t_j) = \text{marg}_{S \times T^\mu_{-j}} \mu (\cdot | t_j).$$

By definition, if $\mu \left( S \times T^\mu_{-j} \times \{t_j\} \right) > 0$ then

$$\mu (s_{-j}, t_{-j} | t_j) = \frac{\mu ((s_{-j}, t_{-j}) \times S_j \times \{t_j\})}{\sum_{(s'_{-j}, t'_{-j}) \in S_{-j} \times T_{-j}} \mu ((s'_{-j}, t'_{-j}) \times S_j \times \{t_j\})}$$
For every \((s_{-j}, t_{-j}) \in S_{-j} \times T_{-j}\). If \(\nu \left( S \times T_{-j}^{\mu} \times \{t_j\} \right) > 0\) then \(t_j \in \text{proj}_{T_j} E^k (t_{-j}^*)\) for some \(k\). For every \((s_{-j}', t_{-j}') \in S_{-j} \times T_{-j}\), if

\[\mu \left( (s_{-j}', t_{-j}') \times S_j \times \{t_j\} \right) > 0\]

then \( (s_{-j}', t_{-j}') \times S_j \times \{t_j\} \subseteq E^{k+1} \left( t_{-j}^* \right) \subseteq E \left( t_{-j}^* \right)\), thus

\[\nu \left( (s_{-j}', t_{-j}') \times S_j \times \{t_j\} \right) = \frac{\mu \left( (s_{-j}', t_{-j}') \times S_j \times \{t_j\} \right)}{\mu (E \left( t_{-j}^* \right))}\]

Therefore

\[
\mu (s_{-j}, t_{-j} | t_j) = \frac{\mu (E \left( t_{-j}^* \right))}{\mu (E \left( t_{-j}^* \right))} \nu \left( (s_{-j}, t_{-j}) \times S_j \times \{t_j\} \right)
\]

We can conclude that for every \(j \in I\), \(\beta^\nu (t_j) = \beta_j^\mu (t_j)\) for each \(t_j \in T_j^\nu\).

It remains to prove that \(\varphi_j^1 (T^\nu) (t_j^*) = \varphi_j (T^\mu) (t_j^*)\). For every \(j\), \(t_j \in T_j^\nu\) and \(k \geq 0\), let \(\varphi_j^k (T^\nu) (t_j)\) be the \(k\)-th order belief of type \(t_j\) in the type structure \(T^\nu\). Define \(\varphi_j^k (T^\mu)\) analogously. For every \(j \in I\) and \(t_j \in T_j^\nu\), we have \(\beta^\nu (t_j) = \beta_j^\mu (t_j)\), hence \(\varphi_j^1 (T^\nu) (t_j) = \varphi_j^1 (T^\mu) (t_j)\). Suppose \(\varphi_j^k (T^\nu) (t_j) = \varphi_j^k (T^\mu) (t_j)\) for all \(j\), \(k \leq K\) and \(t_j \in T_j^\nu\). Then

\[\beta^\nu (t_j) (\{(s_{-j}, t_{-j}) : \varphi_{-j}^K (T^\nu) (t_{-j}) = h_{-j}^K \}) = \beta_j^\mu (t_j) (\{(s_{-j}, t_{-j}) : \varphi_{-j}^K (T^\mu) (t_{-j}) = h_{-j}^K \})\]

for every \(h_{-j}^K \in \Delta \left( X_{-j}^{K-1} \right)\). Therefore \(\varphi_j^{K+1} (T^\nu) (t_j) = \varphi_j^{K+1} (T^\nu) (t_j)\). Since this is true for every \(K\), we have \(\varphi_j (T^\nu) (t_j) = \varphi_j (T^\nu) (t_j)\) for every \(t_j \in T_j^\nu\), in particular, for \(t_j^*\). This concludes the proof that \(t_j^*\) is in \(CP_i (\nu)\). \(\square\)

An analogous result holds for type profiles. We omit the proof, which is an almost exact replica of the proof of Proposition 1.

**Proposition.** Let \(t^*\) be in \(CP (\mu)\) and define \(\nu = \mu (\cdot | E (t^*))\). Define \(T^\nu_i = \text{proj}_{T_i} E (t^*)\) for every \(i \in I\) and let \(T^\nu = (I, (S_{-i}, T^\nu_i, \beta^\nu_i)_{i \in I})\) be the type space generated by the common prior \(\nu\). Then \(t^*\) is in \(CP (\nu)\). In particular, if \(\mu\) is minimal for \(t^*\) then \(\nu = \mu\).
Events across type structures

Let $R^\mu$, $B^k R^\mu$ and $CBR^\mu$ be the events corresponding to, respectively, “rationality”, “$k$-th order belief in rationality” and “common belief in rationality” in the type structure $T^\mu$. In the proofs we will not formally distinguish between $CBR$ and $CBR^\mu$. This is justified by the next result.

**Proposition.** If $(s_i, t_i) \in CP_i (\mu) \cap CBR_i$, then $(s_i, t_i) \in CBR^\mu_i$.

**Proof.** Let $R^*$, $B^k R^*$ and $CBR^*$ be the events corresponding to, respectively, rationality, $k$-th order belief in rationality and common belief in rationality in the universal type structure $H = (I, (S_{-i}, H_i, f_i)_{i \in I})$. For every $i$, let $\psi_i (T) : S_i \times T_i \to S_i \times H_i$ be the map defined as

$$
\psi_i (T) (s_i, t_i) = (s_i, \varphi (T) (t_i))
$$

for every $(s_i, t_i)$. As is well known, $B^k R^*$ and $R^*$ are measurable events, and $\psi_i (T^\mu)$ and $\psi_i (T)$ are measurable maps. Furthermore, for every $i$, every event $E_{-i} \subseteq S_{-i} \times H_{-i}$ and every type $t_i \in T_i$,

$$
f_i (\varphi_i (T) (t_i)) (E_{-i}) = \beta_i (t_i) \left( \psi_{-i} (T)^{-1} (E_{-i}) \right)
$$

where $\psi_{-i} (T) = \prod_{j \neq i} \psi_j (T)$. Define analogously the functions $(\psi_i (T^\mu))_{i \in I}$.

Let $\psi = \prod_{i \in I} \psi_i$. It can be easily checked that $R = \psi (T)^{-1} (R^*)$ and $R^\mu = \psi (T^\mu)^{-1} (R^*)$. Suppose for every $k \leq K$ we have $B^k R = \psi (T)^{-1} (B^k R^*)$ and $B^k R^\mu = \psi (T^\mu)^{-1} (B^k R^*)$. It follows from

$$
\beta_i (t_i) (B^K R) = \beta_i (t_i) \left( \psi_{-i} (T)^{-1} (B^K R^*) \right) = f_i (\varphi_i (T) (t_i)) (B^K R^*)
$$

that $(s_i, t_i) \in B_i (B^K R)$ if and only if $(s_i, \varphi_i (T) (t_i)) \in B_i (B^K R^*)$. Equivalently, $B_i (B^K R) = \psi_i (T)^{-1} (B_i (B^K R^*))$ for every $i$. Therefore $B (B^K R) = \psi (T)^{-1} (B (B^K R^*))$.

Hence

$$
B^{K+1} R = B^K R \cap B (B^K R) = \psi (T)^{-1} (B^K R^*) \cap \psi (T)^{-1} (B (B^K R^*)) = \psi (T)^{-1} (B^{K+1} R^*)
$$

By induction, we can conclude that $B^K R^\mu = \psi (T^\mu)^{-1} (B^K R^*)$ for every $k$. Moreover,

$$
CBR = \cap_k B^K R = \cap_k \psi (T)^{-1} (B^K R^*) = \psi (T)^{-1} (\cap_k B^K R^*) = \psi (T)^{-1} (CBR^*)
$$

The exact same arguments apply to the type structure $T^\mu$, therefore we have $CBR^\mu =$
\[ \psi(T^\mu)^{-1}(CBR^*). \]

Let \((s_i, t_i) \in CP_i(\mu) \cap CBR_i\). By applying the results above and the assumption \(\varphi(T_i(t_i)) = \varphi(T^\mu_i(t_i))\), we can conclude

\[
1 = \beta_i(t_i)(CBR^*)
= \beta_i(t_i)\left(\psi^{-1}_{-i}(T^\mu_i)^{-1}(CBR^*)\right)
= f_i(\varphi_i(T)(t_i))(CBR^*)
= f_i(\varphi_i(T^\mu)(t_i))(CBR^*)
= \beta^\mu_i(t_i)\left(\psi^{-1}(T^\mu)^{-1}(CBR^*)\right)
= \beta^\mu_i(t_i)(CBR^*)
\]

together with \((s_i, t_i) \in CBR^*_i\). \(\square\)

Other events of interest which appear in the next proofs are \(CB([\phi])\) and \(CB([n])\). The argument behind the previous proposition can be easily adapted to show that we do not need to distinguish between these events and their counterparts in the type structure \(T^\mu\).

**Proof of Theorem 4**

(1)

**Claim.** For every \(k\), \(E^k(t^*_i) \subseteq CBR\).

**Proof.** For every profile \((s_{-i}, t_{-i})\), if \(\mu(s_{-i}, t_{-i}|t^*_i) > 0\) then \(\beta^\mu_i(t^*_i)(s_{-i}, t_{-i}) > 0\) and since \(t^*_i\) is in \(CBR_i \subseteq B_i(CBR)\) then \((s_{-i}, t_{-i}) \in CBR_{-i}\). Therefore \(E^1(t^*_i) \subseteq CBR\).

Suppose the claim is proved for every \(k \leq K\). If \((s, t) \in E^{K+1}(t^*_i)\) there exist \((s', t') \in E^K(t^*_i)\) and a player \(j\) such that \(t_j = t'_j\) and \(\beta^\mu_j(t'_j)(s_{-j}, t_{-j}) > 0\). Since \(t'_j\) is in \(B_j(CBR)\) then \((s_{-j}, t_j) \in CBR_{-j}\). Therefore \((s, t) \in CBR\). Therefore, by induction, we conclude that for every \(k\), \(E^k(t^*_i) \subseteq CBR\). \(\square\)

We now show that \(\mu \in \Delta(S \times T)\) defines a correlated equilibrium. Let \(\mu(s_j, t_j) > 0\) for some player \(j\) and pair \((s_j, t_j)\). Then \((s_{-j}, t_{-j}, s_j, t_j) \in E^k(t^*_i)\) for some \(k\) and some \((s_{-j}, t_{-j}) \in S_{-j} \times T_{-j}\). Pick \(l \neq j\). Then

\[
\mu(s_j, t_j|t_l) = \beta^\mu_i(t_l)(s_j, t_j) > 0
\]
Since $t_i$ is in $CBR_t \subseteq B_t R$ then $(s_j, t_j) \in R_j$. Therefore $s_j$ is a best response to 

$$\text{marg}_{S_{-j}} \beta^\mu_{ij} (t_j) = \text{marg}_{S_{-j}} \mu (\cdot|t_j) = \text{marg}_{S_{-j}} \mu (\cdot|s_j, t_j)$$

where the last equality follows from AI independence. Therefore $\mu \in \Delta (S \times T)$ is a correlated equilibrium.

(2)

Let $\nu \in \Delta (S)$ be a correlated equilibrium distribution. Then 

$$\sum_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) \nu (s_{-i}|s_i) \geq \sum_{s_{-i} \in S_{-i}} u(s_i', s_{-i}) \nu (s_{-i}|s_i)$$

for every $s_i' \in S_i$. Let $T^\mu_i = \{ s_i : \nu (s_i) > 0 \}$ and $T^\mu = \prod_{i \in I} T^\mu_i$. Define the prior $\mu \in \Delta (S \times T)$ as 

$$\mu (s, t) = \nu (s)$$

if $s = t$ and 

$$\mu (s, t) = 0$$

otherwise. Define $\beta^\mu$ to be generated by $\mu$, that is 

$$\beta^\mu_i (t_i) (s_{-i}, t_{-i}) = \mu (s_{-i}, t_{-i}|t_i)$$

for every $i$ and every $(s, t)$. We have a well defined type structure $T^\mu = (I, (S_{-i}, T^\mu_i, \beta^\mu_i)_{i \in I})$ admitting $\mu$ as a common prior. The prior satisfies Condition AI trivially, since for every $s_i$ and $t_i$ if $\mu (s_i, t_i) > 0$ then $s_i = t_i$.

If $\mu (s_i, t_i) > 0$ then $s_i = t_i$ and $s_i$ is a best response to $\nu (\cdot|s_i)$, hence $(s_i, t_i) \in R_i$. Moreover, if $\beta_i (t_i) (s_{-i}, t_{-i}) > 0$ then $\mu (s_{-i}, t_{-i}) > 0$ hence $(s_{-i}, t_{-i}) \in R_{-i}$. Therefore, if $\mu (s, t) > 0$ then $(s, t) \in RCBR$.

Proof of Theorem 8

It is enough to prove that if $(s_i, t^*_i) \in CP (\mu) \cap CB ([n])$ and $\mu$ is minimal for $t^*_i$ then $\mu$ satisfies AI. As before, it is immediate to check that for every $k$, $E^k (t^*_i) \subseteq CB ([n])$.

Let $\mu (s_j, t_j) > 0$. There exist $(s_{-j}, t_{-j})$ such that $(s, t) \in E^k (t^*_i)$ for some $k$ and 

$$\mu (s_{-j}, t_{-j}|s_j, t_j) > 0.$$
Let \( l \neq j \). Then \( \mu(s_j, t_j | t_l) > 0 \) and since \((s_j, t_j) \in CB([n])\), then \( t_l \) is in \( B([n]) \), hence \( s_j = n_j(t_j) \). To conclude, if \( \mu(s_j, t_j) > 0 \) then \( s_j = n_j(t_j) \). Therefore \( \mu \) satisfies AI.

**Proof of Theorem 7**

It is convenient to prove here a slightly stronger result.

**Theorem.** (7b) If there is a probability \( \mu \in \Delta(S \times T) \), a tuple \( t^* \in CP(\mu) \cap [\phi] \cap CB([\phi]) \cap B(R) \) and \( \nu = \mu(:,E(t^*)) \) satisfies AI, then there exist \( \sigma_i \in \Delta(S_i) \) for all \( i \) such that \( \sigma = (\sigma_i)_{i \in I} \) is a Nash Equilibrium and \( \phi_i = \prod_{k \neq i} \sigma_k \).

As in the proof of Theorem 2, if \( t^* \in CB([\phi]) \) then for every \( i \) and every \( k \), \( E_k(t^*) \subseteq CB([\phi]) \). The rest of the proof is based on Aumann and Brandenburger (1995).

**Claim.** For every \((s_i, t_i)\) if \( \nu(s_i, t_i) > 0 \) then \( \nu(s_{-i}) = \nu(s_{-i}|s_i, t_i) = \phi_i(s_{-i}) \).

**Proof.** For every \((s_i, t_i)\), if \( \nu(s_i, t_i) > 0 \) then \((s_{-i}, t_{-i}, s_i, t_i) \in E^k(t^*) \) for some \( k \). Since \( E^k(t^*) \subseteq CB([\phi]) \) and \( E^k(t^*) \subseteq E^{k+1}(t^*) \) then \((s_i, t_i) \in [\phi]_i \). Hence

\[
\nu(s_{-i}|s_i, t_i) = \nu(s_{-i}|t_i) = \beta^\nu(t_i)(s_{-i}) = \phi_i(s_{-i})
\]

where the first equality follows from AI. Therefore

\[
\nu(s_{-i}) = \sum_{(s_i, t_i)} \nu(s_{-i}|s_i, t_i) \nu(s_i, t_i) = \sum_{(s_i, t_i)} \phi_i(s_{-i}) \nu(s_i, t_i) = \phi_i(s_{-i}).
\]

\[\Box\]

**Claim.** For every \( s \), \( \nu(s) = \prod_{i=1}^I \nu(s_i). \)

**Proof.** Suppose for \( K < |I| \) and every \( s \in S \) and \( i \in I \),

\[
\nu(s_1, s_2, \ldots, s_K, \ldots, s_I) = \prod_{i=1}^K \nu(s_i) \nu(s_{K+1}, \ldots, s_I)
\]

We know from the previous claim that this is true for \( K = 1 \). Suppose it is true for
some $K > 1$. Then

$$
\nu(s_1, \ldots, s_I) = \text{marg}_{S_{(K+1)^{-1}}} \nu(s_1, \ldots, s_K, s_{K+2}, \ldots, s_I | s_{K+1}) \nu(s_{K+1})
= \text{marg}_{S_{(K+1)^{-1}}} \nu(s_1, \ldots, s_K, s_{K+2}, \ldots, s_I) \nu(s_{K+1})
= \frac{1}{\text{marg}_{S_{(K+1)^{-1}}} \nu(s_1, \ldots, s_K, s_{K+2}, \ldots, s_I)} \nu(s_{K+1})
= \nu(s_1) \cdots \nu(s_K) \nu(s_{K+1}) \sum_{s_{K+1} \in S_{K+1}} \text{marg}_{S_{K+1} \times \ldots \times S_I} \nu(s'_{K+1}, s_{K+2}, \ldots, s_I) \nu(s_{K+1})
= \nu(s_1) \cdots \nu(s_K) \nu(s_{K+1}) \nu(s_{K+2}, \ldots, s_I)
$$

Therefore the claim holds for every $K \leq I$. \hfill \Box

Claim. If $\nu(s_{-i}) > 0$ then $\phi_i(s_{-i}) = \prod_{k \neq i} \nu(s_k)$

Proof. By combining the previous two claims, if $\nu(s_i, t_i) > 0$ then

$$
\nu(s_{-i} | s_i, t_i) = \phi_i(s_{-i}) = \nu(s_{-i}) = \prod_{k \neq i} \nu(s_k).
$$

\hfill \Box

Define $\sigma_i = \text{marg}_{S_i} \nu$. Let $\sigma_i(s_i) > 0$. Fix a player $j \neq i$ and the type $t^*_j$ in the tuple $t^*$. By assumption $t^*_j \in [\phi]_j$. By the claims above and AI independence,

$$
\nu(s_i | t^*_j) = \nu(s_i | s_j, t^*_j) = \phi_j(s_i) = \sigma(s_i)
$$

hence $\nu(s_i | t^*_j) > 0$. Let $t_i$ be a type such that $\nu(s_i, t_i | t^*_j) > 0$. Since $t^*_j \in B(R)_j$, then $t_i$ is a best response to the first order belief of type $t_i$. Because $t^*_j \in CB([\phi])_j$, then $t_i \in [\phi]_i$, i.e. the first order belief of $t_i$ is given by the conjecture $\phi_i = \prod_{k \neq i} \sigma_k$. To conclude, for every player $i$, every strategy in the support of $\sigma_i$ is a best response to the conjecture $\prod_{k \neq i} \sigma_i$. Therefore, $\sigma$ is a Nash Equilibrium.

**Proof of Theorem 9**

Let $t^*$ belong to

$$
CP(\mu) \cap [\phi] \cap CB([\phi]) \cap B(R) \cap CB([n])
$$
Notice that $\mu$ is not assumed to be minimal. Let $\nu = \mu (\cdot | E (t^{*}))$. From Proposition 2, we have that if $t^{*} \in CP (\mu)$ then $t^{*} \in CP (\nu)$. Therefore, $t^{*}$ is in

$$CP (\nu) \cap [\phi] \cap CB ([\phi]) \cap B (R) \cap CB ([n])$$

As before, it is immediate to check that for every $i$ and every $k$, $E^k (t_i^*) \subseteq CB ([n])$. Let $\nu(s_j, t_j) > 0$. There exist $(s_{-j}, t_{-j})$ such that $(s_j, t_j, s_{-j}, t_{-j}) \in E^k (t_i^*)$ for some $k$ and $i$, and

$$\nu(s_{-j}, t_{-j}|s_j, t_j) > 0.$$

Let $l \neq j$. Then $\nu(s_j, t_j|t_l) > 0$ and since $(s_j, t_j) \in CB ([n])$, then $t_i$ is in $B ([n])$, hence $s_j = n_j (t_j)$. To conclude, if $\nu(s_j, t_j) > 0$ then $s_j = n_j (t_j)$. Therefore $\nu$ satisfies AI. We can now apply Theorem 7b.

**Proof of Theorem 14**

By standard arguments, we can find two types $(\bar{t}_1, \bar{t}_2) \in [\mathcal{T}^\Theta] \cap CB ([\mathcal{T}^\Theta]) \cap CB ([\psi]) \cap CB (\Theta \times R)$. Let $\bar{t}_i^\Theta = \varphi_i, \Theta (\bar{t}_i)$ for every $i$.

**Definition.** A type $t_i^\Theta$ of player $i$ is reachable in $N$ steps if there exists a sequence $t_{i(1)}, \ldots, t_{i(N)}^\Theta$ such that:

- $t_{i(1)}^{\Theta} = \bar{t}_i^\Theta$
- $i(N) = i$ and $t_{i(N)}^\Theta = t_i^\Theta$
- For all $n \leq N$, $\beta_{i(n)}^\Theta \left( t_{i(n-1)}^\Theta \right) \left( \left[ t_{i(n)}^\Theta \right] \right) > 0$

Let $RE^N$ be the set of types reachable in $N$ steps. Since the type structure $\mathcal{T}^\Theta$ is minimal, every type is reachable in a finite number of steps.

We need to show that for every $N$ every player $i$ and type $t_i^\Theta$ in $RE^N$, if $\psi_i (t_i^\Theta) (s_i) > 0$ then $s_i$ is optimal to the conjecture $\phi (t_i^\Theta)$ defined as

$$\phi (t_i^\Theta) (s_{-i}) = \sum_{t_{-i}^\Theta \in T_{-i}^\Theta} \beta_{i}^\Theta (t_i^\Theta) (t_{-i}^\Theta) \psi (t_{-i}^\Theta) (s_{-i})$$

for every $s_{-i} \in S_{-i}$.

Let $t_i^\Theta$ be in $RE^N$. Since $\mathcal{T}^\Theta$ is minimal, it is without loss of generality to assume $N > 2$. Let $t_{i(1)}^\Theta, \ldots, t_{i(N)}^\Theta$ be a sequence reaching $t_i^\Theta$ in $N$-steps.

**Claim.** There exist a sequence $t_{i(1)}, t_{i(2)}, \ldots, t_{i(N)}$ in $T$ such that $i(N) = i$, $\varphi_{i(n)} (t_{i(n)}) = \varphi_{i(n)}^\Theta \left( t_{i(n)}^\Theta \right)$ for all $n \leq N$ and $\beta (t_{i(n)}) \left( \left[ t_{i(n+1)} \right] \right) > 0$ for every $n \leq N - 1$. 

8
Proof. Since $\tilde{\Theta}_{i(1)} = \varphi_i, \tilde{t}_{i(1)}$ and $\beta_{i(1)} \left[ t_{i(1)} \right] > 0$ then there must exist a type $t_{i(2)}$ such that $\varphi_i(t_{i(2)}) = \varphi_{i(2)} \left[ t_{i(2)} \right]$ and $\beta_{i(1)} \left( t_{i(1)} \right) (t_{i(2)}) > 0$. A simple argument by induction concludes the proof.

Claim. For every $2 < n \leq N$, $t_{i(n)}$ is in $[\psi] \cap R \cap CB ([\Theta]) \cap CB ([\psi]) \cap CB (\Theta \times R)$.

Proof. It can be easily proved by induction.

Suppose $\psi_i(t_i) > 0$. Since $t_{i(N-1)}$ is in $B ([\psi])$, then $\beta(t_{i(N-1)}, s_i) > 0$. Since $t_{i(N-1)}$ is in $B(R)$ then $s_i$ is a best response to the first order belief over strategies of type $t_i$, defined as the conjecture

$$\phi(t_i)(s_{-i}) = \sum_{t_{-i} \in T_{-i}} \beta(t_i)(t_{-i}, s_{-i})$$

For every $s_{-i} \in S_{-i}$.

Since $t_i$ is in $B ([\Theta]) \cap B ([\psi])$, for every $(t_{-i}, s_{-i})$ such that $\beta(t_i)(t_{-i}, s_{-i}) > 0$ there is a type $t_{i} \in T_{i}$ such that $\varphi_{i} \tilde{t}_{i} = \varphi_{i} \tilde{t}_{i}$ and $s_{-i} = \psi_{-i} (t_{i})$.

Therefore

$$\phi(t_i)(s_{-i}) = \sum_{t_{-i} \in T_{-i}} \beta(t_{i(N)}) (t_{-i}, s_{-i})$$

$$= \sum_{t_{i} \in T_{i} \cap B(t_{i}) \cap \varphi_{i} \tilde{t}_{i} \cap \psi_{i} \tilde{t}_{i}} \beta(t_i)(t_{-i}, \psi(t_{i}))$$

$$= \sum_{t_{i} \in T_{i} \cap \varphi_{i} \tilde{t}_{i} = \varphi_{i} \tilde{t}_{i}} \beta(t_i)(t_{i}) \psi(t_{i})(s_{-i})$$